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ON A LAGUERRE INVERSION

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ABSTRACT

The existence of some automorphisms on certain Laguerre plane L is used to determine the algebraic structure of the projective plane, which is a completion of the affine plane L_A derived at a known point A. In this paper, firstly a special Laguerre automorphism, that is a Laguerre inversion, is defined; and then the set of fixed points and the set of fixed circles of the Laguerre inversion are investigated.

1. INTRODUCTION

A Laguerre plane is a system $(\mathcal{P},\mathcal{C},\in)$ which consists of a nonempty set $\mathcal{P}$ of points, a nonempty set $\mathcal{C}$ of subsets (circles) of $\mathcal{P}$ and $\in$ the set theoretical inclusion satisfying the following axioms [2], [4]:

L.1. For every three pairwise nonparallel points A,B,C, there is a unique circle containing them. This circle is denoted by (ABC).

(Two points $A,B\in\mathcal{P}$ are said to be parallel if and only if A=B or there is no such a circle ç that $A,B\in$ ç. This relation is an equivalance relation on the point set $\mathcal{P}$. Its equivalance classes are called generators. If A and B are parallel points, then we write A//B.)

L.2. For every point there exists a unique generator, containing this point.

L.3. Every circle intersects every generator, containing this point.

L.3. Every circle intersects every generator in exactly one point.

L.4. For each circle ç, each point A on ç, and each point B$\not\parallel$A in L\ç, there is a unique circle ç′ containing A and B such that ç∩ç′ = {A} ç and ç′ are called tangent.

L.5. Every circle ç contains at least three points and L\ç $\neq \emptyset$.

The origin of the Laguerre geometry is the geometry of oriented lines and oriented circles with nonnegative radius of the Euclidean plane [2]. In [1], Benz constructed the following class of Laguerre planes. Let F be an arbitrary field and $V=F^3$ denote the three dimension vector space over F.

Let $\mathcal{O}$ be an oval in the plane $\{(x,y,z)\in V\colon z=0\}$, i.e., is a subset of a projective plane such that i) each line cuts $\mathcal{O}$ in at most two points, and ii) through each point $\mathcal{P}\in\mathcal{O}$ there exists exactly one tangent, i.e. a line intersecting $\mathcal{O}$ in exactly one point. Then $(\mathcal{P},\mathcal{C},\in)$ with $\mathcal{P} = \{(x,y,z)\in V\colon (x,y,0)\in\mathcal{O}\}$ and $\mathcal{C} = \{(x,y,z)\in\mathcal{P}\colon z = ax+by+c,\ a,b,c\in F\}$ is a Laguerre plane in the above sense. Here $\mathcal{P}$ is the set of points of an ovoidal cylinder and a circle is intersection of $\mathcal{P}$ with a plane which is not parallel to the axis of the cylinder (For a general definition, see [4]). In [2], Benz shows that if $\mathcal{O}$ is given by $x^2+y^2 = 1$ then the corresponding Laguerre plane is isomorphic to the parabolic model $(\mathcal{P},\mathcal{C},\in)$ over $\mathcal{R}$ where $\mathcal{P} = \mathcal{R}^2\cup\mathcal{R}$;

$$\mathcal{C} = \left\{\{(x,y)\in\mathcal{R}^2\colon y = ax^2+bx+c\} \cup \{a\}\colon a,b,c,\in\mathcal{R}\right\}$$

and $\mathcal{R}$ is the set of real numbers. In order to obtain some new Laguerre planes, Hartman replaced the parabolas in the above model by some particularly chosen curves [3].

The following theorem explains the main property of the Laguerre planes.

Theorem 1. Let $\overline{X}$ be the generator on the point X. Then the geometrical structure,

$$L_A = (\mathcal{P}\backslash\overline{A},\ \{ç\backslash\{A\}\colon A\in ç,\ ç\in\mathcal{C}\} \cup \{\overline{X}\colon X\in\mathcal{P},\ X\# A\},\ \in)$$

is an affine plane.

This affine plane is called the affine plane derived by the point A of the Laguerre plane, [2].

