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SOME RESULTS ON THE SHEAF OF THE HOMOLOGY GROUPS

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ABSTRACT

Let X be a connected and locally path connected topological space. Constructing the sheaf of homology groups over X, its some characterizations are given. Furthermore, defining the Generalized Whitney Sum and Direct Sum of the sheaves $\overline{H_1}$ and $\overline{H_2}$, it is shown that the sheaf $(\overline{H})^* = \overline{H}_1 \oplus \overline{H}_2$ is isomorphic to the sheaf $\overline{H}_1 \times \overline{H}_2$. Where, $\overline{H}_1$ and $\overline{H}_2$ are sheaves of homology groups over X_1 and X_2, respectively.

1. INTRODUCTION

Let X be a connected, locally arcwise connected topological space. For an arbitrary fixed point $c \in X$, let us consider $X = (X, c)$, where (X, c) be a pointed space. Let H be the sheaf of fundamental groups constructed over X, let $\Gamma(X, H)$ be the group of global sections of H over X, let $K \subset \Gamma(X, H)$ be the commutator subgroup and let $N \subset \Gamma(X, H)$ be any normal subgroup such that $K \subset N$. Then N defines the subsheaf $H(N)$ of H such that for any $x \in X$ stalk $H(N)_x = \{s(x) : s \in N\}$ is a normal subgroup of H_x. It is known that the sheaves H and $H(N)$ are regular covering space of X [1,4]. Also, the quotion group $H_x / H(N)_x$ is abelian. Let us denote the quotion group $H_x / H(N)_x$ by $\overline{H}(N)_x$.

Let $\overline{H}(N) = \bigvee_{x \in X} \overline{H}(N)_x$. $\overline{H}(N)$ is a set over X. Let us define a mapping $\psi : \overline{H}(N) \to X$ with $\psi(\overline{\sigma_x}) = x$, for any $\overline{\sigma}_x = \overline{s(x)} \in \overline{H}(N)_x \subset \overline{H}(N)$. Also, let us define a mapping $\overline{s} : W \to \overline{H}(N)$ such that $\overline{s}(x) = \overline{s(x)} = \overline{[(y^{-1}\alpha)y]}_x$ for any

$x \in W$, where $s \in \Gamma(X, H)$. $\bar{s} = \bar{s}(\bar{\sigma}_c)$, because the homotopy class $[\gamma]$ is choosen as arbitrarily fixed for every $x \in W$. Clearly, $\psi o \bar{s} = 1_w$. Let us denote the totality of all mapping $\bar{s}$ by $\Gamma(W, \bar{H}(N))$.

Now, if B is a basis of arcwise connected open neighborhoods for each $x \in X$, then

$$\bar{B} = \{\bar{s}(W) : W \in B,\ \bar{s} \in \Gamma(W, \bar{H}(N))\}$$

is a topology base on $\bar{H}(N)$ [3,6]. Thus, $\bar{H}(N)$ is a topological space and the mappings ψ and $\bar{s}$ are continuous in this topology. Furthermore, ψ is a locally topological mapping. Hence, $(\bar{H}(N), \psi)$ is a sheaf over X.

As a definition, the mapping $\bar{s} \in \Gamma(W, \bar{H}(N))$ is said to be a section of $\bar{H}(N)$ over W. Particularly, $\Gamma(X, \bar{H}(N))$ is called the set of the global sections of $\bar{H}(N)$ over X. Also, the set $\bar{H}(N)_x$ is said to be the stalk of the sheaf $\bar{H}(N)$ over X. It should be noticed that the stalk $\bar{H}(N)_x$ is an abelian group. Thus, $\bar{H}(N)$ is a sheaf of abelian groups over X, since the set $\Gamma(X, \bar{H}(N))$ is an abelian group for any open set $W \subset X$.

Finally, if $N = K$, then $\bar{H}(K)_x$ is the Homology Group of X at x for any $x \in X$. Thus, $\bar{H}(K)$ is called "The Sheaf of Homology Groups over X " [9].

