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ON THE RECTILINEAR CONGRUENCES GENERATED BY THE INSTANTANEOUS SCREWING AXES CONNECTED WITH A SURFACE

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ABSTRACT

In this paper, the congruences generated by the instantaneous screwing axes of the moving trihedrons (DARBOUX's trihedrons) moving along the lines of curvature on a surface which is referred to its lines of curveture have been investigated.

1. INTRODUCTION

Let $\vec{G}$ denote the instantaneous screwing axis of the moving trihedron $(\vec{x}', \vec{\eta}, \vec{\xi})$ (DARBOUX's trihedron) connected with the point $\vec{x}$ on the surface $\vec{x}$. $\vec{G}$ is given by

$$G = \frac{(\tau_g + \varepsilon)\,\vec{X}_1 - \rho_n\vec{X}_2 + \rho_g\vec{X}_3}{\sqrt{\tau^2{}_g + \rho^2{}_g + \rho^2{}_n + 2\varepsilon\tau_g}} \tag{1.1}$$

[1], where the dual vectors $\vec{X}_i = \vec{x}_i + \varepsilon\vec{x}_{i0}$, $(i = 1, 2, 3)$, $(\varepsilon^2 = 0)$ are the axes of the moving trihedron. The real and the dual part of three vectors are defined, respectively, by $\vec{x}_1 = \vec{x}'$, $\vec{x}_2 = \vec{\eta}$, $\vec{x}_3 = \vec{\xi}$ and $\vec{x}_{10} = \vec{x}'_0$, $\vec{x}_{20} = \vec{\eta}_0$, $\vec{x}_{30} = \vec{\xi}_0$. ρ_n, ρ_g and τ_g are the normal curvature, the geodesic curvature and the geodesic torsion of the curve with the arc length s through that point $\vec{x}$. Let the parametric curves v = const, and u = const, on the surface $\vec{x}$ be its lines of

curvature $(F = M = 0)$. Let $S_1 = S_u/\sqrt{E}$, $S_2 = S_v\sqrt{G}$ be the invariant derivatives of $S(u, v)$ which is an invariant function on the surface $\vec{x}$ ([2], [3]). These derivatives are formed according to the arc length s of the lines of curvature on the surface $\vec{x}$ of S. By means of the invariant derivatives, the moving trihedron (x, η, ξ) is the trihedron $(\vec{x}_1, \vec{x}_2, \vec{\xi})$ belonging to the line of curvature $v =$ const. on the surface $\vec{x}$, where $\vec{x}_1$ and $\vec{x}_2$ are the invariant derivatives of $\vec{x}$. $\vec{\xi}$ is the unit normal vector to the surface $\vec{x}$. Since q, $\bar{q}$ are the geodesic curvature of the lines of curvature $v =$ const., $u =$ const. and r, $\bar{r}$ are the normal curvatures of the lines of curvatures $v =$ const., $u =$ const., respectively, some equations between the invariant quantities q, $\bar{p}$, r, $\bar{r}$ defining the surface $\vec{x}$ and their invariant derivatives and integrability conditions namely GAUSS–CODAZZI equations characterize important surface types [2].

A rectilinear conqruence may be represented analytically by an equation of the form

$$\vec{y} = \vec{x} + t\vec{d}, \quad (\vec{d}^{\,2} = 1) \tag{1.2}$$

where $\vec{x}$ and $\vec{d}$ are functions of two independent parameters u, v. The point $\vec{x}$ may be taken as a point on a surface of reference $\vec{x}$, which is cut by all the lines of the congruence. We may take $\vec{d}$ as a unit vector giving the direction of the line, and t is then the distance from the surface of reference $\vec{x}$ to the moving point $\vec{y}$ on the line [4].

Some equantities between the first fundamental form and the second fundamental form of the spherical representation of the congruence, namely in the KUMMER sens, the function $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ the fundamental coefficients of the first order and the functions, e, f, f′, g the fundamental coefficients of the second order characterize some properties of the congruence and some important congruence type [3].