2. HYPERBOLIC AND PARABOLIC PENCILS

The set of all circles containing nonparallel points A and B is called a hyperbolic pencil and denoted by

$$[A,B] = \{ç: A,B \in ç,\ ç \in \mathcal{C}\}$$

The set of all circles, which are tangent to the circle ç at the given point A is called a parabolic pencil and denoted by [A,ç],

$$[A,ç] = \{ç': ç' \in \mathcal{C},\ ç' \cap ç = \{A\}\} \cup \{ç\},$$

[5].

Definition 1. Let $L = (\mathcal{P},\mathcal{C},\in)$ and $L' = (\mathcal{P}',\mathcal{C}',\in')$ be two Laguerre planes. If there exists a one to one and onto function f, which maps points and circles of L to points and circles of L'; respectively, and if, $A \in ç \Rightarrow f(A) \in f(ç)$ for $\forall A \in \mathcal{P}$ and $\forall ç \in \mathcal{C}$, then these Laguerre planes are said to be isomorphic and such a function f is said to be isomorphism.

If $L = L'$ then the function f is called an automorphism of the plane L, [5]. The automorphism f maps parallel points to parallel points. Therefore an automorphism preserves the generators invariant in whole. Every automorphism f maps a hyperbolic pencil to a hyperbolic pencil and a parabolic pencil to a parabolic pencil, that is,

$$f([A,B]) = [f(A),f(B)]$$
$$f([A,ç]) = [f(A),f(ç)].$$

Restriction of an automorphism f to the affine plane is an automorphism $f|_{L_A}$ of L_A. Consider the affine plane L_A as an embedding into Laguerre plane L. If the restriction $f|_{L_A}$ acts on the points of the affine plane, then all points, which are parallel to A, are invariant under the mapping f.

Definition 2. An automorphism $f \neq I$ of a Laguerre plane is said to be Laguerre Inversion, if for all points $A \in \mathcal{P}$ with $A \neq f(A)$ it follows that A and f(A) are not parallel and all the circles passing through A and f(A) are invariant under f, where I denotes the identity automorphism.

Proposition 1. Every Laguerre inversion is an involution.

Proof. Let $A \neq f(A)$, then $A \nparallel f(A)$. Since the Laguerre inversion f maps every circle of the hyperbolic pencil [A,f(A)] into itself, the points A, f(A) of all circles of the hyperbolic pencil [A,f(A)] are invariant with respect to f. Then $A = f^2(A)$, that is f is an involution.

Proposition 2. Let A and B are two different points of a Laguerre plane L and f is a Laguerre inversion. Then the set {A, f(A), B, f(B)} is either concircular or a subset of union of two generators.

Proof. Let A denote a point which is not fix under f. If some points B are not parallel to A and f(A), then there is a circle ç containing the points A,f(A) and B. Since ç∈[A,f(A)], f(ç) = ç as according to the Definition 2. That is f(B)∈ç and {A,f(A),B,f(B)} consists of the points of one circle. Now, let B//A. Then f(B)//f(A). If B//f(A), then f(B)//f^2(A) = A, and by the Proposition 1, the set {A,f(A),B,f(B)} is the union of two different generators. Finally, if A and B are two invariant points of f, then obviously the set {A,B} is a subset of the union of two generators.

Proposition 3. If a generator $\overline{A}$ is invariant under the Laguerre inversion f, then $\overline{A}$ is pointwise invariant under f.

Proof. Let $\overline{A}$ = f($\overline{A}$). Then every point A of the generator $\overline{A}$ can be transformed to the point f(A), which is parallel to A. According to the Definition 2, f does not change any points of the generator $\overline{A}$. In the case of f(A) = A we have $\overline{f(A)}$ = $\overline{A}$ = f($\overline{A}$). That is the generator $\overline{A}$ is pointwise invariant under f.

Proposition 4. The set of fixed points of a Laguerre inversion is empty or consists of the points of one or two generators.

