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SOME EXTENSIONS OF GALLOP'S FORMULAS INCLUDING HOMOGENEOUS OPERATORS

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ABSTRACT

In this study, some relational formulas between homogeneous differential operators, Lorentzian metric and ultra-hyperbolic operators defined in p + q dimensional space are obtained.

1. INTRODUCTION

It is shown by E.G. Gallop [2] that

$$r^{m-n}\phi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) r^{2n+1}\psi_m\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \lambda\mu\upsilon\left(\lambda^2+\mu^2+\upsilon^2\right)^{m-n-\frac{1}{2}}\phi_m\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \upsilon}\right)\left(\lambda^2+\mu^2+\upsilon^2\right)^{n+\frac{1}{2}}$$

$$\cdot\ \psi_m\left(\frac{\partial}{\partial \lambda}, \frac{\partial}{\partial \mu}, \frac{\partial}{\partial \upsilon}\right)\frac{1}{\lambda\mu\upsilon} \tag{1}$$

where Φ_m and Ψ_n are homogeneous polynomial of differential operators with degree m and n, respectively, and $r^2 = x^2 + y^2 + z^2$. Here, λ, μ, and ν are the operators which transform the spherical harmonic P(a, b, c), defined in [2], to the forms P(a-1, b, c), P(a, b-1, c) and P(a, b, c-1) respectively.

In this study, we obtain a new form of the formula (1) by extending it to p + q dimensional space.

Let

$$Lu = \sum_{i=1}^{p}\frac{\partial^2 u}{\partial x_i^2} - \sum_{j=1}^{q}\frac{\partial^2 u}{\partial y_j^2} \tag{2}$$

$$s^2 = \sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 = |x|^2 - |y|^2 > 0 \tag{3}$$

$$s_1^2 = \sum_{i=1}^{p} \lambda_i^2 - \sum_{j=1}^{q} \mu_j^2 = |\lambda|^2 - |\mu|^2 > 0 \tag{4}$$

and

$$P\left(a_1,...,a_p,b_1,...,b_q\right) = (-1)^n \frac{s^{n+p+q-2}}{a_1!...a_p!b_1!...b_q!} \left(\frac{\partial}{\partial x_1}\right)^{a_1} ... \left(\frac{\partial}{\partial x_p}\right)^{a_p} \left(\frac{\partial}{\partial y_1}\right)^{b_1} ... \left(\frac{\partial}{\partial y_q}\right)^{b_q} \frac{1}{s^{p+q-2}} \tag{5}$$

where $a_1,...,a_p,b_1,...,b_q$ are non-negative integers and $a_1+...+a_p+b_1+...+b_q = n$. It is shown in [5] that the function P is a solution of the equation $Lu = 0$ and satisfies the recurrence relations:

$$\sum_{i=1}^{p} (a_i + 1)(a_i + 2) P\left(a_1,...,a_i+2,...,a_p,b_1,...,b_q\right) - \sum_{j=1}^{q} (b_j + 1)(b_j + 2) P\left(a_1,...,a_p,b_1,...,b_j+2,...,b_q\right) = 0 \tag{6}$$

and

$$\frac{\partial}{\partial x_1}\left[S^n P\left(a_1,...,a_p,b_1,...,b_q\right)\right] = -s^{n-1} \left\{ \sum_{i=2}^{p} (a_1 + 1) P\left(a_1+1,...,a_i-2,...,a_p,b_1,...,b_q\right) - \sum_{j=1}^{q} (a_1 + 1) P\left(a_1+1,...,a_p,b_1,...,b_j-2,...,b_q\right) - [2n+p+q-3-a_1] P\left(a_1-1,...,a_p,b_1,...,b_q\right) \right\}. \tag{7}$$

2. SOME LEMMAS

Lemma 1. Let P be defined as in (5). Then,

$$P(0, 0, ..., 0) = 1. \tag{8}$$

Proof. It is clear from the definition of P.