After this, we will show $\bar{H}(K)$ with $\bar{H}$.

2. CHARACTERISTIC FEATURES OF THE SHEAF $\bar{H}$

1. Let $W \subset X$ be an open set. Then, any section over W can be extended to a global section over X [2].

2. Any two stalks of $\bar{H}$ are isomorphic with each other.

3. Let $W_1, W_2 \subset X$ be any two open sets, $W_1 \cap W_2 \neq \varnothing$ and $\bar{s}_1 \in \Gamma(W_1, \bar{H}), \bar{s}_2 \in \Gamma(W_2, \bar{H})$ be any two sections. If $\bar{s}_1(x_0) = \bar{s}_2(x_0)$ for any point $x_0 \in W_1 \cap W_2$, then $\bar{s}_1 = \bar{s}_2$ over the whole $W_1 \cap W_2$.

4. Let $W \subset X$ be an open set and $\bar{s}_1, \bar{s}_2 \in \Gamma(W, \overline{H})$. If $\bar{s}_1(x_0) = \bar{s}_2(x_0)$ for any point $x_0 \in W$, then $\bar{s}_1 = \bar{s}_2$ over the whole W.

5. Let $x \in X$ be any point and let $W = W(x)$ be an open set. Then $\psi^{-1}(W) = \bigvee_{i \in I} \bar{s}_i(W)$ for every $\bar{s}_i \in \Gamma(W, \overline{H})$ and $\psi / \bar{s}_i(W) : \bar{s}_i(W) \to W$ is a topological mapping, for each $i \in I$. Thus, each open set W of X is evenly covered by ψ. Therefore, $\overline{H}$ is a covering space of X and ψ is a covering projection. As a covering space, $\overline{H}$ is said to be "The Homology Covering Space of X" [8,9]. Moreover, $\overline{H}$ is a regular covering space.

Theorem 2.1. Let $W \subset X$ be an open set. The Homology Group $\overline{H}_x$ is isomorphic to the group $\Gamma(W, \overline{H})$ for every $x \in W$.

Proof. Let $W \subset X$ be an open set and $\bar{s} \in \Gamma(W, \overline{H})$. Then, there exists a unique element $\bar{\sigma}_x \in \overline{H}_x \subset \overline{H}$ such that

$$\bar{s}(x) = \overline{[(\gamma^{-1}\alpha_1)\gamma]} = \bar{\sigma}_x$$

for every $x \in W$. That is, to each element of $\overline{H}_x$, there correspondence only one element in $\Gamma(W, \overline{H})$. Let us denote this correspondence by $\Phi : \overline{H}_x \to \Gamma(W, \overline{H})$ such that $\Phi(\bar{\sigma}_x) = \bar{s}$ for any $\bar{\sigma}_x \in \overline{H}_x$. Let $\overline{(\sigma_1)_x}, \overline{(\sigma_2)_x} \in \overline{H}_x$. Then $\overline{(\sigma_1)_x}, \overline{(\sigma_2)_x}$ determine the sections $\bar{s}_1, \bar{s}_2 \in \Gamma(W, \overline{H})$, respectively.

Thus,

$$\bar{s}_1(x) = \overline{[(\gamma^{-1}\alpha_1)\gamma]_x} = \overline{(\sigma_1)_x}$$

and

$$\bar{s}_2(x) = \overline{[(\gamma^{-1}\alpha_2)\gamma]_x} = \overline{(\sigma_2)_x}$$

for every $x \in W$. Φ is clearly one to one. Furthermore, as a result of the definition of Φ, Φ is onto.

