Commun. Fac. Sci. Univ. Ank. Series A_1
V. 44. pp. 25-33 (1995)

MATRIX TRANSFORMATIONS OF SOME SEQUENCE SPACES OVER NON-ARCHIMEDIAN FIELDS

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(Received Aug. 1, 1994; Accepted April 25, 1995)

ABSTRACT

Defining the sequence spaces $C_o(F)$, $C(F)$, $m(F)$, $l_p(F)$; $(p > 1)$ and $\chi(F)$ over a field F with non-trivial non-archimedian valuation, inclusion theorems have been established for an infinite matrix defined over a field F to transform (i) l_p (F) into χ (F), (ii) V into χ (F) where V is either C_o (F) or C (F) or m (F), and (iii) χ (F) into m (F).

1. INTRODUCTION

Inclusion theorems on matrix transformations of sequence spaces deal with finding necessary and sufficient conditions for an infinite matrix to transform one sequence space into the same or another sequence space. In all such theorems we usually restrict ourselves to the sequences and series composed of real or complex entries. In this paper, replacing the field of scalars into a field with non-trivial, non-archimedian valuation, we shall establish some inclusion theorems of matrix transformations of some sequence spaces which are not studied by authors like Somasundaram. [4, 5, 6]

§ 2 deals with pre-requisites containing the definitions of the relevant sequences paces, some of their properties, proofs of some theorems and a known result quoted as Lemma, which will be used in § 3 to prove our main results.

2. PRE-REQUISITES

Let F be a non-trivial, non-archimedian field which is complete under the metric of valuation. If $x = (x_k) = (x_1, x_2, \ldots x_k, \ldots)$, $x_k \in F$ is a sequence defined over F, this assumption ensures not only the completion of the sequence spaces we consider but also the absolute convergence of a series in F implies convergence in F. In what follows Σx_k denote $\sum_{k=1}^{\infty} x_k$ and the notion of convergence and boundedness will be in relation to the metric of valuation of the field.

Let us list the relevant sequence spaces as follows:

$C_o(F)$: The set of all null sequences $x = (x_k)$

$C(F)$: The set of all convergence sequences $x = (x_k)$

$m(F)$: The set of all bounded sequences $x = (x_k)$

$l_p(F)$: $\{x = (x_k): \Sigma |x_k|^p$ is convergent, $p > 1\}$

$\chi(F)$: $\{x = (x_k): (k! |x_k|)^{1/k} \to 0$ as $k \to \infty\}$

Remark: $\chi(F)$ can be regarded as the collection of all entire functions $f(z) = \Sigma x_k z^k$ of exponential order 1 and type 0 (Sirajudeen[2]).

$C_o(F)$, $C(F)$ and $m(F)$ are non-archimedian Banach spaces with non-archimedian norm, $\|x\| = \sup_k |x_k|$. If $x = (x_k)$ is an element of χ (F) then $|x| = \sup \{(k! |x_k|^{1/k}, k \geq 1\}$ satisfies the following conditions.

i) $|x| > 0$, $|x| = 0$ if and only if $x = (0, 0 \ldots)$ where 0 is the zero element of the field F.

ii) $|x + y| \leq$ Max $\{|x|, |y|\}$.

iii) $|tx| \leq A(t) |x|$, $t \in F$, $A(t) = \max \{1, |t|\}$.

Hence $\chi(F)$ is a metric space defined over F with a metric d (x, y) $= |x-y|$.

If X is a complete metric space over F, then a continuous linear functional is a continuous linear operator on X with values belonging to the field F. Then as in the archimedian case, we can establish the following theorems.

Theorem 1. $\chi(F)$ is a complete linear metric space over the non-archimedian field F.

Theorem 2. Every continuous linear functional f(x) defined for $x \in \chi$ (F) is of the form $f(x) = \Sigma c_n x_n$, $x = (x_n)$ where $\left(\frac{1}{\eta!} |C_n|\right)^{1/n}$ is a bounded sequence.

Now let us quote a known result as the following Lemma.

Lemma: [Theorem 3 in Somasundaram]

Let $T_n(x)$ be a sequence of continuous linear functionals defined on a complete linear metric space E over F. Let $\lim |T_n(x)| < \infty$ for each $x \in E$. Then there exists a fixed number M and a closed sphere $S \subset E$ such that $|T_n(x)| < M$ for all $x \in S$ and for all $n > 1$.