Definition 1. λ_i, $(i = 1, 2, ..., p)$ and μ_j, $(j = 1, 2, ..., q)$ are such operators that when applied to P the following relations hold:

$$\lambda_i P\left(a_1,...,a_{i-1},a_i,a_{i+1},...,a_p,b_1,...,b_{j-1},b_j,b_{j+1},...,b_q\right) = P\left(a_1,...,a_{i-1},a_i-1,a_{i+1},...,a_p,b_1,...,b_{j-1},b_j,b_{j+1},...,b_q\right) \tag{9}$$

$$\mu_j\, P\left(a_1,...,a_{i-1},a_i,a_{i+1},...,a_p,b_1,...,b_{j-1},b_j,b_{j+1},...,b_q\right)$$
$$= P\left(a_1,...,a_{i-1},a_i,a_{i+1},...,a_p,b_1,...,b_{j-1},b_j-1,b_{j+1},...,b_q\right). \tag{10}$$

Lemma 2. The following equality holds:

$$P(a_1,...,a_p,b_1,...,b_q) = \lambda_1^{-a_1}...\lambda_p^{-a_p}.\mu_1^{-b_1}...\mu_q^{-b_q} \tag{11}$$

Proof. In view of (9) and (10), by applying the operators $\lambda_1,...,\lambda_p$ to the function $P(a_1,...,a_p,b_1,...,b_p)$ respectively $a_1,...,a_p$ times and then by applying the operators $\mu_1,...,\mu_q$ again to the function $P(a_1,...,a_p,b_1,...,b_q)$ respectively $b_1,...,b_q$ times, we get

$$\lambda_1^{a_1}...\lambda_p^{a_p}.\mu_1^{b_1}...\mu_q^{b_q}P(a_1,...,a_p,b_1,...,b_q) = P(a_1-a_1,...,a_p-a_p,b_1-b_1,...,b_q-b_q)$$
$$= P(0,...,0).$$

Hence (11) follows from Lemma 1.

Lemma 3. Let Ψ_n be a homogeneous polynomial of degree n, and s^2 be as in (3). Then,

$$s^{n+p+q-2}\psi_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{1}{s^{p+q-2}} = (-1)^n a_1!...a_p!b_1!...b_q!\psi_n\left(\lambda_1^{-1},...,\lambda_p^{-1},\mu_1^{-1},...,\mu_q^{-1}\right)$$
$$= \lambda_1...\lambda_p\mu_1...\mu_p\psi_n\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\mu}\right)\frac{1}{\lambda_1...\lambda_p\mu_1...\mu_p} \tag{12}$$

where $x = (x_1, ..., x_p)$, $y = (y_1, ..., y_q)$, $\lambda = (\lambda_1, ..., \lambda_p)$, $\mu = (\mu_1, ..., \mu_q)$ and

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_p}\right), \frac{\partial}{\partial y} = \left(\frac{\partial}{\partial y_1},...,\frac{\partial}{\partial y_q}\right), \frac{\partial}{\partial\lambda} = \left(\frac{\partial}{\partial\lambda_1},...,\frac{\partial}{\partial\lambda_p}\right), \frac{\partial}{\partial\mu} = \left(\frac{\partial}{\partial\mu_1},...,\frac{\partial}{\partial\mu_q}\right).$$

Proof. By using the representation (11) in formula (5), we get

$$s^{n+p+q-2}\left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{p+q-2}} = (-1)^n a_1!...a_p!b_1!...b_q!\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}. \tag{13}$$

By using $\left(\frac{\partial}{\partial\lambda_i}\right)^{a_i}\lambda_i^{-1} = (-1)^{a_i}a_i!\lambda_i^{-a_i-1}$ on the right hand side of (13), we have

$$s^{n+p+q-2}\left(\frac{\partial}{\partial x_1}\right)^{a_1}...\left(\frac{\partial}{\partial x_p}\right)^{a_p}\left(\frac{\partial}{\partial y_1}\right)^{b_1}...\left(\frac{\partial}{\partial y_q}\right)^{b_q}\frac{1}{s^{p+q-2}} = (-1)^n a_1!...a_p!b_1!...b_q!\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}$$
$$= \lambda_1...\lambda_p\mu_1...\mu_q\left(\frac{\partial}{\partial\lambda_1}\right)^{a_1}...\left(\frac{\partial}{\partial\lambda_p}\right)^{a_p}\left(\frac{\partial}{\partial\mu_1}\right)^{b_1}...\left(\frac{\partial}{\partial\mu_q}\right)^{b_q}\lambda_1^{-1}...\lambda_p^{-1}\mu_1^{-1}...\mu_q^{-1}. \tag{14}$$