Now, we will prove that Φ is a homomorphism. Let $\overline{(\sigma_1)_x}, \overline{(\sigma_2)_x} \in \overline{H}_x$. Then, $\overline{(\sigma_1)_x} . \overline{(\sigma_2)_x} \in \overline{H}_x$ defines a section $\bar{s}' \in \Gamma(W, \overline{H})$ such that

$$\overline{s}'(x)=\overline{(s_1.s_2)}(x)=\overline{(\sigma_1)}_x.\overline{(\sigma_2)}_x$$

for every $x\in W$. On the other hand, for every $x\in W$,

$$\begin{aligned}\overline{s}_1(x).\overline{s}_2(x)&=\overline{[(\gamma^{-1}\alpha_1)\gamma]_x}.\overline{[(\gamma^{-1}\alpha_2)\gamma]_x}\\&=\overline{[(\gamma^{-1}\alpha_1\alpha_2)\gamma]_x}\\&=\overline{s_1.s_2}(x).\end{aligned}$$

Thus,

$$\Phi\left(\overline{(\sigma_1)}_x.\overline{(\sigma_2)}_x\right)=\overline{s}_1.\overline{s}_2=\Phi\left(\overline{(\sigma_1)}_x\right).\Phi\left(\overline{(\sigma_2)}_x\right).$$

Now, we can state the following Corollary.

Corollary 2.1. Particularly, $\left(\overline{H}\right)_x\cong\Gamma(X,\overline{H})$.

3. SOME RESULTS ON THE SHEAF OF THE HOMOLOGY GROUPS

Let X_1,X_2 be any connected and locally path connected topological spaces and $\overline{H}_1,\overline{H}_2$ be the corresponding sheaves, respectively. Let us denote these as the pairs $(X_1,\overline{H}_1)$ and $(X_2,\overline{H}_2)$.

Let the pairs $(X_1,\overline{H}_1),(X_2,\overline{H}_2)$ be given. Consider the sets of $\Gamma_1(W_1,\overline{H}_1)$ and $\Gamma_2(W_2,\overline{H}_2)$ being $W_1\subset X_1$ and $W_2\subset X_2$ are open sets. Let

$$M_W=\Gamma_1(W_1,\overline{H}_1)\times\Gamma_2(W_2,\overline{H}_2)$$

such that $W=W_1\times W_2\subset X_1\times X_2$. For an element $\overline{s}=(\overline{s}_1,\overline{s}_2)\in M_W$ and an open set $V\subset W$, where $V=V_1\times V_2$; $V_1\subset W_1$, $V_2\subset W_2$ are open sets, let

$$\begin{aligned}\gamma_{W,V}(\overline{s})&=\gamma_{W,V}((\overline{s}_1,\overline{s}_2))\\&=(\gamma_{W_1,V_1}(\overline{s}_1),\gamma_{W_2,V_2}(\overline{s}_2))\\&=(\overline{s}_1|_{V_1},\overline{s}_2|_{V_2}).\end{aligned}$$

Then, the system $\{X_1 \times X_2, M_W, \gamma_{W,V}\}$ is a pre-sheaf. Thus, forming inductive limit, a sheaf is obtained from the pre-sheaf $\{X_1 \times X_2, M_W, \gamma_{W,V}\}$ [5,7]. This sheaf is called "The Generalized Whitney Sum" of the sheaves $\overline{H}_1$ and $\overline{H}_2$, and denoted by $(\overline{H})^* = \overline{H}_1 \oplus \overline{H}_2$.

It is easily shown that; for each $(x_1, x_2) \in X_1 \times X_2$, the set

$$(\overline{H})^*_{(x_1,x_2)} = \{(W, (\bar{s}_1, \bar{s}_2))_{(x_1,x_2)} : W = W(x_1, x_2) \subset X_1 \times X_2 \; open\ set\}$$

is a group with respect to the operation of multiplication defined by

$$(W, (\bar{s}_1, \bar{s}_2))_{(x_1,x_2)} . (W', (\overline{s'_1}, \overline{s'_2}))_{(x_1,x_2)} = (W'', (\overline{s_1 s'_1}, \overline{s_2 s'_2}))_{(x_1,x_2)},$$

where, $W'' = W''_1 \cap W''_2$ and $W''_1 = W_1 \cap W'_1$, $W''_2 = W_2 \cap W'_2$.

