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ON CONVERGENCE OF SINGULAR INTEGRALS WITH NON-ISOTROPIC KERNELS

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ABSTRACT

In this article convergence of singular integrals with non-isotropic kernels and an application of integrals of Gauss-Weierstrass and Abel-Poisson type with non-isotropic kernels are studied.

INTRODUCTION

It is well known that the problem of convergence of the sequences or families of integral operators with positive kernels have many applications in different problems, in the theory of differential equation, approximation theory, harmonic analysis etc. ([3],[4]) We refer to the original monographies (Stein [5], Stein-Weiss [6], Altomare-Compity [1]). Note that the integral operators of convolution type, that is integrals with the kernels, depending on difference between the variables have principal applications. In multidimensional case, this type of kernels are function of euclidean distance between two points.

In this paper, we introduce multidimensional integral operators with the kernels, depending on non-isotropic distance and study the problem of approximation of function by the families of a such type of integrals. As an application of our result Gauss-Weierstrass and Abel-Poisson type integrals with non isotropic kernels are also given.

First, we define a non isotropic distance in n dimensional euclidean space $\mathbf{R}^n$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be positive numbers and let $|\lambda| = \lambda_1 + \lambda_2 + ... + \lambda_n$. For $x \in \mathrm{R}^n$,

$$\|x\|_\lambda = \left(|x_1|^{\frac{1}{\lambda_1}} + |x_2|^{\frac{1}{\lambda_2}} + ... + |x_n|^{\frac{1}{\lambda_n}} \right)^{\frac{|\lambda|}{n}}$$

is the non-isotropic distance or λ-distance between x and origin.

It is easy to see that the distance has the following non-isotropic homogeneity properties for $t > 0$:

$$\left(\left|t^{\lambda_1}x_1\right|^{\frac{1}{\lambda_1}} + \left|t^{\lambda_2}x_2\right|^{\frac{1}{\lambda_2}} + \ldots + \left|t^{\lambda_n}x_n\right|^{\frac{1}{\lambda_n}}\right)^{\frac{|\lambda|}{n}} = t^{\frac{|\lambda|}{n}}\|x\|_\lambda .$$

We can also see that this distance has triangle inequality in the form

$$\|x+y\|_\lambda \le 2^{(1+\frac{1}{\lambda_{\min}})\frac{|\lambda|}{n}}\left(\|x\|_\lambda + \|y\|_\lambda\right).$$

Now we introduce analogies of some known kernels in non-isotropic case.

1. λ – Gauss-Weierstrass Kernel

Let α be a positive parameter. For $x \in R^n$, we define λ – Gauss-Weierstrass kernel in the following form:

$$W_\lambda(x,\alpha) =: c_n \alpha^{-|\lambda|} e^{\frac{-|x|_\lambda^{\frac{n}{|\lambda|}}}{4\alpha}} \tag{1}$$

where $c_n = \dfrac{1}{w_{\lambda,n-1} 2^{2|\lambda|-1}\Gamma(|\lambda|)}$.

Lemma 1.

$$\int_{R^n} W_\lambda(x,\alpha)dx = 1 \quad \text{for all } \alpha > 0.$$

Proof. By a change of variables with $x = \alpha^\lambda t$, we first note that

$$\int_{R^n} W_\lambda(x,\alpha)dx = \int_{R^n} W_\lambda(x,1)dx.$$

Thus, it is sufficient to calculate the integral for $\alpha = 1$. The integral can easily be calculated by passing to generalized spherical coordinates ;

$$\begin{aligned}
x_1 &= (\rho\cos(\theta_1))^{2\lambda_1}.\\
x_2 &= (\rho\sin(\theta_1)\cos(\theta_2))^{2\lambda_2}.\\
&\vdots\\
x_{n-1} &= (\rho\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{n-2})\cos(\theta_{n-1}))^{2\lambda_{n-1}}.\\
x_n &= (\rho\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{n-2})\sin(\theta_{n-1}))^{2\lambda_n}.
\end{aligned}$$

where $0 \le \theta_1, \theta_2, \ldots, \theta_{n-2} \le \pi$ and $0 \le \theta_{n-1} \le 2\pi$.