By replacing $a_1,...,a_p,b_1,...,b_q$ by $a_1^{\,i},...,a_p^{\,i},b_1^{\,i},...,b_q^{\,i}$ respectively, and by multiplying all sides by the coefficients of the homogeneous differential operator $\Psi_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)$ and taking the sum over i covering all of the coefficients, we obtain (12).

Lemma 4. Let f_N be a homogeneous polynomial of degree N. Then,

$$\lambda_1...\lambda_p\mu_1...\mu_q\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial\lambda_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial\mu_j^2}\right)f_N\left(\lambda_1^{-1},...,\lambda_p^{-1},\mu_1^{-1},...,\mu_q^{-1}\right)=0\ . \tag{15}$$

Proof. The recurrence relation given by (6) can be written as

$$\lambda_1...\lambda_p\mu_1...\mu_q\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial\lambda_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial\mu_j^2}\right)\lambda_1^{-(a_1+1)}...\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}...\mu_q^{-(b_q+1)}=0\ . \tag{16}$$

To see this, take the derivatives on the left hand side in (16) and use the representation (11). Now, in (16), by replacing $a_1,...,a_p,b_1,...,b_q$ and $a_1^{\,i},...,a_p^{\,i},b_1^{\,i},...,b_q^{\,i}$ respectively, and by multiplying both sides by the coefficients of the polynominal f_N and then by taking the sum over i covering all of the coefficients, we obtain (15).

3. MAIN THEOREMS

Theorem 1. Let Φ_m and Ψ_n be homogeneous polynomials of diferential operators with degree m and n, respectively, and let s^2 and $s_1^{\,2}$ be defined as in (3) and (4). Then

$$s^{m-n}\phi_m\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)s^{2n+p+q-2}\psi_n\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\frac{1}{s^{p+q-2}}$$

$$=\lambda_1...\lambda_p\mu_1...\mu_q\left(s_1^2\right)^{m-n-\frac{p+q-2}{2}}\phi_m\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\mu}\right)\left(s_1^2\right)^{n+\frac{p+q-2}{2}}\psi_n\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\mu}\right)\lambda_1^{-1}...\lambda_p^{-1}\mu_1^{-1}...\mu_q^{-1} \tag{17}$$

where $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial\lambda}$ and $\frac{\partial}{\partial\mu}$ as defined in Lemma 3.

Proof. To prove the theorem, we use (11) in the open form of (7). Hence, we obtain

$$\frac{\partial}{\partial x_1}\left[s^n\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}\right]$$

$$= -s^{n-1}\Big\{(a_1+1)\Big[\lambda_1^{-(a_1+1)}\lambda_2^{-(a_2-2)}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q} +...+ \lambda_1^{-(a_1+1)}\lambda_2^{-a_2}...\lambda_p^{-(a_p-2)}\mu_1^{-b_1}...\mu_q^{-b_q}\Big]$$
$$-(a_1+1)\Big[\lambda_1^{-(a_1+1)}...\lambda_p^{-a_p}\mu_1^{-(b_1-2)}\mu_2^{-b_2}...\mu_q^{-b_q} +...+ \lambda_1^{-(a_1+1)}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-(b_q-2)}\Big]$$
$$-(2n+p+q-3-a_1)\lambda_1^{-(a_1-1)}...\lambda_p^{-a_p}\mu_1^{-b_1}\mu_2^{-b_2}...\mu_q^{-b_q}\Big\}. \tag{18}$$