2. DEFINITION OF THE CONGRUENCES TO BE INVESTIGATED

2.1) Since $\tau_{gu} = \rho_{gu} = -q$, $\rho_{nu} = r$, the insantaneous screwing axis $\vec{G}$ of the trihedron $(\vec{x}_1, \vec{x}_2, \vec{\xi})$ which is moving along the lines of curvature $v =$ const. on the surface $\vec{x}(u, v)$, we find

$$\vec{G} = \frac{\varepsilon\vec{X}_1 - r\vec{X}_2 - q\vec{X}_3}{\sqrt{r^2 + q^2}} \quad (r \neq 0,\ q \neq 0). \tag{2.1}$$

Seperating $\vec{G}$ into real and dual parts, we get

$$\left\{\begin{aligned} G &= -\frac{r\vec{x}_2 + q\vec{\xi}}{\sqrt{r^2 + q^2}} + \varepsilon\frac{\vec{x}_1 - r\vec{x}_{20} - q\vec{\xi}_0}{\sqrt{r^2 + q^2}} = \vec{g} + \varepsilon\vec{g}_0, \\ g &= -\frac{r\vec{x}_2 + \rho\vec{\xi}}{\sqrt{r^2 + q^2}},\ \vec{g}_0 = \frac{\vec{x}_1 - r\vec{x}_{20} - q\vec{\xi}_0}{\sqrt{r^2 + g^2}} . \end{aligned}\right. \tag{2.1'}$$

The congruence generated by $\vec{G}$ is expressed as

$$\vec{y} = \vec{r} + t\vec{g}, \quad (\vec{g} = 1) \tag{2.2}$$

form (1.2), where $\vec{r}$ is the surface of reference. Using the properties of the line $\vec{G}$ given by (2.1), that it cuts the $\vec{X}_3$ axis and is perpendicular to $\vec{X}_1$, the displacement vector of the intersection point on $\vec{X}_3$ is found as a result of dual and real calculations as

$$\vec{r} = \vec{x} + \frac{1}{r}\vec{\xi}. \tag{2.3}$$

Therefore, the instantaneous screwing axis $\vec{G}$ of the moving trihedron $(\vec{x}_1, \vec{x}_2, \vec{\xi})$ which moves along the line of curvature $v =$ const.

on the surface $\vec{x}$, cuts the surface normal at the center point corresponding to the line of curvature $v =$ const. From this, it can be seen that the reference surface $\vec{r}$ is the center surface belonging to the line of curvature $v =$ const. on the surface $\vec{x}$. Therefore, the instantaneous screwing axis $\vec{G}$ is tangent to the center surface $\vec{r}$ of $\vec{x}$.

The vectorial equation of the congruence which will be investigated according to the considerations above, may be stated as

$$\vec{y} = \vec{y}\,(u, v, t) = \vec{x}\,(u, v) + \frac{1}{r\,(u, v)}\,\vec{\xi}\,(u, v)$$

$$-\, t\,\frac{r\,(u, v)\,\vec{x}_2\,(u, r) + q\,(u, v)\,\vec{\xi}\,(u, v)}{\sqrt{r^2(u, v) + q^2(u, v)}}\,.$$

2.2) Since $\tau_{g_v} = o$, $\rho_{g_v} = \bar{q}$, $\rho_{nv} = \bar{r}$, the instantaneous screwing axis $\vec{G}$ of the Thrihedron $(\vec{x}_2, -\vec{x}_1, \vec{\xi})$ moving along the lines of curvature $u =$ const. on the surface $\vec{x}\,(u, v)$, we find

$$\vec{G} = \frac{\varepsilon\vec{X}_2 + \bar{r}\vec{X}_1 + \bar{q}\vec{X}_3}{\sqrt{\bar{r}^2 + \bar{q}^2}}\,, \quad (\bar{r} \neq 0,\ \bar{q} = 0).$$

