Commun. Fac. Sci. Univ. Ank. Series A1
V. 46. pp. 165-171 (1997)

THE BOUNDS FOR PERRON ROOTS OF GCD, GMM, AND AMM MATRICES

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(Received Sep. 15, 1997; Revised Dec. 12, 1997; Accepted Dec. 22, 1997)

ABSTRACT

In this paper we define the greatest common divisor matrix (or GCD), the geometric mean matrix (or GMM) and the arithmetic mean matrix (or AMM) on the set E = {1, 2 3, ..., n} and we obtain the bounds for the Perron root of these matrices.

INTRODUCTION AND MAIN RESULTS

Definition 1. Let $S = \{x_1, x_2, ..., x_n\}$ be a finite ordered set of distinct positive integers. The greatest common divisor matrix (GDC) defined on S is given by

$$\begin{bmatrix} (x_1,x_1) & (x_1,x_2) & \cdots & (x_1,x_n) \\ (x_2,x_1) & (x_2,x_2) & \cdots & (x_2,x_n) \\ \cdots & \cdots & \cdots & \cdots \\ (x_n,x_1) & (x_n,x_2) & \cdots & (x_n,x_n) \end{bmatrix}$$

and is denoted by $[S]_{gcd}$. In order words, for $S = \{x_1, x_2, ..., x_n\}$, $[S]_{gcd} = (s_{ij})_{nxn}$, where $S_{ij} = (x_i, x_j) = \gcd(x_i, x_j)$.

Definition 2. $S = \{x_1, x_2, ..., x_n\}$ be a finite ordered set of distinct positive integers. The geometric mean matrix (GMM) defined on S is given by

$$\begin{bmatrix} \sqrt{x_1,x_1} & \sqrt{x_1,x_2} & \cdots & \sqrt{x_1,x_n} \\ \sqrt{x_2,x_1} & \sqrt{x_2,x_2} & \cdots & \sqrt{x_2,x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{x_n,x_1} & \sqrt{x_n,x_2} & \cdots & \sqrt{x_n,x_n} \end{bmatrix}$$

and is denoted by $[S]_{gmm}$. In other words, for $S = \{x_1, x_2, ..., x_n\}$, $[S]_{gmm} = (g_{ij})_{nxn}$, where $g_{ij} = \sqrt{x_i x_j}$.

Definition 3. Let $S = \{x_1, x_2, ..., x_n\}$ be a finite ordered set of distinct positive integers. The arithmetic mean matrix (AMM) defined on S is given by

$$\begin{bmatrix} \frac{x_1 + x_1}{2} & \frac{x_1 + x_2}{2} & \cdots & \frac{x_1 + x_n}{2} \\ \frac{x_2 + x_1}{2} & \frac{x_2 + x_2}{2} & \cdots & \frac{x_2 + x_n}{2} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x_n + x_1}{2} & \frac{x_n + x_2}{2} & \cdots & \frac{x_n + x_n}{2} \end{bmatrix}$$

and is denoted by $[S]_{amm}$. In other words, for $S = \{x_1, x_2, ..., x_n\}$, $[S]_{amm} = (a_{ij})_{nxn}$, where $a_{ij} = \frac{x_i + x_j}{2}$.

Theorem 1 [1]. Let $A, B \in M_n$. If $0 \leq A \leq B$, then

$$\rho(A) \leq \rho(B),$$

where $\rho(.)$ denotes spectral radius i.e.,

$$\rho(A) = \max \{|\lambda_i(A)|\}.$$

Definition 4. A real n-square matrix $A = (a_{ij})$ is called nonnegative, if $a_{ij} \geq 0$ for $i,j = 1, 2, ..., n$. We write $A \geq 0$.

Definition 5. Let A be a square nonnegative matrix. Then a nonnegative eigenvalue r(A) which is not less than the absolute value of any other eigenvalue of A is called Perron root.