After making the necessary simplifications on the right hand side of (18), by using definition (4), we get

$$\frac{\partial}{\partial x_1}\Big[s^n\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}\Big]$$
$$= -s^{n-1}\Big\{(a_1+1)\lambda_1^{-1}s_1^2 - (2n+p+q-2)\lambda_1\Big\}\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}. \tag{19}$$

Since the right hand side of the above equality can be written as

$$-s^{n-1}\Big\{(a_1+1)\lambda_1^{-1}s_1^2 - (2n+p+q-2)\lambda_1\Big\}\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}$$

$$= s^{n-1}\lambda_1...\lambda_p\mu_1...\mu_q\left(s_1^2\right)^{-\left(n+\frac{p+q-2}{2}-1\right)}\frac{\partial}{\partial\lambda_1}\left(s_1^2\right)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}...\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}...\mu_q^{-(b_q+1)} \tag{20}$$

from (19) and (20), we have

$$\frac{\partial}{\partial x_1}\Big[s^n\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}\Big]$$

$$= s^{n-1}\lambda_1...\lambda_p\mu_1...\mu_q\left(s_1^2\right)^{-\left(n+\frac{p+q-2}{2}-1\right)}\frac{\partial}{\partial\lambda_1}\left(s_1^2\right)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}...\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}...\mu_q^{-(b_q+1)} \tag{21}$$

Now the derivative with respect to x_1 in (18) gives us

$$\frac{\partial^2}{\partial x_1^2}\Big[s^n\lambda_1^{-a_1}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}\Big] = -(a_1+1)\frac{\partial}{\partial x_1}\Big[s^{n-1}\lambda_1^{-(a_1+1)}\lambda_2^{-(a_2-2)}...\lambda_p^{-a_p}\mu_1^{-b_1}...\mu_q^{-b_q}\Big]$$
$$-...-(a_1+1)\frac{\partial}{\partial x_1}\Big[s^{n-1}\lambda_1^{-(a_1+1)}...\lambda_p^{-(a_p-2)}\mu_1^{-b_1}...\mu_q^{-b_q}\Big]$$
$$+(a_1+1)\frac{\partial}{\partial x_1}\Big[s^{n-1}\lambda_1^{-(a_1+1)}...\lambda_p^{-a_p}\mu_1^{-(b_1-2)}\mu_2^{-b_2}...\mu_q^{-b_q}\Big]$$
$$+...+(a_1+1)\frac{\partial}{\partial x_1}\Big[s^{n-1}\lambda_1^{-(a_1+1)}...\lambda_p^{-a_p}\mu_1^{-b_1}\mu_2^{-b_2}...\mu_q^{-(b_q-2)}\Big]$$
$$+(2n+p+q-3)\frac{\partial}{\partial x_1}\Big[s^{n-1}\lambda_1^{-(a_1-1)}...\lambda_p^{-a_p}\mu_1^{-b_1}\mu_2^{-b_2}...\mu_q^{-b_q}\Big].$$

In view of (21), by making the necessary arrangements and simplifications, we obtain

$$\frac{\partial^2}{\partial x_1^2}\left[s^n\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q}\right]$$

$$= s^{n-2}\lambda_1\ldots\lambda_p\mu_1\ldots\mu_q(s_1^2)^{-\left(n+\frac{p+q-2}{2}-2\right)}\frac{\partial^2}{\partial\lambda_1^2}\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_q^{-(b_q+1)}\right].$$

Proceeding in this way, the k_1-th derivative of (18) yields

$$\frac{\partial^{k_1}}{\partial x_1^{k_1}}\left[s^n\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q}\right]$$

$$= s^{n-k_1}\lambda_1\ldots\lambda_p\mu_1\ldots\mu_q(s_1^2)^{-\left(n+\frac{p+q-2}{2}-k_1\right)}\frac{\partial^{k_1}}{\partial\lambda_1^{k_1}}\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_q^{-(b_q+1)}\right].$$