$\| x \| = (\Sigma |x_k|^p)^{1/p}$ is evidently a non-archimedian norm in the sense that, it satisfies the stronger form of triangular inequality $\| x + y \| \leq \text{Max} \{\| x \|, \| y \|\}$. With this as norm as in the archimedian case, we can establish the following theorem.

Theorem 3:

i) $l_p(F)$, $p > 1$ is a non-archimedian Banach space.

ii) If $p > 1$, so that $p^{-1} + q^{-1} = 1$ and $\Sigma a_k x_k$ converges for every $x = (x_k) \in l_p(F)$, then $\Sigma |a_k|^q$ is convergent.

3. MAIN RESULTS

Let (X, Y) denote the set of all matrices $A = (a_{nk})$, $n, k = 1, 2, \ldots$ that transform a sequence $x = (x_k) \in X$ into a sequence $A(x) = (A_n(X)) = y = (y_n) \in Y$ defined by

$$Y_n = \Sigma a_{nk} x_k, \; n = 1, 2, 3, \ldots, \text{ and } a_{nk} \in F.$$

Theorem 4:

When $p > 1$ and $p^{-1} + q^{-1} = 1$, $A \in (l_p(F), \chi(F))$ if and only if $\sup_{1 \leq k < \infty} (n! |a_{nk}|^q)^{1/n} \to 0$ as $n \to \infty$ (1)

Proof: Sufficiency:

Let $(x_k) \in l_p(F)$ and (1) holds so that $\Sigma |x_k|^p$ converges, converging to L (say). Then

$$(n! |y_n|)^{1/n} = (n! |\Sigma a_{nk} x_k|)^{1/n}$$

$$\leq (n! \Sigma |a_{nk}|^q)^{1/nq} (n! \Sigma |x_k|^p)^{1/np}$$

(by Hölder's inequality)

$$\leq \sup_{1 \leq k < \infty} (n! |a_{nk}|^q)^{1/nq} (n! L)^{1/np}$$

$$\leq \sup_{1 \leq k < \infty} (n! \mid a_{nk} \mid^q)^{1/n} (n! \, L)^{1/n}.$$

Hence using (1), we get $(\eta! \mid y_n \mid)^{1/n} \to 0$ as $n \to \infty$ so that $(y_n) \in \chi(F)$.

Necessity: Let $A \in (l_p(F), \chi(F))$. If condition (1) does not hold, then for some $\varepsilon > 0$, there exists subsequences of n, such that $\sup_{1 \leq k < \infty} (n! \mid a_{nk} \mid^q)^{1/n} > \varepsilon$ for sufficiently large n. (2)

Since $y_n = \Sigma a_{nk} x_k$ is defined for all $(x_k) \in l_p(F)$, from Theorem 3 (ii), $\Sigma \mid a_{nk} \mid^q$ is convergent, so that we have $\mid a_{nk} \mid^q \to 0$ as $k \to \infty$ for every fixed n.

Hence we have $(n! \mid a_{nk} \mid^q)^{1/n} \to 0$ as $k \to \infty$ for every fixed n. (3)

Since $(0, 0, \ldots 1, 0 \ldots)$, 1, the identity element of the field F in the k^{th} place, is a sequence belonging to $l_p(F)$,

$(y_n) = (a_{nk}) \in \chi(F)$ gives

$(n! \mid a_{nk} \mid^q)^{1/n} \to 0$ as $n \to \infty$ for every fixed k, so that

$(n! \mid a_{nk} \mid^q)^{1/n} < \frac{\varepsilon}{2}$ for $n > n_k$ for every fixed k. (4)

Now we shall construct a sequence $(x_k) \in l_p(F)$ and prove that the corresponding $(y_n) \notin \chi(F)$ using (2), (3) and (4). Then that will suffice to prove the necessity of the condition (1).