Seperating $\vec{G}$ into its real and dual parts, we get

$$\left\{\begin{aligned} \vec{\bar{G}} &= \frac{\bar{r}\vec{x}_1 + \bar{q}\vec{\xi}}{\sqrt{\bar{r}^2 + \bar{q}^2}} + \varepsilon\,\frac{\vec{x}_2 + \bar{r}\vec{x}_{10} + \bar{q}\vec{\xi}_0}{\sqrt{\bar{r}^2 + \bar{q}^2}} = \vec{\bar{g}} + \varepsilon\vec{\bar{g}}_0 \\ \vec{\bar{g}} &= \frac{\bar{r}\vec{x}_1 + \bar{q}\vec{\xi}}{\sqrt{\bar{r}^2 + \bar{q}^2}}\,,\ \vec{\bar{g}}_0 = \frac{\vec{x}_2 + \vec{x}_{10}\,\bar{q}\vec{\xi}_0}{\sqrt{\bar{r}^2 + \bar{q}^2}}\,. \end{aligned}\right. \tag{2.4}$$

The congruence generated by $\vec{G}$ is expressed as

$$\vec{y} = \vec{\bar{r}} + t\vec{\bar{g}}, \quad (\vec{\bar{g}}^2 = 1). \tag{2.5}$$

Similarly, the reference surface $\vec{\bar{r}}$ is the centers surface belonging to the lines of curvature u = const. on the surface $\vec{x}$, that is

$$\vec{\bar{r}} = \vec{x} + \frac{1}{\bar{r}} \vec{\xi}. \tag{32.6}$$

Therefore, the equation of the conguence $\vec{\bar{y}}$ may be expressed as

$$\vec{\bar{y}} = \vec{\bar{y}}(u, v, t) = \vec{x}(u, v) + \frac{1}{\bar{r}(u, v)} \vec{\xi}(u, v) +$$

$$\bar{t} \frac{\bar{r}(u, v) \vec{x}_1(u, v) + \bar{q}(u, v) \vec{\xi}(u, v)}{\sqrt{\bar{r}^2(u, v) + q^2(u, v)}}.$$

3. PROPERTIES OF THE CONGRUENCES $\vec{y}$, $\vec{\bar{y}}$

3.1) In order to investigate the properties of the congruence $\vec{y}$, at first, its first and second fundamental forms, in the KUMMER sense, are calculated

$$\begin{cases} \mathcal{E} = \vec{g}_u^{\,2} = E\vec{g}_1^{\,2} = E\left(\dfrac{r_1 q - q_1 r}{r^2 + q^2}\right)^2 \\ \mathcal{F} = <\vec{g}_u, \vec{g}_v> = \sqrt{EG} <\vec{g}_1, \vec{g}_2> = \sqrt{EG}\, \dfrac{r(r_2 q - q_1 r)(\bar{q}_1 + \bar{q}^2)}{(r^2 + q^2)^2} \\ \mathcal{G} = \vec{g}_v^{\,2} = G\vec{g}_2^{\,2} = G\, \dfrac{r^2 [\bar{q}^2 (r^2 + q^2) + (\bar{q}_1 + \bar{q}^2)^2]}{(r^2 + q^2)^2}, \end{cases} \tag{3.1}$$

$$d\sigma^2 = (d\vec{g})^2 = E\left(\frac{r_1 q - q_1 r}{r^2 + q^2}\right)^2 du^2 +$$

$$2\sqrt{EG}\, \frac{r(r_1 q - q_1 r)(\bar{q}_1 + \bar{q}_2)}{(r^2 + q^2)^2} du dv$$

$$+ \frac{r^2 [\bar{q}^2 (r^2 + q^2) + (\bar{q}_1 + \bar{q}^2)^2]}{(r^2 + q^2)^2} dv^2 = [I] \tag{3.2}$$

and

$$\begin{cases} e = <\vec{r}_u, \vec{g}_u> = E<\vec{r}_1 \cdot \vec{g}_1> = E \dfrac{r_1(q_1 r - r_1 q)}{r(r^2+q^2)^{3/2}}, f = <\vec{r}_v, \vec{g}_u> = \\ \sqrt{EG} <\vec{r}_2, \vec{\xi}_1> = 0 \\ f' = <\vec{r}_u, \vec{g}_v> = \sqrt{EG} <\vec{r}_1, \vec{g}_2> = -\sqrt{EG} \dfrac{r_1(\bar{q}_1 + \bar{q}^2)}{(r^2+q^2)^{3/2}}, \\ g = <\vec{r}_v, \vec{g}_v> = G<\vec{r}_2, \vec{g}_2> = 0, \end{cases} \tag{3.3}$$

$$<\vec{dr}, \vec{dg}> = E \frac{r_1(q_1 - r_1 q)}{r(r^2+q^2)^{3/2}} du^2 - \sqrt{EG} \frac{r_1(\bar{q}_1 + \bar{q})}{(r^2+q^2)^{3/2}} dudv = (II] \tag{3.4}$$

are found.