By taking the derivaties of both sides of the above equality with respect to $x_2,\ldots,x_p,y_1,\ldots,y_q$ respectively $k_2,\ldots,k_p,\ell_1,\ldots,\ell_q$ times $(k_1+\ldots+k_p+\ell_1+\ldots+\ell_q=m)$, we obtain

$$\left(\frac{\partial}{\partial x_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial x_p}\right)^{k_p}\left(\frac{\partial}{\partial y_1}\right)^{\ell_1}\cdots\left(\frac{\partial}{\partial y_q}\right)^{\ell_q}\left[s^n\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q}\right]$$

$$= s^{n-m}\lambda_1\ldots\lambda_p\mu_1\ldots\mu_q(s_1^2)^{m-n-\frac{p+q-2}{2}}\,.$$

$$\left(\frac{\partial}{\partial\lambda_1}\right)^{k_1}\cdots\left(\frac{\partial}{\partial\lambda_p}\right)^{k_p}\left(\frac{\partial}{\partial\mu_1}\right)^{\ell_1}\cdots\left(\frac{\partial}{\partial\mu_q}\right)^{\ell_q}\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_q^{-(b_q+1)}\right].$$

Replacement of $k_1,\ldots,k_p,\ell_1,\ldots,\ell_q$ by k_1^ν, $k_2^\nu,\ldots,k_p^\nu,\ell_1^\nu,\ldots,\ell_q^\nu$ above and multiplication of both sides by the coefficients of the homogeneous differential operator Φ_m yields us

$$\phi_m\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)\left[s^n\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q}\right]$$

$$= s^{n-m}\lambda_1\ldots\lambda_p\mu_1\ldots\mu_q(s_1^2)^{m-n-\frac{p+q-2}{2}}\phi_m\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\mu}\right)\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-a_1}\lambda_1^{-1}\lambda_2^{-a_2}\lambda_2^{-1}\ldots\mu_1^{-b_1}\mu_1^{-1}\ldots\mu_q^{-b_q}\mu_q^{-1}\right].$$

For each i with $a_1^i+\dots+a_p^i+b_1^i+\dots+b_q^i = n$, replacing $a_1,\dots,a_p,b_1,\dots,b_q$ by $a_1^i,\dots,a_p^i,b_1^i,\dots,b_q^i$ and multiplying both sides of the above equality by the coefficients of the function Ψ_n and then taking the sum over i covering the coefficients of Ψ_n, we obtain

$$\phi_m\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) s^n\psi_n\left(\lambda_1^{-1},\dots,\lambda_p^{-1},\mu_1^{-1},\dots,\mu_q^{-1}\right)$$

$$=s^{n-m}\lambda_1\dots\lambda_p\mu_1\dots\mu_q(s_1^2)^{m-n-\frac{p+q-2}{2}}\phi_m\left(\frac{\partial}{\partial\lambda},\frac{\partial}{\partial\mu}\right)\left[\frac{(s_1^2)^{n+\frac{p+q-2}{2}}}{\lambda_1\dots\lambda_p\mu_1\dots\mu_q}\psi_n\left(\lambda_1^{-1},\dots,\lambda_p^{-1},\mu_1^{-1},\dots,\mu_q^{-1}\right)\right]. \quad (22)$$

Finally, by multiplying both sides of (22) by $(-1)^n a_1!a_2!\dots a_p!b_1!\dots b_q!$ and by using (12), we obtain (17) which proves the theorem.

Theorem 2. Let L, s and P be defined as in (2), (3) and (11), respectively. Then,

$$L(s^nP) = 0. \quad (23)$$

Proof. By using the definitions of L, s and P, we can rewrite (23) as

$$\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial y_j^2}\right)s^n\lambda_1^{-a_1}\dots\lambda_p^{-a_p}\mu_1^{-b_1}\dots\mu_q^{-b_q}=0.$$

Now, to prove the theorem, in (22) let specifically

$$\phi_m\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)=\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial y_j^2}=L$$

and

$$\psi_n(\lambda_1^{-1},\dots,\lambda_p^{-1},\mu_1^{-1},\dots,\mu_q^{-1})=\lambda_1^{-a_1}\dots\lambda_p^{-a_p}\mu_1^{-b_1}\dots\mu_q^{-b_q}=P.$$

