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ON THE CLOSED MOTIONS AND CLOSED SPACE-LIKE RULED SURFACES

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ABSTRACT

The theorems about closed motion in Euclidean space is given in [2] and [3]. In this paper, using the real Steiner rotation and translation vectors, we have given some theorems about the real pitch and the real angle of pitch which are the integral invariants of the closed space-like ruled surfaces. Taking a space-like line x in Frenet trihedron $\{t,n,b\}$, the real angle of pitch of space-like ruled surface, which is drawn by the space-like line x, is calculated by the real angle of pitchs of space-like ruled surfaces those are drawn by the space-like lines t and b. Then, we give some theorems about dral and harmonic curvature in $\mathbf{R}_1^3$.

Let us consider Minkowski 3-space $\mathbf{R}_1^3 = [R^3,(+,+,-)]$ and let the Lorentzian inner product of $x=(a_1,a_2,a_3)$ and $y=(b_1,b_2,b_3) \in R^3$ be $\langle x,y\rangle = a_1b_1 + a_2b_2 - a_3b_3$ .Thus, following conditions can be given:

i) if $\langle x,x\rangle > 0$, x is said to be space-like,

ii) if $\langle x,x\rangle < 0$, x is said to be time-like,

iii) if $\langle x,x\rangle = 0$, x is said to be light-like (null)

If $\sqrt{a_1^2+a_2^2} < a_3$ (or $\sqrt{a_1^2+a_2^2} > a_3$), thus x is a future pointing (or past pointing) vector,[6].

Lemma 1 : Let X and Y be future pointing time-like unit vectors. If θ is a hyperbolic angle between X and Y then we have

$$\cosh\theta = -\langle X,Y\rangle ,$$

[1].

Let a moving space-like line space H be represented by the moving frame $\{O; v_1, v_2, v_3\}$ and H' be represented by the fixed frame $\{O'; e_1, e_2, e_3\}$. Where, H is moving on a differentiable space-like closed curve r=r(s). We know that, any space-like line in H is drawing a closed space-like ruled surface in H' along the motion. Thus, the relation of the closed space-like ruled surface can be written by

$$x(s,v) = r(s) + v\, v_1(s) \quad , \quad x(s+T,v) = x(s,v) \qquad \|v_1\| = 1 \tag{1}$$

During the motion, we assume that v_1 and v_2 are space-like vectors and v_3 is a time-like vector. This closed ruled surface is generated by the axes- v_1. By taking differential, from (1), we may write the differential equation of the ortogonal trajectory of v_1-closed space-like ruled surface as follows:

$$\langle dx, v_1 \rangle = 0 \quad , \quad \|v_1\| = 1 \tag{2}$$

Using the equation (2) , we have

$$dv = -\langle dr, v_1 \rangle \tag{3}$$

Definition 1: The pitch (öffnungsstrecke) of $v_1(s)$ – closed space-like ruled surface is defined by

$$\ell_{v_1} = \oint dv = -\oint \langle dr, v_1 \rangle \tag{4}$$

This definition means that, after one periodic , an orthogonal trajectory of $v_1(s)$ – closed space-like ruled surface intersects the axis v_1 at the P_1 different from P_0. Thus , $\ell_{v_1} = \overline{P_0 P_1}$

Now in order to rewrite (4) in terms of the elements of the dual Steiner vector, we use the following expression for the diffential velocity of the fixed point $r(t_0)$ of the moving space H with respect to the fixed space H':

$$dr = \psi^* + \psi \wedge r \tag{5}$$

Where, ψ^* is the moment vector with respect to a fixed point. In (5), since ψ and ψ^* are respectively the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of the motion H/H', they form the instantaneous dual Pfaffian vector $\Psi = \psi + \varepsilon \psi^*$ of the corresponding dual spherical motion K/K'. Then, replacing (5) in (4) we obtain

$$\ell_{v_1} = -\oint \langle v_1, \psi^* \rangle - \oint \langle v_1, \psi \wedge r \rangle \tag{6}$$