3.2) To investigate the properties of the congruence $\vec{\bar{y}}$, similarly, its first and second fundamental forms, in the KUMMER sens, are

$$\begin{cases} \bar{\mathscr{E}} = \vec{\bar{g}}_u^2 = E\vec{\bar{g}}_1^2 = E \dfrac{\bar{r}^2 [q^2(\bar{r}^2 + \bar{q}^2) + (q_2 + q^2)^2]}{(\bar{r}^2 + \bar{q}^2)^2} \\ \bar{\mathscr{F}} = <\vec{\bar{g}}_u, \vec{\bar{g}}_v> = \sqrt{EG} <\vec{\bar{g}}_1, \vec{\bar{g}}_2> = \sqrt{EG} \dfrac{\bar{r}(\bar{q}_2 \bar{r} - \bar{r}_2 \bar{q})(q_2 + q^2)}{(\bar{r}^2 + q^2)^2} \\ \bar{\mathscr{G}} = \vec{\bar{g}}_v^2 = G\vec{\bar{g}}_2^2 = G \left(\dfrac{\bar{q}_2 \bar{r} - \bar{r}_2 \bar{q}}{\bar{r}^2 + \bar{q}^2} \right)^2, \end{cases} \tag{3.5}$$

$$d\xi^2 = d\vec{\bar{g}}^2 = E \frac{\bar{r}^2 [q^2(\bar{r}^2 + \bar{q}^2) + (q_2 + q^2)^2]}{(\bar{r}^2 + \bar{q}^2)^2} du^2 +$$

$$+ 2\sqrt{EG} \frac{\bar{r}(\bar{q}_2 \bar{r} - \bar{r}_2 \bar{q})(q_2 + q^2)}{(\bar{r}^2 + \bar{q}^2)^2} dudv +$$

$$G \left(\frac{\bar{q}_2 \bar{r} - \bar{r}_2 \bar{g}}{\bar{r}^2 + \bar{q}^2} \right)^2 dv^2 = [I] \tag{3.6}$$

and

$$\begin{cases} \bar{e} = <\vec{\bar{r}}_u, \vec{\bar{g}}_u> = E<\vec{\bar{r}}_1, \vec{\bar{g}}_1> = 0,\ \bar{f} = <\vec{\bar{r}}_v, \vec{\bar{g}}_u> = \sqrt{EG}\ <\vec{\bar{r}}_2, \vec{\bar{g}}_1> \\ = -\sqrt{EG}\ \dfrac{\bar{r}_2\,(q_2 + q^2)}{(\bar{r}^2 + \bar{q}^2)^{3/2}} \\ \bar{f}' = <\vec{\bar{r}}_u, \vec{\bar{g}}_v> = \sqrt{EG}\ <\vec{\bar{r}}_1, \vec{\bar{g}}_2> = 0,\ \bar{g} = <\vec{\bar{r}}_v, \vec{\bar{g}}_v> = G<\vec{\bar{r}}_2, \vec{\bar{g}}_2> \\ = -G\ \dfrac{\bar{r}_2\,(\bar{q}_2\bar{r} - \bar{r}_2\bar{q})}{\bar{r}\,(\bar{r}^2 + \bar{q}^2)^{3/2}} \end{cases} \tag{3.7}$$

$$<d\vec{\bar{r}}, d\vec{\bar{g}}> = \sqrt{EG}\ \frac{\bar{r}_2\,(q_2 + q^2)}{(\bar{r}^2 + \bar{q}^2)^{3/2}}\ dudv - G\ \frac{\bar{r}_2\,(\bar{q}_2\bar{r} - \bar{r}_2\bar{q})}{\bar{r}\,(\bar{r}^2 + \bar{q}^2)^{3/2}}\ dv^2$$

$$= [II]. \tag{3.8}$$

The other properties of the congruences $\vec{y}$ and $\vec{\bar{y}}$ may be listed as: a.1) For $\mathcal{H}$

$$\mathcal{H}^2 = EG\ \frac{r^2\bar{q}^2\,(r_1q - q_1r)^2}{(r^2 + q^2)^3}\ , \tag{3.9}$$

and

2) For $\overline{\mathcal{H}}$

$$\overline{\mathcal{H}}^2 = EG\ \frac{\bar{r}^2q^2\,(\bar{q}_2\bar{r} - \bar{r}_2\bar{q})^2}{(\bar{r}^2 + \bar{q}^2)^3} \tag{3.10}$$

are found.