By (2), first choose n_1 for n such that

$$\sup_{1 \leq k < \infty} (n_1! \mid a_{n_1 k} \mid^q)^{1/n_1} > \varepsilon \tag{5}$$

Having fixed an n_1, by (3) we can choose a k_{n_1} for k such that

$$\sup_{k_{n_1} + 1 \leq k < \infty} (n_1! \mid a_{n_1 k} \mid^q)^{1/n_1} < \frac{\varepsilon}{2} \tag{6}$$

Hence from (5) and (6) we get $\sup_{1 \leq k < k_{n_1}} (n_1! \mid a_{n_1 k} \mid^q)^{1/n_1} > \varepsilon$

Therefore there is a k_1, $1 \leq k_1 \leq k_{n_1}$ such that

$$(n_1! \mid a_{n_1 k_1} \mid^q)^{1/n_1} > \varepsilon \tag{7}$$

Next by (2) and (4) choose $n_2 > n_1$ such that

$$\sup_{1 \leq k < \infty} (n_2! \mid a_{n_2 k} \mid^q)^{1/n_2} > \varepsilon \tag{8}$$

and

$$\sup_{1 \leq k < k_{n_1}} (n_2! \mid a_{n_2 k} \mid^q)^{1/n_2} > \varepsilon / 2 \tag{9}$$

This is possible if n_2 is large enough that $n_2 > \max (n_k)$ when $1 \leq k \leq k_{n_1}$ defined in (4).

Having chosen an n_2 by (3), there exists a $k_{n_2} > k_{n_1}$ such that

$$\sup_{k_{n_2} + 1 \leq k < \infty} (n_2! \mid a_{n_2 k} \mid^q)^{1/n_2} < \varepsilon / 2 \tag{10}$$

Now from (8) and (10) we get $\sup_{1 \leq k < k_{n_2}} (n_2! \mid a_{n_2 k} \mid^q)^{1/n_2} > \varepsilon$

Therefore there exists a $k_2 > k_1$ in $1 \leq k \leq k_{n_2}$, that is in $k_{n_1} + 1 \leq k \leq k_{n_2}$ such that

$$(n_2! \mid a_{n_2 k_2} \mid^q)^{1/n_2} > \varepsilon \tag{11}$$

Proceeding like this, by (2), (3) and (4) we can find $n_m > n_{m-1}$ and $k_m > k_{m-1}$ in $1 \leq k \leq k_{n_m}$ such that

$$\sup_{1 \leq k < k_{n_{m-1}}} (n_m! \mid a_{n_m k} \mid^q)^{1/n_m} < \varepsilon / 2 \tag{12}$$

$$\sup_{k_{n_m} + 1 \leq k < \infty} (n_m! \mid a_{n_m k} \mid^q)^{1/n_m} < \varepsilon / 2 \tag{13}$$

and $(n_m! \mid a_{n_m k_m} \mid^q)^{1/n_m} > \varepsilon$ (14)

Now defining the sequence (x_k) for all n as

$$x_k = |a_{nk}|^{q-1} \text{ for } k = k_1, k_2, \ldots$$
$$= 0 \qquad \text{for } k \neq k_1, k_2 \tag{15}$$

so that $(x_k) \in l_p(F)$, then

$$|n_1!\, y_{n_1}| = |n_1! \sum_{1}^{k_{n_1}} a_{n_1 k} x_k + n_1! \sum_{k_{n_1}+1}^{\infty} a_{n_1 k} x_k|$$

gives

$$|n_1! \sum_{1}^{k_{n_1}} a_{n_1 k} x_k| = |n_1!\, y_{n_1} - n_1! \sum_{k_{n_1}+1}^{\infty} a_{n_1 k} x_k|$$

$$\leq \text{Max } \{n_1!\, |y_{n_1}|,\ |n_1! \sum_{k_{n_1}+1}^{\infty} a_{n_1 k} x_k|\} \tag{16}$$

Now

$$|n_1! \sum_{1}^{k_{n_1}} a_{n_1} x_k| = n_1!\, |a_{n_1 k_1}|\,|x_{k_1}|$$

$$= n_1!\, |a_{n_1 k_1}|^q \qquad \text{(using 15)}$$

$$< \varepsilon^{n_1} \quad \text{(using (7)} \tag{17}$$

$$|n_1! \sum_{k_{n_1}+1}^{\infty} a_{n_1 k} x_k| < \sup_{k_{n_1}+1 \leq k < \infty} (n_1!\, |a_{n_1 k}|\,|x_k|)$$

$$\leq \sup_{k_{n_1}+1 \leq k < \infty} (n_1!\, |a_{n_1 k}|^q) \text{ (using (15)}$$

$$< (\varepsilon/2)^{n_1} \text{ (using 6)} \tag{18}$$

Using (17), (18) in (16) we have

$$\varepsilon^{n_1} < \text{Max } \{n_1!\, |y_{n_1}|,\ (\varepsilon/2)^{n_1}\}$$

Hence $n_1!\, |y_{n_1}| > \varepsilon^{n_1}$ so that $(n_1!\, |y_{n_1}|)^{1/n_1} > \varepsilon$