Theorem 2. If $[S]_{gcd}$, $[S]_{gmm}$ and $[S]_{amm}$ denote GCD, GMM and AMM matrices on $S = \{x_1, x_2, ..., x_n\}$, respectively, then

$$r([S]_{gcd}) < r([S]_{gmm}) < r([S]_{amm}).$$

Proof. In the following inequality is always true:

$$(x_i, x_j) \leq \sqrt{x_i x_j} \leq \frac{x_i + x_j}{2} \tag{1}$$

the equality hold if and only if $x_i = x_j$. So from the inequality (1) we have

$$[S]_{gcd} \leq [S]_{gmm} \leq [S]_{amm}.$$

Thus considering Theorem 1, it follows that the proof of theorem, is complete

Theorem 3. If A is an nxn symmetric matrix, then

$$r(A) \geq \frac{E^T Ae}{e^T e}, \tag{2}$$

where r(A) denotes Perron root of A and $e^T = (1, 1, ..., 1)$.

Proof. We recall first the classical lower Frobenius bound of the Perron root an nxn nonnegative matrix A (see, e.g., [2]),

$$r(A) \geq \min_i P_i , \tag{3}$$

where $P_i = P_i(A) = \sum_{i=1}^{n} a_{ij}$ is the i-th row sum of A. Obviously when A is symmetric [since the Rayleigh quotient is a lower bound for r(A)] the bound (3) can be improved as follows:

$$r(A) \geq \frac{E^T Ae}{e^T e} = \frac{1}{n} \sum_{i=1}^{n} P_i$$

Thus the proof is complete.

Remark. Unfortunately, for unsymmetric matrix A, the bound (2) can be wrong. Indeed, for

$$A = \begin{bmatrix} 2 & 2 \\ a & 2 \end{bmatrix}, \ a > 0$$

we have

$$\frac{E^T Ae}{e^T e} = \frac{6 + a}{2}$$

On the other hand since $r(A) = 2 + \sqrt{2a}$, the lower bound (2) is valid if and only if

$$2 + \sqrt{2a} \geq \frac{6 + a}{2}$$

i.e., if a = 2 or, in other words, if A is symmetric.

Theorem 4. Let $[E]_{amm}$ be arithmetic mean matrix (AMM) on E = {1, 2 3, ..., n}. Then

$$\frac{E^T[E]_{amm}e}{e^T e} = \frac{n(n + 1)}{2}$$

where $e = (1, 1, ..., 1)^T$.

Proof. It is easily seen that $e^T e = n$. On the other hand considering

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

we have

$$e^T[E]_{amm}e = \sum_{i,j=1}^{n} \frac{x_i + x_j}{2} = \sum_{i,j=1}^{n} \frac{x_i}{2} + \sum_{i,j=1}^{n} \frac{x_j}{2}$$

$$= \sum_{i=1}^{n} \frac{x_i}{2}\sum_{j=1}^{n} 1 + \sum_{j=1}^{n} \frac{x_j}{2}\sum_{i=1}^{n} 1$$

$$= \frac{n^2(n+1)}{2}$$

Consequently since $e^T e = n$, we write

$$\frac{e^T[E]_{amm}e}{e^T e} = \frac{n(n+1)}{2}$$

Thus the proof is complete.

Lemma 1. Let $[E]_{gmm}$ be geometric mean matrix (GMM) on $E = \{1, 2, 3, ..., n\}$. Then

(i) $\det([E]_{gmm}) = 0$

(ii) $\mathrm{rank}([E]_{gmm}) = 1$.

Proof. If $r_1, r_2 ..., r_n$ denote the rows of the matrix $[E]_{gmm}$, then we have

$$r_k = \sqrt{k}\ r_1 \qquad (k = 2, 3, ..., n) \tag{4}$$

So by the properties of the determinants it follows that (i). on the other hand by the elemantary row operations it follows that (ii).

Thus lemma is proved.

Theorem 5. Let $[E]_{gmm}$ be geometric mean matrix (GMM) on $E = \{1, 2, 3, ..., n\}$. Then

$$r\left([E]_{gmm}\right) = \frac{n(n+1)}{2}$$

where r(.) denotes Perron root.