b.1) The limit points of the line of the congruence $\vec{y}$

$$\begin{cases} l_I = \dfrac{r_1}{2r\bar{q}\,(r_1q - q_1r)}\ [\bar{q}\sqrt{r^2 + q^2} + \sqrt{\bar{q}^2(r^2 + q^2) + (\bar{q}_1 + \bar{q}^2)^2}] \\ l_{II} = \dfrac{r_1}{2r\bar{q}\,(r_1q - q_1r)}\ [\bar{q}\sqrt{r^2 + q^2} - \sqrt{\bar{q}^2(r^2 + q^2) + (\bar{q}_1 + \bar{q}^2)^2}] \end{cases} \tag{3.11}$$

are found.

2) The limit points of the line of the congruence $\vec{\bar{y}}$ are

$$\left\{\begin{aligned} \bar{I}_I &= \frac{\bar{r}_2}{2\bar{r}q\,(q_2\bar{r}-\bar{q}\bar{r}_2)}\,\left[q\sqrt{\bar{r}^2+\bar{q}^2}+\sqrt{q^2(\bar{r}^2+\bar{q}^2)+(q_2+q^2)^2}\right] \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.12) \\ \bar{I}_{II} &= \frac{\bar{r}_2}{2\bar{r}q\,(\bar{q}_2\bar{r}-\bar{q}\bar{r}_2)}\,\left[q\sqrt{\bar{r}^2+\bar{q}^2}-\sqrt{q^2(\bar{r}^2+q^2)+(q_2+q^2)^2}\right]. \end{aligned}\right.$$

c.1) The principal surfaces of the congruence $\vec{y}$

$$\sqrt{EG}\,\frac{\bar{r}_1}{2\,(r^2+q^2)^{7/2}}\,\{E\,(\bar{q}_1+\bar{q}^2)\,(r_1q-q_1r)^2du^2+$$

$$+\,2\sqrt{EG}\,r\,(r_1q-q_1r)\,[\bar{q}^2\,(r^2+q^2)+(\bar{q}_1+\bar{q}^2)^2]\,dudv+$$

$$+\,Gr^2\,(\bar{q}_1+\bar{q}^2)\,[\bar{q}^2\,(r^2+q^2)+(\bar{q}_1+\bar{q}^2)^2]\,dv^2\}=0 \qquad (3.13)$$

are found as the solutions of the differantial equation.

2) The principal surfaces of the congruence $\vec{\bar{y}}$ are

$$\sqrt{EG}\,\frac{\bar{r}_1}{2\,(\bar{r}^2+\bar{q}^2)^{7/2}}\,\{E\,\bar{r}^2\,(q_2+q^2)\,[q^2\,(\bar{r}^2+\bar{q}^2)+ \qquad (3.14)$$

$$+\,(q_2+q^2)^2]\,du^2+2\sqrt{EG}\,\bar{r}\,(\bar{q}_2\bar{r}-\bar{r}_2\bar{q})\,[q^2\,(\bar{r}^2+\bar{q}^2)+$$

$$+\,(q_2+q^2)^2]\,dudv+G\,(q_2+q^2)\,(\bar{q}_2\bar{r}-\bar{r}_2\bar{q})^2dv^2\}=0$$

d.1) The developable surfaces of the congruence $\vec{y}$

$$EG\,\frac{rr_1\bar{q}^2\,(r_2q-q_1r)}{(r^2+q^2)^{5/2}}\,dudv=0 \qquad (3.15)$$

are found as the solutions of the differential equation.