Then

$$y_{n_2} = \sum_{1}^{k_{n_1}} a_{n_2 k} x_k + \sum_{k_{n_1}+1}^{k_{n_2}} a_{n_2 k} x_k + \sum_{k_{n_2}+1}^{\infty} a_{n_2 k} x_k$$

gives

$$\left| n_2! \sum_{k_{n_1}+1}^{k_{n_2}} a_{n_2 k} x_k \right| \leq \operatorname{Max} \left\{ n_2! \left| y_{n_2} \right|, \left| n_2! \sum_{1}^{k_{n_1}} a_{n_2 k} x_k \right|, \right.$$

$$\left. \left| n_2! \sum_{k_{n_2}+1}^{\infty} a_{n_2 k} x_k \right| \right\} \tag{19}$$

Now

$$\left| n_2! \sum_{k_{n_1}+1}^{k_{n_2}} a_{n_2 k} x_k \right| = n_2! \left| a_{n_2 k_2} \right| \left| x_{k_2} \right|$$

$$= n_2! \left| a_{n_2 k_2} \right|^q \quad \text{(using (15)}$$

$$< \varepsilon^{n_2} \quad \text{(using (11)} \tag{20}$$

$$\left| n_2! \sum_{1}^{k_{n_2}} a_{n_2 k} x_k \right| \leq \sup_{1 \leq k \leq k_{n_1}} \left(n_2! \left| a_{n_2 k} \right|^q \right) \text{ (using (15)}$$

$$< (\varepsilon / 2)^{n_2} \quad \text{(using (9)} \tag{21}$$

$$\left| n_2! \sum_{k_{n_2}+1}^{\infty} a_{n_2 k} x_k \right| \leq \sup_{k_2+1 < k \leq \infty} \left(n_2! \left| a_{n_2 k} \right|^q \right) \text{ (using (15)}$$

$$< (\varepsilon / 2)^{n_2} \text{ (using (10)} \tag{22}$$

using (20), (21) and (22) in (19) we have

$$\varepsilon^{n_2} < \operatorname{Max} \left\{ n_2! \left| y_{n_2} \right|, (\varepsilon / 2)^{n_2}, (\in / 2)^{n/2} \right\}$$

Hence $n_2! \left| y_{n_2} \right| > \varepsilon^{n_2}$ so that $\left(n_2! \left| y_{n_2} \right| \right)^{1/n_2} > \varepsilon$

Proceeding in this manner using (15) and the inequalities (12), (13) and (14) we can show that $(n_m! \, |y_{n_m}|)^{1/n_m} > \varepsilon$

so that $(n_m! \, |y_{n_m}|)^{1/n_m}$ does not tend to zero as $n_m \to \infty$

Hence $(y_n) \notin \chi(F)$ which gives a contradiction so that (1) is necessary.

Using a method similar to that in the above theorem and taking $(x_k) \in V$ as

$$x_k = z^{n_i} \text{ for } k = k_i$$

$= 0$ for $k \neq k_i$, $i = 1, 2, \ldots$ where $|z| = \lambda < 1$ for some $z \in F$, we can establish the following theorem.

Theorem 5

$A \in (V, \chi(F))$, if and only if

$$\sup_{1 < k \leq \infty} (n! \, |a_{nk}|)^{1/n} \to 0 \text{ as } n \to \infty \tag{23}$$

where $V = C(F)$ or $m(F)$

Note: If in addition, $a_{nk} \to 0$ as $k \to \infty$ for each fixed n, then (23) is the necessary and sufficient condition for $A \in (C_0(F), \chi(F))$.

By using the Lemma and following the method given in K. Chandrasekkara Rao [1] in the complex case, we can establish the following theorem.

Theorem 6

$A \in (\chi(F), m(F))$ if and only if

$$\sup_{\substack{1 \leq k < \infty \\ 1 \leq k < \infty}} \left(\frac{1}{k!} \, |a_{nk}| \right)^{1/k} \leq M, \text{ where M is a constant.}$$

In the case of sequences in the complex field, the theorems corresponding to the Theorems 4 and 5 have been studied by Sirajudeen [2, 3] and Sridhar [7].