Denoting the Jacobian of this transformation by $I_\lambda(\rho, \theta_1, \theta_2, \ldots, \theta_{n-1})$, we obtain

$$I_\lambda = \rho^{2|\lambda|-1}\Omega_\lambda(\theta), \tag{2}$$

where

$$\Omega_\lambda(\theta) = 2^n \lambda_1 \lambda_2 \ldots \lambda_n \prod_{j=1}^{n} (\cos\theta_1)^{2\lambda_j - 1} (\sin\theta_j)^{2\sum_{k=j+1}^{n} \lambda_k - 1}$$

Denoting the unit sphere in R^n by S^{n-1}, we see that the integral

$$w_{\lambda,n-1} = \int_{S^{n-1}} \Omega_\lambda(\theta) d\theta \tag{3}$$

is finite. Therefore, we have

$$\int_{R^n} W_\lambda(x,1)dx = c_n \int_{R^n} e^{-\frac{|x|_\lambda^{\frac{n}{|\lambda|}}}{4}} dx$$

$$= c_n \int_0^\infty \int_{S^{n-1}} e^{-\frac{\rho^2}{4}} \rho^{2|\lambda|-1} \Omega_\lambda(\theta) d\theta d\rho$$

$$= w_{\lambda,n-1} c_n \int_0^\infty e^{-\frac{\rho^2}{4}} \rho^{2|\lambda|-1} d\rho$$

$$= w_{\lambda,n-1} c_n 2^{2|\lambda|-1} \int_0^\infty e^{-t} t^{|\lambda|-1} dt$$

$$= w_{\lambda,n-1} c_n 2^{2|\lambda|-1} \Gamma(|\lambda|) = 1.$$

This is the desired result.

2. λ **– Abel-Poisson Kernel**

Let α be a positive parameter. For $x \in R^n$, we define λ-Abel-Poisson kernel in the following form

$$P_\lambda(x,\alpha) =: c_n \frac{\alpha}{\left(\alpha^2 + |x|_\lambda^{\frac{n}{|\lambda|}}\right)^{\frac{2|\lambda|+1}{2}}},$$

where $c_n = \dfrac{1}{w_{\lambda,n-1} \frac{1}{2} B(|\lambda|, \frac{1}{2})}$.

Lemma 2.

$$\int_{R^n} P_\lambda(x,\alpha)dx = 1 \text{ for } \alpha > 0.$$

Proof. By a change of variable with $x = \alpha^{2|\lambda|} t$ we have

$$\int_{R^n} P_\lambda(x,\alpha)dx = \int_{R^n} P_\lambda(x,1)dx \tag{4}$$

Thus, it is sufficient to calculate the integral for $\alpha = 1$. Also, the integral can easily be calculated by passing to generalized spherical coordinates. From (4) we have

$$\int_{R^n} P_\lambda(x,1)dx = \int_{R^n} c_n \frac{1}{\left(1+|x|_\lambda^{\frac{n}{|\lambda|}}\right)^{\frac{2|\lambda|+1}{2}}} dx$$

$$= c_n \int_0^\infty \int_{S^{n-1}} \frac{\rho^{2|\lambda|-1}}{(1+\rho^2)^{\frac{2|\lambda|+1}{2}}} \Omega_\lambda(\theta) d\theta d\rho$$

$$= w_{\lambda,n-1} c_n \int_0^\infty \frac{\rho^{2|\lambda|-1}}{(1+\rho^2)^{\frac{2|\lambda|+1}{2}}} d\rho.$$

If we change the variable by $\rho = \tan(u)$, it gives us

$$\int_{R^n} P_\lambda(x,1)dx = w_{\lambda,n-1} c_n \int_0^{\frac{\pi}{2}} (\sin(u))^{2|\lambda|-1} du$$

$$= w_{\lambda,n-1} c_n \tfrac{1}{2} B\left(|\lambda|, \tfrac{1}{2}\right) = 1.$$

Now, we consider multidimensional singular integral with the general non-isotropic kernel in the form of

$$\left(L_\alpha^\lambda f\right)(x) = \int_{R^n} f(t) K_\alpha\left(|x-t|_\lambda\right) dt, \tag{5}$$

where α is a positive parameter which takes values in some number set, having a limit point 0.

The following theorem gives the condition of convergence of integral operators (5) in the λ-Lebesgue points of function f.

We need the following definition.