2) The developable surface of the congruences $\vec{\bar{y}}$ are

$$EG\,\frac{\bar{r}\bar{r}_2q^2\,(\bar{q}_2\bar{r}-\bar{r}_2\bar{q})}{(\bar{r}^2+\bar{q}^2)^{5/2}}\,dudv=0. \qquad (3.16)$$

e.1) The focal points for the line of the congruence $\vec{y}$

$$\rho_I = \frac{r_1\sqrt{r^2+q^2}}{r(r_1q-q_1r)}, \quad \rho_{II} = 0 \tag{3.17}$$

are found.

2) The folcal points for the line of the congruence $\vec{\bar{y}}$ are

$$\bar{\rho}_I = \frac{\bar{r}_2\sqrt{\bar{r}^2+\bar{q}^2}}{\bar{r}(\bar{r}q_2-\bar{q}\bar{r}_2)}, \quad \bar{\rho}_{II} = 0. \tag{3.18}$$

f.1) The middle point of the line of the congruence $\vec{y}$

$$m = \frac{r_1\sqrt{r^2+q^2}}{2r(r_1q-q_1r)} \tag{3.19}$$

is found.

2) The middle point of the line of the congruence $\vec{\bar{y}}$ is

$$m = \frac{\bar{r}_2\sqrt{\bar{r}^2+\bar{q}^2}}{2\bar{r}(\bar{r}\bar{q}_2-\bar{q}\bar{r}_2)}. \tag{3.20}$$

g.1) The middle surface of the congruence $\vec{y}$

$$\vec{m} = \vec{r} + \frac{\rho}{2}\vec{g}, \quad (p_I - p_{II} = \rho) \tag{3.21}$$

is obtained.

2) The middle surface of the congruence $\vec{\bar{y}}$ is

$$\vec{\bar{m}} = \vec{\bar{r}} + \frac{\rho}{2}\vec{\bar{g}}, \quad (\bar{\rho}_I - \bar{\rho}_{II} = \bar{\rho}). \tag{3.22}$$

h.1) Let the middle surface $\vec{m}$ of the congruence $\vec{y}$ be the reference surface. Its first local surface generated by ρ_I

$$\vec{k} = \vec{m} + \frac{\rho}{2}\cdot\vec{g} = \vec{r} + \rho\vec{g} = \vec{r} + \frac{r_1\sqrt{r^2+q^2}}{r(r_1q-q_1r)}\vec{g} \tag{3.23}$$

and its second focal surface generated by ρ_{II}

$$\vec{p} = \vec{m} - \frac{\rho}{2}\vec{g} = \vec{r} = \vec{x} + \frac{1}{r}\vec{\xi} \tag{3.24}$$

are found.

2) Let the middle surface $\vec{m}$ of the congruence $\vec{y}$ be the reference surface. Its first focal surface generated by $\bar{\rho}_I$

$$\vec{\bar{k}} = \vec{\bar{m}} + \frac{\bar{\rho}}{2}\vec{\bar{g}} = \vec{\bar{r}} + \bar{\rho}\vec{\bar{g}} = \vec{\bar{r}} + \frac{\bar{r}_2\sqrt{\bar{r}^2 + q^2}}{\bar{r}(\bar{r}\bar{q}_2 - \bar{q}\bar{r}_2)}\vec{\bar{g}} \tag{3.25}$$

and its second focal surface generated by $\bar{\rho}_{II}$ are

$$\vec{\bar{p}} = \vec{\bar{m}} = \frac{\bar{\rho}}{2}\vec{\bar{g}} = \vec{\bar{r}} = \vec{\bar{x}} + \frac{1}{\bar{r}}\vec{\xi}. \tag{3.26}$$

i.1) Considering (3.1) and (3.3), the mean ruled surfaces of the congruence $\vec{y}$ are found as the solutions of the differential equations

$$r_1 r\bar{q}(r_1 q - q_1 r)\ \{E(r_1 q - q_1 r)^2 du^2 - \tag{3.27}$$
$$- Gr^2[\bar{q}^2(r^2 + q^2) + (\bar{q}_1 + \bar{q}^2)^2]dv^2\} = 0.$$