Denote the moment vector of **r** , with respect to origin 0 , by **r*** then

$$v_1^* = r \wedge v_1 \tag{7}$$

and the last equation reduces to

$$\ell_{v_1} = -\oint \langle v_1, \psi^* \rangle - \oint \langle \psi, v_1^* \rangle \tag{8}$$

On the other hand the Plückerian normalized line coordinates $v_i,\ v_i^*$ (i=1,2,3) of the fixed line V in H are independent of the motion H/H'. They depend on the choice of V in H.
Then the last expression becomes

$$\ell_{v_1} = -\langle v_1, \oint\psi^* \rangle - \langle v_1^*, \oint\psi \rangle \tag{9}$$

Taking integral of ψ and ψ^*, we may write

$$\oint\psi = d \qquad \oint\psi^* = d^* \tag{10}$$

and then (9) reduces to

$$\ell_{v_1} = -\langle v_1, d^* \rangle - \langle v_1^*, d \rangle \tag{11}$$

Let us consider a unit time-like vector $\mathbf{n_2}$ and space like unit vector $\mathbf{n_3}$ on (v_2, v_3) which is defined as follows:

$$\begin{aligned} n_2 &= \operatorname{sh}\varphi\, v_2 + \operatorname{ch}\varphi\, v_3 \\ n_3 &= \operatorname{ch}\varphi\, v_2 + \operatorname{sh}\varphi\, v_3 \end{aligned} \tag{12}$$

The time like unit vector $\mathbf{n_2}$ generates a time-like ruled surface along the ortogonal trajectory of v_1-closed space-like ruled surface during the closed motion. Where φ is the hyperbolic angle between unit time-like vectors $\mathbf{n_2}$ and v_3. Thus, the equation of the time-like surface is

$$T = x + w\, n_2 \qquad w \in \mathbf{R} \tag{13}$$

Now, let us consider fixed space H' which is represented by ortogonal frame $\{n_1, n_2, n_3\}$. Using the equation (12), we may write

$$\begin{aligned} v_2 &= n_3 \operatorname{ch}\varphi - n_2 \operatorname{sh}\varphi \\ v_3 &= -n_3 \operatorname{sh}\varphi + n_2 \operatorname{ch}\varphi, \qquad \varphi = \varphi(s) \end{aligned} \tag{14}$$

Then, from (14), taking differential according to the parameter s, we obtain

$$\begin{aligned} dv_2 &= dn_3 \operatorname{ch}\varphi - dn_2 \operatorname{sh}\varphi + (n_3 \operatorname{sh}\varphi - n_2 \operatorname{ch}\varphi)\, d\varphi \\ dv_3 &= -dn_3 \operatorname{sh}\varphi + dn_2 \operatorname{ch}\varphi + (-n_3 \operatorname{ch}\varphi + n_2 \operatorname{sh}\varphi)\, d\varphi \end{aligned} \tag{15}$$

where n_2 and n_3 are the edges of fixed frame $\{n_1, n_2, n_3\}$. Therefore,

$$dn_2 = 0 \qquad dn_3 = 0 \tag{16}$$

Thus, we obtain

$$\begin{aligned} dv_2 &= (n_3 \operatorname{sh}\varphi - n_2 \operatorname{ch}\varphi)\, d\varphi \\ dv_3 &= (-n_3 \operatorname{ch}\varphi + n_2 \operatorname{sh}\varphi)\, d\varphi \end{aligned} \tag{17}$$

Then, using the equation (14), we get the following

$$\begin{aligned} dv_2 &= -v_3\, d\varphi \\ dv_3 &= -v_2\, d\varphi \end{aligned} \tag{18}$$