Definition: The point $x \in R^n$ will be called λ-Lebesgue point of function f, at this point, if it has the following properties;

$$\lim_{h\to 0} \frac{1}{h^{2|\lambda|}} \int_{|t|_\lambda^{\frac{n}{2|\lambda|}} \le h} |f(x-t) - f(x)| dt = 0. \tag{6}$$

Theorem1. Let $\int_{R^n} K_\alpha\left(|t|_\lambda\right) dt = 1$, $\Psi_\lambda(x) = ess.\sup\left|K_\alpha\left(|t|_\lambda\right)\right|$ and

$K_\alpha\left(|t|_\lambda\right) = \alpha^{-|\lambda|} K_\alpha\left(\left|\frac{t}{\alpha^\lambda}\right|_\lambda\right)$ for $\alpha > 0$. If $f \in L_p$ $(1 \le p \le \infty)$ then

$$\lim_{\alpha\to 0}\left(L_\alpha^\lambda f\right)(x) = f(x) \tag{7}$$

whenever x is a λ-Lebesgue point of function f.

Proof. Since x is a λ-Lebesgue point of function f, $\forall \varepsilon > 0$ there is a $\delta > 0$ such that for $h < \delta$

$$\int_{|t|_\lambda^{\frac{n}{2|\lambda|}} \le h} |f(x-t)-f(x)|dt < \varepsilon h^{2|\lambda|} . \tag{8}$$

We first observe the properties of function Ψ_λ. This is a radial function and from definition of Ψ_λ we see that this is a decreasing function. Thus

$$\int_{\frac{r^{\frac{1}{2}}}{2} < |x|_\lambda^{\frac{n}{2|\lambda|}} < r^{\frac{1}{2}}} \Psi_\lambda(t) \ge \Psi_\lambda\left(r^{\frac{1}{2}}\right) \int_{\frac{r^{\frac{1}{2}}}{2} < |x|_\lambda^{\frac{n}{2|\lambda|}} < r^{\frac{1}{2}}} dt$$

$$= \Psi_\lambda\left(r^{\frac{1}{2}}\right) \int_{\frac{r^{\frac{1}{2}}}{2}}^{r^{\frac{1}{2}}} \int_{S^{n-1}} \Omega_\lambda(\theta)\rho^{2|\lambda|-1} d\theta d\rho$$

$$= \Psi_\lambda\left(r^{\frac{1}{2}}\right) r^{|\lambda|} \left(\frac{1}{2|\lambda|} - \frac{1}{2|\lambda| 2^{2|\lambda|}}\right) w_{\lambda,n-1}.$$

That is, for $r \to \infty$ and $r \to 0$ we have $\Psi_\lambda\left(r^{\frac{1}{2}}\right) r^{|\lambda|} \to 0$. There is a constant A such that

$$\Psi_\lambda\left(r^{\frac{1}{2}}\right) r^{|\lambda|} \le A \tag{9}$$

for $0 \le r \le \infty$. Using generalized spherical coordinates to calculate the integral (6) we have

$$\lim_{h \to 0} \frac{1}{h^{2|\lambda|}} \int_0^h \left\{ \int_{S_{n-1}} \left| f\left(x - (\rho\theta)^{2|\lambda|}\right) - f(x) \right| \Omega_\lambda(\theta) d\theta \right\} \rho^{2|\lambda|-1} d\rho = 0 .$$

For simplicity, let us write

$$g_\lambda(\rho) =: \int_{S_{n-1}} \left| f\left(x - (\rho\theta)^{2|\lambda|}\right) - f(x) \right| \Omega_\lambda(\theta) d\theta .$$

and

$$G_\lambda(\rho) =: \int_0^\rho g_\lambda(\xi) \xi^{2|\lambda|-1} d\xi.$$

Therefore the integral (8) is equivalent to

$$G_\lambda(\rho) \le \varepsilon^{2|\lambda|}. \tag{10}$$

Since $\int_{R^n} K_\alpha\left(|t|_\lambda\right) dt = 1$ for all $\alpha > 0$, we have

$$\left|\left(L_\alpha^\lambda f\right)(x)-f(x)\right|=\left|\int_{E^n}\{f(x-t)-f(x)\}K_\alpha\left(|t|_\lambda\right)dt\right|$$
$$\leq \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta}|f(x-t)-f(x)|K_\alpha\left(|t|_\lambda\right)dt$$
$$+\int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta}|f(x-t)-f(x)|K_\alpha\left(|t|_\lambda\right)dt$$
$$=I_1+I_2$$