2) Considering (3.5) and (3.7), the mean ruled surfaces of the congruence $\vec{\bar{y}}$ are found as the solutions of the differential equations

$$\bar{r}_2\bar{r}\bar{q}(\bar{r}\bar{q}_2 - \bar{q}\bar{r}_2)\ \{E\bar{r}^2[q^2(\bar{r}^2 + \bar{q}^2) + (q_2 + q^2)^2]du^2 -$$
$$- G(\bar{r}\bar{q}_2 - \bar{q}\bar{r}_2)^2 dv^2\} = 0. \tag{3.28}$$

The surface $\vec{x}$, chosen to be the families of the lines of curvature as parameter curves, should obey $EG \neq 0$ at each point to become regular.

Therefore, from the cylindrical conditions of the congruences $\vec{y}$ and $\vec{\bar{y}}$ or $\mathcal{H} = 0$ and $\overline{\mathcal{H}} = 0$ give respectively from (3.9) and (3.10)

$$r = 0,\ \bar{q} = 0, \quad r_1 q - q_1 r = 0$$

and

$$\bar{r} = 0, \quad q = 0, \quad \bar{q}_2\bar{r} - \bar{r}_2\bar{q} = 0.$$

Here, the conditions $r = 0$, $\bar{r} = 0$ infer the surface $\vec{x}$ to be developable surface, the conditions $q = 0$, $\bar{q} = 0$ infer the surface x to be a Mulür surface, the conditions $r_1q - q_1r = 0$, $\bar{q}_2\bar{r} - \bar{r}_2\bar{q} = 0$ infer the surface $\vec{x}$ to have these families of the lines of curvature $v =$ const. and $u =$ const. formed from plane curves. Whereas, since one of the families of lines of curvature of a developable surface is generator of a surface and since the trihedron $(\vec{x}_1, \vec{x}_2, \xi)$ or $(\vec{x}_2, -\vec{x}_1, \vec{\xi})$ is only displaced along the generator, the instantenous screwing axis does not exist. Consequently, $\vec{G}$ and similarly $\vec{\bar{G}}$ are not defined. Therefore the surface $\vec{x}$ cannot be developable surface, that is $r \neq 0$ and $\bar{r} \neq 0$. (In (2.1) and (2.4), we have already taken $r \neq 0$ and $\bar{r} \neq 0$). Since there families of the lines of curvature will also be plane curves, the surface $\vec{x}$ cannot be the surface which have the lines of curvature $v =$ const., $u =$ const. consicting of plane curves, that is $r_1q - q_1r \neq 0$, $\bar{r}_2\bar{q} - \bar{q}_2\bar{r} \neq 0$. On the other hand, since one of the families of the lines of curvature on Mulür surfaces is plane curves, the surface $\vec{x}$ cannot be a Mulür surface, that is $q \neq 0$, $\bar{q} \neq 0$. (In (2.1) and (2.4) we have alredady taken $q \neq 0$ and $\bar{q} \neq 0$). Therefore, we may stade the theorem below:

3.1. Theorem

The congruence $\vec{y}$ or $\vec{\bar{y}}$ generated by the instantaneous screwing $\vec{G}$ or $\vec{\bar{G}}$ during the motion of the moving trihedrons along the families of the lines of curvature $v =$ const., $u =$ const. on the surface $\vec{x}\,(u, v)$ which cannot be developable surface which cannot have the lines of curvatures consisting of plane curves, which cannot be Mulür surface, cannot be cylindrical congruences. In order to make the congruence $\vec{y}$ and $\vec{\bar{y}}$ normal congruences, from (3.3) and (3.7) we find

$$r_1 (\bar{q}_1 + \bar{q}^2) = 0, \quad (\bar{q} \neq 0)$$

and

$$\bar{r}_2 (q_2 + q^2) = 0, \quad (q \neq 0).$$

Here the conditions $r_1 = 0$, $\bar{r}_2 \neq 0$ or $\bar{r}_2 = 0$, $r_1 \neq 0$ denote that the surface $\vec{x}$ is a canal surface.