From (18), $d\varphi$ is calculated as

$$d\varphi = \langle dv_2, v_3\rangle = -\langle dv_3, v_2\rangle \tag{19}$$

If we take integral, from (19), during the motion H/H', λ_{v_1} is obtained as

$$\lambda_{v_1} = \oint_{(r)} d\varphi \tag{20}$$

or

$$\lambda_{v_1} = \oint_{(r)} d\varphi = \oint \langle dv_2, v_3\rangle = -\oint \langle dv_3, v_2\rangle \tag{21}$$

On the other hand, the differential forms of the edges of the moving trihedron $\{v_1, v_2, v_3\}$ can be written as the following

$$\begin{bmatrix} dv_1 \\ dv_2 \\ dv_3 \end{bmatrix} = \begin{bmatrix} 0 & w_3 & w_2 \\ -w_3 & 0 & w_1 \\ w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{22}$$

Thus, the real instantaneous vector of the motion H/H' is

$$\psi = w_1 v_1 - w_2 v_2 - w_3 v_3 \tag{23}$$

Then, from (22),

$$\langle dv_2, v_3\rangle = -\langle dv_3, v_2\rangle = -w_1 \tag{24}$$

Since we have the relation

$$\lambda_{v_1} = -\oint w_1 \tag{25}$$

Definition 2 : The integral of the real instantaneous vector ψ of the motion will be called Stainer vector of the motion and denoted by

$$d = \oint w_1 v_1 - w_2 v_2 - w_3 v_3 \tag{26}$$

Using definition 2,

$$\langle d, v_1\rangle = \oint w_1 \tag{27}$$

and so we get

$$\lambda_{v_1} = -\langle d, v_1\rangle \tag{28}$$

Thus , the angle of pitch of the closed space-like ruled surface is defined as the total change of the hyperbolic angle φ which is given in the following relation:

$$\lambda_{v_1} : \oint d\varphi = \oint \langle dv_2, v_3\rangle \tag{29}$$

The pitch and the hyperbolic angle of the pitch are integral invariants of a closed space-like ruled surface, [7].

According to the definition of Stainer translation vector

$$v = \oint_{(\alpha)} dr \tag{30}$$

and from here, we get

$$\ell_x = -\langle v, v_1 \rangle \quad (31)$$

Thus, we have following theorem.

Theorem 1. The real pitch and the real angle of pitch are equal the orthogonal projections of the Steiner rotation vector and the translation vector,respectively, at the direction of generating line v_1 of the closed space-like ruled surface. (i.e)

$$\ell_{v_1} = -\langle v, v_1 \rangle \quad (32)$$

$$\lambda_{v_1} = -\langle d, v_1 \rangle \quad (33)$$

Let H: $\{t, n, b\}$ be a moving trihedron during the motion H/H'. Where, t , n are sapace-like vectors and b is a time-like vector. From [5],using the derivative formulas of the Frenet trihedron $\{t, n, b\}$, the real Stainer rotation vector of the motion is

$$d = \oint_{(\alpha)} (k_2 t - k_1 b)\, ds \quad (34)$$

and the real translation vector is

$$v = t \oint_{(r)} ds \quad (35)$$

On the other hand, the real angle of pitchs of the ruled surfaces which are drawn during the motion H/H' by the edges of the trihedron $\{t, n, b\}$ are written in the following relations, respectively.

$$\lambda_t = -\langle d, t \rangle = -\oint k_2\, ds \quad (36)$$

$$\lambda_n = \langle d, n \rangle = 0 \quad (37)$$

$$\lambda_b = \oint k_1 ds \quad (38)$$

Moreover, the real pitch of the edges of these space-like and time-like ruled surfaces are obtained by

$$\ell_t = -\oint \langle dr, t \rangle \quad (39)$$

$$= -\oint \langle t\, ds, t \rangle$$
$$= -\oint ds \quad (40)$$

$$\ell_n = -\oint \langle dr, n \rangle = 0$$
$$\ell_b = 0 \quad (41)$$

Theorem 2. The components of Stainer rotational vector with respect to the trihedron $\{t, n, b\}$ are equal to the real angle of pitchs of the ruled surfaces which are drawn by the edges t, n, b, respectively, so the Stainer vector is written by these components as follows:

$$d = -\lambda_t t - \lambda_b b \quad . \quad (42)$$

Let

$$x = x_1 t + x_2 n + x_3 b \quad , \qquad x_1^2 + x_2^2 - x_3^2 = 1 \tag{43}$$

be a space-like line in $\{t,n,b\}$. While $\{t,n,b\}$ is moving during the motion, x draws a closed space-like ruled surfaces. Thus, we may calculate the real angle of pitch of this closed space-like ruled surfaces as follows:

$$\lambda_x = -\langle d, x \rangle \tag{44}$$

$$= \langle \lambda_t t + \lambda_b b, x_1 t + x_2 n + x_3 b \rangle$$

$$= \lambda_t x_1 - \lambda_b x_3 \tag{45}$$

Theorem 3 : The real angle of pitch of any closed space-like ruled surface (x) , which is drawn by fixed any space-like line during the motion H/H' in fixed space H is ,

$$\lambda_x = \lambda_t x_1 - \lambda_b x_3$$

, where, λ_t and λ_b are the real angle of pitchs of the closed ruled surfaces which are drawn by the edges t and b,respectively.

Moreover, for the real pitch of this space-like ruled surface, which is drawn by the fixed space-like line x , we get

$$\ell_x = -\oint \langle dr, x \rangle \tag{46}$$

$$= -\oint \langle t\, ds, x_1 t + x_2 n + x_3 b \rangle \tag{47}$$

$$= -x_1 \oint ds$$

$$= x_1 \ell_t . \tag{48}$$

Thus what we get is.

Theorem 4. The real pitch of any closed space-like ruled surface (x) , which is drawn by the fixed any space-like line during the motion H/H' in the fixed space H is equal to the multiple of ℓ_t and x_1

Let , x drawn a developable closed space-like ruled surface. In this case, the dral of the closed space-like ruled surface is zero [4]. Thus,

$$\frac{dx}{ds} = x_1 k_1 n + x_2(-k_1 t + k_2 b) + x_3(k_2 n) \tag{49}$$

$$= -x_2 k_1 t + (x_1 k_1 + x_3 k_2) n + x_2 k_2 b \tag{50}$$

$$\frac{dr}{ds} \wedge x = t \wedge x \tag{51}$$

$$= x_2 b + x_3 n \tag{52}$$

so

$$\det\left[\frac{dr}{ds}, x, \frac{dx}{ds}\right] = -\left\langle t \wedge x, \frac{dx}{ds} \right\rangle \tag{53}$$

$$\det\left[\frac{dr}{ds}, x, \frac{dx}{ds}\right] = -\left\langle x_2 b + x_3 n, \frac{dx}{ds}\right\rangle \tag{54}$$

$$\det\left[\frac{dr}{ds}, x, \frac{dx}{ds}\right] = x_2^2 k_2 - x_3(x_1 k_1 + x_3 k_2) = 0 \tag{55}$$

is obtained. Using the equation (55) , we get the following

$$(x_2^2 - x_3^2)k_2 - x_1 x_3 k_1 = 0 \tag{56}$$

$$\Rightarrow \quad \frac{k_1}{k_2} = \frac{x_2^2 - x_3^2}{x_1 x_3} \tag{57}$$

Then, from (43), (57) becomes

$$\frac{k_1}{k_2} = \frac{1 - x_1^2}{x_1 x_3} \tag{58}$$

Solving x_1 and x_3 from the equation (48) and (45) , we get the following

$$x_1 = \frac{\ell_x}{\ell_t} \tag{59}$$

$$\lambda_x = \frac{\ell_x}{\ell_t}\lambda_t - x_3\lambda_b \tag{60}$$

$$x_3 = \frac{\ell_x \lambda_t - \lambda_x \ell_t}{\ell_t \lambda_b} \tag{61}$$

Then, if we insert (59) and (61) into (58),

$$\frac{k_1}{k_2} = \frac{(\ell_t^2 - \ell_x^2)\lambda_b}{(\ell_x\lambda_t - \lambda_x\ell_t)\ell_x} \tag{62}$$

is obtained.