Now, we estimate I_1. Using above notations and the observations, we get

$$I_1\leq\int_0^\delta\left\{\int_{S_{n-1}}\left|f(x-(\rho\theta)^{2|\lambda|})-f(x)\right|\Omega_\lambda(\theta)d\theta\right\}\rho^{2|\lambda|-1}\alpha^{-|\lambda|}\Psi_\lambda\left(\tfrac{\rho}{\alpha^{\frac{1}{2}}}\right)d\rho$$
$$=\int_0^\delta g_\lambda(\rho)\rho^{2|\lambda|-1}\alpha^{-|\lambda|}\Psi_\lambda\left(\tfrac{\rho}{\alpha^{\frac{1}{2}}}\right)d\rho$$
$$=G_\lambda(\rho)\alpha^{-|\lambda|}\Psi_\lambda\left(\tfrac{\rho}{\alpha^{\frac{1}{2}}}\right)d\rho\Big|_0^\delta-\int_0^\delta G_\lambda(\rho)d\left(\alpha^{-|\lambda|}\Psi_\lambda\left(\tfrac{\rho}{\alpha^{\frac{1}{2}}}\right)\right)$$
$$\leq\varepsilon\rho^{2|\lambda|}\alpha^{-|\lambda|}\Psi_\lambda\left(\tfrac{\rho}{\alpha^{\frac{1}{2}}}\right)d\rho\Big|_0^\delta-\int_0^{\frac{\delta}{\alpha^{\frac{1}{2}}}}G_\lambda(\alpha^{\frac{1}{2}}u)\alpha^{-|\lambda|}d(\Psi_\lambda(u))$$
$$=\varepsilon\left(\frac{\delta^2}{\alpha}\right)^{|\lambda|}\Psi_\lambda\left(\frac{\delta}{\alpha^{\frac{1}{2}}}\right)-\int_0^{\frac{\delta}{\alpha^{\frac{1}{2}}}}\varepsilon\alpha^{|\lambda|}u^{2|\lambda|}\alpha^{-|\lambda|}d(\Psi_\lambda(u))$$
$$=\varepsilon\left(A-\int_0^\infty u^{2|\lambda|}d(\Psi_\lambda(u))\right).$$

We calculate right hand side of the integral as

$$-\int_0^\infty u^{2|\lambda|}d(\Psi_\lambda(u))=\lim_{r\to\infty}\left(-r^{2|\lambda|}\Psi_\lambda(r)\right)+2|\lambda|\int_0^\infty u^{2|\lambda|}\Psi_\lambda(u)du$$
$$=\frac{2|\lambda|}{w_{\lambda,n-1}}\int_{R^n}\Psi_\lambda(x)dx.$$

Since the integral is finite, we have

$$I_1\leq\varepsilon B, \tag{11}$$

where B depens only on Ψ_λ.

Let us consider I_2. We denote ψ_δ which is the characteristic function of the set of all $x \in R^n$ such that $|x|_\lambda^{\frac{n}{2|\lambda|}} > \delta$.

If $\frac{1}{p}+\frac{1}{p'}=1$ then using Hölder inequality we have

$$I_2 \le \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} |f(x-t)|K_\alpha(|t|_\lambda)dt + \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} |f(x)|K_\alpha(|t|_\lambda)dt$$

$$\le \|f\|_p \|\psi_\delta K_\alpha(|t|_\lambda)\|_{p'} + |f(x)| \|\psi_\delta K_\alpha(|t|_\lambda)\|_1$$

Now, we calculate right hand side of the integrals;

$$\|\psi_\delta K_\alpha(|t|_\lambda)\|_1 = \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} K_\alpha(|t|_\lambda)dt$$

$$= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} \alpha^{-|\lambda|} K\left(\left|\frac{t}{\alpha^\lambda}\right|_\lambda\right)dt$$

$$= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\frac{\delta}{\sqrt{\alpha}}} \alpha^{-|\lambda|} K(|t|_\lambda)\alpha^{|\lambda|}dt$$

$$= \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\frac{\delta}{\sqrt{\alpha}}} K(|t|_\lambda)dt.$$