The families of the lines of curvature of the canal surfaces are planar curves since they are circular. Furthermore the central surfaces of the canal surfaces, (2.3) and (2.6), become curves. Since there curves are also the reference surfaces of the congurences $\vec{y}$ and $\vec{\bar{y}}$, ruled surfaces generate instead of the congruences $\vec{y}$ and $\vec{\bar{y}}$, congruences are not generate. The conditions $\bar{q}_1 + \bar{q}^2 = 0$, $(\bar{q} \neq 0)$ or $q_2 + q^2 = 0$, $(q \neq 0)$ on the other hand, indicate that for the congruences $\vec{y}$ or $\vec{\bar{y}}$, their limit points (3.11) or (3.12) coincide with their focal points (3.17) or (3.18) and that their principal surfaces (3.13) or (3.14) are developable (3.15) or (3.16). Thus the theorem bellow .s obtained.

3.2. **Theorem**

The necessary and sufficient conditions that the congruences $\vec{y}$ and $\vec{\bar{y}}$ generated by the instantaneous screwing axes $\vec{G}$ and $\vec{\bar{G}}$ of the trihedrons connected with the families of the lines of curvature $v =$ const., $u =$ const, on a surface $\vec{x}(u, v)$ which is not a canal surface and Mulür surface, can be normal congruences are

$$\bar{q}_2 + \bar{q}^2 = 0, \quad (\bar{q} \neq 0)$$

and

$$q_1 + q^2 = 0, \quad (q \neq 0).$$

The necessary and sufficient conditions that the parametric surfaces of the congruences $\vec{y}$ and $\vec{\bar{y}}$ be their principal surfaces are, respectively, considering (3.1), (3.3) and (3.5), (3.7),

$$r\,(r_1q - q_1r)\,(\bar{q}_1 + \bar{q}^2) = 0, \qquad r_1(\bar{q}_1 + \bar{q}^2) = 0$$

and

$$\bar{r}(\bar{q}_2\bar{r} - \bar{q}\bar{r}_2)\,(q_2 + q^2) = 0, \qquad \bar{r}_2(q_2 + q^2) = 0—$$

also using $r \neq 0$, $r_1 \neq 0$, $r_1q - q_1r \neq 0$ and $\bar{r} \neq 0$, $\bar{r}_2 \neq 0$, $\bar{q}_2\bar{r} - \bar{r}_2q \neq 0$ we find

$$\bar{q}_1 + \bar{q}^2 = 0, \qquad (\bar{q} \neq 0)$$

and

$$\bar{q}_2 + \bar{q}^2 = 0, \qquad (\bar{q} \neq 0).$$

The conditions indicate that the congruences $\vec{y}$ and $\vec{\bar{y}}$ are normal congruences and also that their parametric surfaces are their principal surfaces. Since these congruences are normal, their principal surfaces are developable. Therefore the following theorem may be stated:

3.3. **Theorem**

The principal surfaces of the normal congruences $\vec{y}$ and $\vec{\bar{y}}$, which are developable are their parametric suffaces.

If the congruences $\vec{y}$ and $\vec{\bar{y}}$ are normal conguences, their mean ruled surfaces (3.27) and (3.28) may be written as

$$E\;du^2 - G\;dv^2 = 0$$

and

$$\bar{E}\;du^2 - \bar{G}\;dv^2 = 0.$$

Considering (3.3) and (3.7), to make the congruences $\vec{y}$ and $\vec{\bar{y}}$ isotrpîic, we find consecutivly

$$r_1\,(q_1r - r_1q) = 0, \qquad r_1\,(\bar{q}_1 + \bar{q}^2) = 0$$

and

$$\bar{r}_2\,(\bar{q}_2\bar{r} - \bar{r}_2\bar{q}) = 0, \qquad \bar{r}_2\,(q_2 + q^2) = 0.$$

Here, we get $r_1 = 0$ and $\bar{r}_2 = 0$ as the common solution. On the contrary $\bar{r}_1 \neq 0$, $\bar{r}_2 \neq 0$ according to Theorem (3.2). Hence forth the following theorem is obtained.

3.4. Theorem

The congruences $\vec{y}$ and $\vec{\bar{y}}$ generated by the instantaneous screwing axes of the moving trihedrons along the lines of curvatures v = const. and u = const. on the surface $\vec{x}$ (u, v) which is not a canal surface, can not be izotropic congruences.

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