Proof. Assume that a Laguerre inversion f has the fixed point A. Then, according to Proposition 3, $\overline{A}$ is pointwise invariant under f. If B∉$\overline{A}$ is another fixed point of f, then the generators $\overline{A}$ and $\overline{B}$ ($\overline{B}$≠$\overline{A}$) are pointwise invariant under f. If R∉$\overline{A}$∪$\overline{B}$ and f(R)=R, then the automorphism $f|_{L_R}$ preserves pointwise invariant two different lines of the affine plane L_R. Then $f|_{L_R}$ = $I|_{L_R}$ and f = I. This is a contradiction to the condition of f ≠ I.

Proposition 5. Let f be a Laguerre inversion. The set of all circles, that are invariant under a Laguerre inversion f and include a fixed point A, is a hyperbolic pencil when f(A) ≠ A and a parabolic pencil when f(A) = A.

Proof. Let f(A) ≠ A. According to the Definition 2, the points A and f(A) are supports of the hyperbolic pencil [A,f(A)], which are preserved under f. Inversely, if the circle ç is invariant under f and contains the point A, then for A∈ç we have f(A)∈f(ç)=ç, then ç is an element of hyperbolic pencil [A,f(A)] and a fixed circle of f.

Now, let f(A) = A. In this case there will be two different situations.

a) The generator $\bar{A}$ is unique generator which is pointwise invariant under f. If $B \notin \bar{A}$, then $B \neq f(B)$, that is, if $B \# f(B)$, then A, B and f(B) are not pairwise parallel points. Denote by ç the circle which contains the points A, B and f(B) and define the following set

$$K := \{ ç \in \mathcal{C} : ç = (ABf(B)),\ \forall B \notin \bar{A} \}.$$

K is the set of circles which are invariant under f and covers the set $\mathcal{P} \setminus \bar{A}$. All circles of the set K which $f(B) \neq B$ necessarily contains a point $(B, f(B))$. Suppose that there are two different circles v and w from K, which have another common point $D \neq A$, besides A. Since v and w are invariant under f and A is a fixed point of f, then D is also a fixed point of f. But this contradicts to $D \# A$. Therefore K is a parabolic pencil of fix circles under f with basepoint A.

b) f has two nonparallel fixed points A and D. Let $B \notin \bar{A} \cup \bar{D}$. Then $B \# f(B)$ and the circle $ç = (ABf(B))$ is invariant under f. Let $S := [A, ç]$. Because of $f(A) = A$ and $ç = f(ç)$ the set S is invariant in whole under f. Since each circle $w \in S$ contains the fixed point $\bar{D} \cap w$, then the set S is elementwise invariant. In this case, A carries a parabolic pencil of fixed circles of f. Now let w be a circle, which contains A and invariant under f. Suppose that $[A, ç]$ is a parabolic pencil of fixed circles with carrier point A. We have proved above the existence of such a pencil.

Suppose that $w \notin [A, ç]$ and that $A \in w$, then $|w \cap v| = 2$ for every $v \in [A, ç]$ and $f(w \cap v) = w \cap v$. Therefore, both of elements of $w \cap v$ are also fixed points under f. Since this one is valid for every $v \in [A, ç]$, then w is pointwise invariant under f. According to the Proposition 3 and 4 a Laguerre inversion can not possess three pairwise nonparallel fixed points. Therefore $w \in [A, ç]$ and the Proposition 5 is proved.

REFERENCES

[1] BENZ, W., Pseudo-Ovale und Laguerre Ebenen. Abh. Math. Sem. Univ. Hamburg 27 (1964), 80-84.

[2] BENZ, W., MÄURER, H., Über die Grundlagen der Laguerre-Geometrie. Jahresberg. Deutsch Math. Verein 67 (1964), 14-42.

[3] HARTMAN, E., Eine Klasse nicht einbettbarer reeller Laguerre-Ebenen. Journal of Geometry Vol. 13/1 (1979), 49-67.

[4] KAYA, R., A Class of Ovoidal Laguerre Planes, Communications de la Faculté des Sciences de l'université d'Ankara. Série A1 Mathématiques, Tome 28 (1979), 1-5.

[5] KLEINEWILLINGHÖFER, R., Eine Klassifikation der Laguerre-Ebenen nach L-Streckungen und L-Translationen, Arch. Math. Vol. 34 (1980), 469-480.