For $\alpha \to 0$, it is convergent to 0. So

$$\|\psi_\delta K_\alpha(|t|_\lambda)\|_{p'} = \left(\int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} [K_\alpha(|t|_\lambda)]^{p'} dt \right)^{\frac{1}{p'}}$$

$$= \left(\int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} K_\alpha(|t|_\lambda)[K_\alpha(|t|_\lambda)]^{\frac{p'}{p}} dt \right)^{\frac{1}{p'}}$$

$$\le \|\psi_\delta K_\alpha\|_\infty^{\frac{1}{p}} \|\psi_\delta K_\alpha\|_1^{\frac{1}{p'}}.$$

$$\|\psi_\delta K_\alpha\|_\infty = \sup_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} K_\alpha(|t|_\lambda) = \alpha^{-|\lambda|} \sup_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\frac{\delta}{\sqrt{\alpha}}} K(|t|_\lambda) \to 0 \quad (\alpha \to 0).$$

Therefore, we have shown that for sufficiently small ε

$$\left|\left(L_\alpha^\lambda f\right)(x)-f(x)\right|$$

is bounded by a constant depending only on Ψ_λ. Then, Theorem 1 is proved.

Corollary 1. Let a function f satisfies at fixed x the condition of Theorem 1. Then at this point x the following statements hold for non-isotropic Gauss-Weierstrass and Abel-Poisson integrals,

a)

$$\lim_{\alpha\to 0}\int_{R^n} f(x-t)P_\lambda(t,\alpha)dt=f(x)$$

b)

$$\lim_{\alpha\to 0}\int_{R^n} f(x-t)W_\lambda(t,\alpha)dt=f(x)$$

Now, we give a theorem about the order of convergence of integral operators family (5);

Theorem 2. Let β be real parameter, $\int_{R^n} K_\alpha(|t|_\lambda)dt=1$ and

$\Delta_\alpha(\lambda,\beta)=:\int_0^\infty \rho^{2|\lambda|+\beta-1}K_\alpha(\rho)d\rho\to 0\ \ (\alpha\to 0)$. Let

i) $K_\alpha(\rho)$ be a decreasing function

ii) For $\forall\delta>0\ \ K_\alpha(\delta)=o(\Delta_\alpha(\lambda,\beta))\ \ (\alpha\to 0)$

iii) $\int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} K_\alpha(|t|_\lambda)dt=o(\Delta_\alpha(\lambda,\beta))\ \ (\alpha\to 0)$

If the function $f\in L_1(R^n)$ satisfies the condition

$$\lim_{h\to 0}\frac{1}{h^{2|\lambda|+\beta}}\int_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le h}\left|f(x-t)-f(x)\right|dt=0 \tag{13}$$

at the point x then

$$\left|\left(L_\alpha^\lambda f\right)(x)-f(x)\right|=o(\Delta_\alpha(\lambda,\beta))\quad(\alpha\to 0).$$

Proof: Since the point x satisfies the condition (13), we have

$$\lim_{h\to 0}\frac{1}{h^{2|\lambda|+\beta}}\int_0^h\left\{\int_{S^{n-1}}\left|f(x-(\theta\rho)^{2|\lambda|})-f(x)\right|\Omega_\lambda(\theta)d\theta\right\}\rho^{2|\lambda|-1}d\rho=0.$$

If we denote

$$F_\lambda(\rho)=:\int_{S^{n-1}}\left|f(x-(\theta\rho)^{2|\lambda|})-f(x)\right|\Omega_\lambda(\theta)d\theta \tag{14}$$

then we have

$$\lim_{h\to 0}\frac{1}{h^{2|\lambda|+\beta}}\int_0^h F_\lambda(\rho)\rho^{2|\lambda|-1}d\rho = 0.$$

For simplicity let us write

$$A_\lambda(\rho) =: \int_0^\rho F_\lambda(\xi)\xi^{2|\lambda|-1}d\xi.$$

Then, $\forall\varepsilon > 0$ there is a $\delta > 0$ such that for $\rho < \delta$,

$$A_\lambda(\rho) \le \varepsilon\rho^{2|\lambda|+\beta}. \tag{15}$$

Obviously, for $\delta > 0$ we have

$$\begin{aligned}\left|\left(L_\alpha^\lambda f\right)(x) - f(x)\right| &= \left|\int_{R^n}\{f(x-t) - f(x)\}K_\alpha(|t|_\lambda)dt\right| \\ &\le \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}\le\delta}|f(x-t) - f(x)|K_\alpha(|t|_\lambda)dt \\ &\quad + \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta}|f(x-t) - f(x)|K_\alpha(|t|_\lambda)dt \\ &= I_1 + I_2.\end{aligned} \tag{16}$$