Thus, we have

$$\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial y_j^2}\right)s^n\lambda_1^{-a_1}\dots\lambda_p^{-a_p}\mu_1^{-b_1}\dots\mu_q^{-b_q}=s^{n-2}\lambda_1\dots\lambda_p\mu_1\dots\mu_q(s_1^2)^{2-n-\frac{p+q-2}{2}}$$

$$\cdot\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial\lambda_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial\mu_j^2}\right)\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}\dots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\dots\mu_q^{-(b_q+1)}\right]. \quad (24)$$

On the other hand, for $1\le i\le p$ and $1\le j\le q$,

$$\frac{\partial^2}{\partial\lambda_i^2}\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right]$$

$$= (2n+p+q-2)(2n+p+q-4)(s_1^2)^{n+\frac{p+q-2}{2}-2}\lambda_1^{-a_1+1}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$- a_i(2n+p+q-2)(s_1^2)^{n+\frac{p+q-2}{2}-1}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$- (a_i+1)(2n+p+q-2)(s_1^2)^{n+\frac{p+q-2}{2}-1}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$+ (s_1^2)^{n+\frac{p+q-2}{2}}\frac{\partial^2}{\partial\lambda_i^2}\left[\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right] \tag{25}$$

and

$$\frac{\partial^2}{\partial\mu_j^2}\left[(s_1^2)^{n+\frac{p+q-2}{2}}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right]$$

$$= (2n+p+q-2)(2n+p+q-4)(s_1^2)^{n+\frac{p+q-2}{2}-2}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$+ b_j(2n+p+q-2)(s_1^2)^{n+\frac{p+q-2}{2}-1}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$+ (b_j+1)(2n+p+q-2)(s_1^2)^{n+\frac{p+q-2}{2}-1}\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$+ (s_1^2)^{n+\frac{p+q-2}{2}}\frac{\partial^2}{\partial\mu_j^2}\left[\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right] \tag{26}$$

By substituting (25) and (26) in the right hand side of (24), after necessary simplifications, we get

$$\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial y_j^2}\right)s^n\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q} = s^{n-2}\left\{-4(2n+p+q-2)s_1^2\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q}\right.$$

$$\left.+\lambda_1\ldots\lambda_p\mu_1\ldots\mu_q(s_1^2)^2\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial\lambda_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial\mu_j^2}\right)\lambda_1^{-(a_1+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_q^{-(b_q+1)}\right\}. \tag{27}$$

Now, since

$$\frac{\partial^2}{\partial\lambda_i^2}\left[(s_1^2)^2\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right]$$

$$= 8\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-a_i+1}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$-\ 4(2a_i+1)(s_1^2)\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$+\ (s_1^2)^2\ \frac{\partial^2}{\partial\lambda_i^2}\left[\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right].$$

and

$$\frac{\partial^2}{\partial\mu_j^2}\left[(s_1^2)^2\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right]$$

$$= 8\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-b_j+1}\ldots\mu_q^{-(b_q+1)}$$

$$+\ 4(2b_j+1)(s_1^2)\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}$$

$$+\ (s_1^2)^2\ \frac{\partial^2}{\partial\mu_j^2}\left[\lambda_1^{-(a_1+1)}\ldots\lambda_i^{-(a_i+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_j^{-(b_j+1)}\ldots\mu_q^{-(b_q+1)}\right].$$

the equation (27) becomes, on substitution,

$$\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial y_j^2}\right)s^n\lambda_1^{-a_1}\ldots\lambda_p^{-a_p}\mu_1^{-b_1}\ldots\mu_q^{-b_q}$$

$$= s^{n-2}\lambda_1\ldots\lambda_p\mu_1\ldots\mu_q\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial\lambda_i^2}-\sum_{j=1}^{q}\frac{\partial^2}{\partial\mu_j^2}\right)(s_1^2)^2\lambda_1^{-(a_1+1)}\ldots\lambda_p^{-(a_p+1)}\mu_1^{-(b_1+1)}\ldots\mu_q^{-(b_q+1)}.$$

The right hand side vanishes by Lemma 4. Hence the proof is complete.

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