On the otherhand , x is fixed line in $\{t, n, b\}$.Hence, the components of x in $\{t, n, b\}$ are fixed. Furthermore, from (57), $\frac{k_1}{k_2}$ is constant. Thus, we may write the following theorem:

Theorem 5. The closed space-like ruled surface which is drawn by a space-like line x during the motion H/H' is developable if and only if the harmonic *curvature, which is calculated following formula, (63),* of the base curve of the closed space-like ruled surface (x), is constant.

$$h = \frac{k_1}{k_2} = \frac{(\ell_t^2 - \ell_x^2)\lambda_b}{(\ell_x\lambda_t - \lambda_x\ell_t)\ell_x} \tag{63}$$

Using the definition of dral in [4], we may calculate the dral of the closed space-like ruled surface (x) which is drawn by space-like tangent line t, as follows

$$P_t = \frac{\det(r', t, t')}{\langle t', t' \rangle} \tag{64}$$

$$= \frac{\det(t, t, t')}{\langle k_1 n, k_1 n \rangle} \tag{65}$$

$$= 0 \tag{66}$$

In the Similar way, the dral of the closed space-like ruled surface (x), which is drawn by the space-like principle normal line, is

$$P_n = \frac{\det(\alpha', n, n')}{\langle n', n' \rangle} \tag{67}$$

$$= \frac{\det(t, n, -k_1 t + k_2 b)}{\langle -k_1 t + k_2 b, -k_1 t + k_2 b \rangle} \tag{68}$$

$$= \frac{-\langle b, -k_1 t + k_2 b \rangle}{k_1^2 - k_2^2} \tag{69}$$

$$= \frac{k_2}{k_1^2 - k_2^2} \tag{70}$$

and the dral of the closed space-like ruled surface (x), which is drawn by the time-like binormal line, is

$$P_b = \frac{\det(r', b, b')}{\langle b', b' \rangle} \tag{71}$$

$$= \frac{-\langle t \wedge b, k_2 n \rangle}{\langle k_2 n, k_2 n \rangle} \tag{72}$$

$$= \frac{-\langle n, k_2 n \rangle}{k_2^2} \tag{73}$$

$$= -\frac{k_2}{k_2^2} \tag{74}$$

$$= -\frac{I}{k_2} \tag{75}$$

Furthermore, (74) is written as

$$P_n = \frac{1/k_2}{\left(\frac{k_1}{k_2}\right)^2 - 1} \tag{76}$$

Then, if we consider (75),

$$P_n = \frac{-P_b}{\left(\frac{k_1}{k_2}\right)^2 - 1} \tag{77}$$

so

$$\left(\frac{k_1}{k_2}\right)^2 - 1 = -\frac{P_b}{P_n} \tag{78}$$

and

$$\left(\frac{k_1}{k_2}\right)^2 = -\frac{P_b}{P_n} + 1 \tag{79}$$

are obtained. Thus, we have the following theorem:

Theorem 6. The harmonic curvature of the closed space-like curve r(t) of the space-like ruled surface (x), during the space motion H/H', is calculated as follows:

$$\left(\frac{k_1}{k_2}\right)^2 = -\frac{P_b}{P_n} + 1 \tag{80}$$

Where P_b and P_n are the drals of the closed space-like and timelike ruled surfaces which are drawn by t and n.

Corollary 1: r(t) is a helix, if and only if, $\frac{P_b}{P_n}$ is constant.

Proof: If r(t) is a helix, then, $\frac{k_1}{k_2}$ is constant. Thus, from (80), $\frac{P_b}{P_n}$ is constant.

If $\frac{P_b}{P_n}$ is constant, from(80), $\frac{k_1}{k_2}$ is constant. Thus r(t) is a helix.

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