Let us consider I_1. Passing to generalized spherical coordinates and using (14) and (15), we have

$$\begin{aligned}I_1 &\le \int_0^\delta\left\{\left|f(x-(\theta\rho)^{2|\lambda|}) - f(x)\right|\Omega_\lambda(\theta)d\theta\right\}K_\alpha(\rho)\rho^{2|\lambda|-1}d\rho \\ &= \int_0^\delta F_\lambda(\rho)K_\alpha(\rho)\rho^{2|\lambda|-1}d\rho \\ &= \int_0^\delta K_\alpha(\rho)d(A_\lambda(\rho)) \\ &= K_\alpha(\rho)A_\lambda(\rho)\Big|_0^\delta + \int_0^\delta A_\lambda(\rho)d\left[-K_\alpha(\rho)\right] \\ &\le \varepsilon\delta^{2|\lambda|+\beta}K_\alpha(\delta) + \varepsilon\left\{-\rho^{2|\lambda|+\beta}K_\alpha(\rho)\Big|_0^\delta + (2|\lambda|+\beta)\int_0^\delta K_\alpha(\rho)\rho^{2|\lambda|+\beta-1}d\rho\right\} \\ &= \varepsilon(2|\lambda|+\beta)\int_0^\delta K_\alpha(\rho)\rho^{2|\lambda|+\beta-1}d\rho.\end{aligned}$$

Therefore, we have

$$\begin{aligned}I_1 &\le \varepsilon\left(2|\lambda|+\beta\right)\Delta_\alpha(\lambda,\beta) \\ &\le C\varepsilon\Delta_\alpha(\lambda,\beta).\end{aligned} \tag{17}$$

Now let us consider I_2.

$$I_2 \le \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} |f(x-t)| K_\alpha(|t|_\lambda)dt + |f(x)| \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} K_\alpha(|t|_\lambda)dt$$

$$\le \|f\| K_\alpha(\delta) + |f(x)| \int_{|t|_\lambda^{\frac{n}{2|\lambda|}}>\delta} K_\alpha(|t|_\lambda)dt$$

and by hypothesis i) and ii)

$$I_2 = o(\Delta_\alpha(\lambda,\beta)) \quad (\alpha \to 0). \tag{18}$$

Hence, from (17) and (18) we have

$$\left|(L_\alpha^\lambda f)(x) - f(x)\right| \le \varepsilon \Delta_\alpha(\lambda,\beta) + o(\Delta_\alpha(\lambda,\beta))$$

$$= o(\Delta_\alpha(\lambda,\beta)) \quad (\alpha \to 0).$$

Corollary 2. Let a function f satisfies, at fixed point x, the condition (13). Then at this point x for order of convergence of non-isotropic Gauss-Weierstrass and Abel-Poisson integrals the following statements hold:

a)

$$\int_{R^n} f(x-t)P_\lambda(t,\alpha)dt = f(x) = o(\alpha^\beta) \quad (\alpha \to 0) \text{ and } (\beta > 0).$$

b)

$$\int_{R^n} f(x-t)W_\lambda(t,\alpha)dt = f(x) = o(\alpha^{-\frac{\beta}{2}}) \quad (\alpha \to 0) \text{ and } (\beta > 0).$$

Proof. Since non-isotropic Gauss-Weierstrass and Abel-Poisson integrals satisfy the condition of Theorem 2, we must calculate $\Delta_\alpha(\lambda,\beta)$.

First, we calculate for Gauss-Weierstrass kernel;

$$\Delta_\alpha(\lambda,\beta) = c_n \alpha^{|\lambda|} \int_0^\infty \rho^{2|\lambda|+\beta-1} e^{-\alpha\rho^2} d\rho$$

$$= \frac{c_n \alpha^{-\frac{\beta}{2}}}{2} \Gamma\left(|\lambda| + \tfrac{\beta}{2}\right)$$

For Poisson kernel; by a change of variables with $\rho = \alpha \tan(u)$, we have

$$\Delta_\alpha(\lambda,\beta) = c_n \alpha \int_0^\infty \frac{\rho^{2|\lambda|+\beta-1}}{(\alpha^2+\rho^2)^{\frac{2|\lambda|+1}{2}}} d\rho$$

$$= c_n \alpha^\beta \int_0^{\frac{\pi}{2}} (\sin u)^{2|\lambda|+\beta-1} (\cos u)^{-\beta} du.$$

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