Proof. If α_s is the sum of all principal minors of order s of $[E]_{gmm}$, $1 \leq s \leq n$, then we have

$$\det(\lambda I - [E]_{gmm}) = \lambda^n - \alpha_1\lambda^{n-1} + \alpha_2\lambda^{n-2} + \dots + (-1)^n\alpha_n.$$

In particular, we note that

$$\alpha_1 = \sum_{i=1}^{n} x_i = \frac{n(n+1)}{2} \quad \text{and} \quad \alpha_n = \det\left([E]_{gmm}\right).$$

So by Lemma 1. (i) we have $\alpha_n = 0$. On the other hand by the Lemma 1. (ii) we write

$$\alpha_2 = \alpha_3 = \dots = \alpha_n = 0.$$

Thus we obtain

$$\lambda^n - \frac{n(n+1)}{2}\lambda^{n-1} = 0$$

or

$$\lambda^{n-1}\left(\lambda - \frac{n(n+1)}{2}\right) = 0.$$

Therefore the eigenvalues of the matrix $[E]_{gmm}$ are $\lambda_1 = \lambda_2 = \dots = \lambda_{n-1} = 0$ and

$$\lambda_n = r(A) = \frac{n(n+1)}{2}.$$

Thus the theorem is proved.

Theorem 6. Let $S = \{x_1, x_2, \dots, x_n\}$ be an factor-closed set, and let $[S]_{gcd}$ be the GCD matrix defined on S. Then

$$\det([S]_{gcd}) = \varphi(x_1)\,\varphi(x_2)\,\dots\,\varphi(x_n),$$

where $\varphi(.)$ denotes Euler's totient function.

Corollary 1. If $[E]_{gcd}$ is the GCD matrix defined on $E = \{1, 2, 3, \dots, n\}$, then

$$\det([E]_{gcd}) = \varphi(1)\,\varphi(2)\,\dots\,\varphi(n),$$

Proof. Since the set $E = \{1, 2, 3, \dots, n\}$ is factor-closed, the proof is immediately seen by Theorem 6.

Theorem 7. If $[E]_{gcd}$ is the GCD matrix defined on $E = \{1, 2, 3, ..., n\}$ then

$$r([E]_{gcd}) \geq \left[\prod_{i=1}^{n} \varphi(i)\right]^{1/n}$$

where r(.) denotes Perron root and φ(.) denotes Euler's totient function.

Proof. If λ_i (i = 1, 2, ..., n) are eigenvalues of the matrix $[E]_{gcd}$, then we have

$$\det([E]_{gcd}) = \prod_{i=1}^{n} \lambda_i \leq \prod_{i=1}^{n} r([E]_{gcd}) = r([E]_{gcd})^n .$$

On the other hand by the Corollary 1., we write

$$\varphi(1)\ \varphi(2)\ ...\varphi(n) \leq r([E]_{gcd})^n$$

or

$$\left[\prod_{i=1}^{n} \varphi(i)\right]^{1/n} \leq r([E]_{gcd})$$

Thus the proof is complete.

NUMERICAL EXAMPLES

Example 1. Consider the set E = {1, 2, 3}. Then we write

$$[E]_{amm} = \begin{bmatrix} 1 & \frac{3}{2} & 2 \\ \frac{3}{2} & 2 & \frac{5}{2} \\ 2 & \frac{5}{2} & 3 \end{bmatrix}$$

and we find

$$r([E]_{amm}) = 3 + \frac{1}{2}\sqrt{42} \approx 6.24.$$

Indeed, since $\frac{n(n+1)}{2} = 6$, we obtain $6.24 \geq 6$.

Similarly for n = 4 $r([E]_{amm}) = 10.47 \geq 10$

for n = 5 $r([E]_{amm}) = 15.79 \geq 15$

etc.

Example 2. For E = {1, 2, 3}, since

$$[E]_{gmm} = \begin{bmatrix} 1 & \sqrt{2} & \sqrt{3} \\ \sqrt{2} & 2 & \sqrt{6} \\ \sqrt{3} & \sqrt{6} & 3 \end{bmatrix}$$

we obtain $r([E]_{amm}) = 6$.

Similarly for $n = 4$ $r([E]_{gmm}) = 10$

for $n = 5$ $r([E]_{gmm}) = 15$

etc.

REFERENCES

[1] HORN, R.A. and JOHNSON, C.R., *Matrix Analysis*, Cambridge University Press., 1990.

[2] MARCUS, M. and MINC, H., A *Survey of Matrix Theory ad Inequalities*, Allyn and Bacon, Boston, 1964.

[3] SMITH, H.J.S. *On the value of a certain arithmetical determinant*, Proc. London Math. Soc. 7: 208-212 (1975-76).