On the other hand, the set $(\overline{H}_1)_{x_1} \times (\overline{H}_2)_{x_2}$ is also a group with respect to the operation of multiplication defined by

$$(\overline{\sigma_1}, \overline{\sigma_2}).(\overline{\sigma'_1}, \overline{\sigma'_2}) = (\overline{\sigma_1 \sigma'_1}, \overline{\sigma_2 \sigma'_2}).$$

We can now give the following theorem.

Theorem 3.1. Let $(\overline{H})^* = \overline{H}_1 \oplus \overline{H}_2$. Then, for each $(x_1, x_2) \in X_1 \times X_2$, the mapping $f : (\overline{H})^*_{(x_1,x_2)} \to (\overline{H_1})_{x_1} \times (\overline{H_2})_{x_2}$ defined by

$$(W, (\bar{s}_1, \bar{s}_2))_{(x_1,x_2)} \to (\bar{s}_1(x_1), \bar{s}_2(x_2)) = (\overline{s_1(x_1)}, \overline{s_2(x_2)})$$

is an isomorphism [4].

From now on, we identify $(\overline{H})^*_{(x_1,x_2)}$ with $(\overline{H_1})_{x_1} \times (\overline{H_2})_{x_2}$.

Let the pairs $(X_1, \overline{H}_1)$ and $(X_2, \overline{H}_2)$ be given. Then $\overline{H}_1 = \bigvee_{x_1 \in X_1} (\overline{H}_1)_{x_1}, \overline{H}_2 = \bigvee_{x_2 \in X_2} (\overline{H}_2)_{x_2}$. Hence,

$$\overline{H}_1 \times \overline{H}_2 = \bigvee_{(x_1,x_2) \in X_1 \times X_2} (\overline{H}_1)_{x_1} \times (\overline{H}_2)_{x_2}$$

is a set over the topological space $X_1 \times X_2$. Moreover, since $\overline{H}_1, \overline{H}_2$ are topological spaces, $\overline{H}_1 \times \overline{H}_2$ is a topological space.

Now, let us define a mapping $\Phi : \overline{H}_1 \times \overline{H}_2 \to X_1 \times X_2$ such that

$$\Phi((\overline{\sigma_1}, \overline{\sigma_2})) = (\psi_1(\overline{\sigma_1}), \psi_2(\overline{\sigma_2})) = (x_1, x_2).$$

$(\overline{H}_1 \times \overline{H}_2, \Phi)$ is a sheaf over $X_1 \times X_2$. Also, $\overline{H}_1 \times \overline{H}_2$ is a sheaf with algebraic structure.

Definition 3.1. Let the pairs $(X_1, \overline{H}_1)$ and $(X_2, \overline{H}_2)$ be given. Then, the sheaf $\overline{H}_1 \times \overline{H}_2$ is called the Direct Sum of the sheaves $\overline{H}_1$ and $\overline{H}_2$.

Thus, we can give the following theorem.

Theorem 3.2. Let the pairs $(X_1, \overline{H}_1)$ and $(X_2, \overline{H}_2)$ be given. Then, the sheaves $(\overline{H})^*$ and $\overline{H}_1 \times \overline{H}_2$ are isomorphic.
that $(\overline{H})^* = ((\overline{H})^*, \psi^*)$ and

Proof. Let us assume $\overline{H}_1 \times \overline{H}_2 = (\overline{H}_1 \times \overline{H}_2, \theta = (\psi_1, \psi_2))$. We first show that the mapping

defined by
$F : (\overline{H})^* \to \overline{H}_1 \times \overline{H}_2 \; (W, (\bar{s}_1, \bar{s}_2))_{(x_1, x_2)} \to \left(\overline{s_1(x_1)}, \overline{s_2(x_2)}\right)$ is continuous.

Now, let $U \subset F((\overline{H})^*)$ is an open, i.e., $U = U_1 \times U_2$ and

$$U_1 = \bigcup_{i \in I} \bar{s}_i(W_i) \; , \; U_2 = \bigcup_{j \in J} \bar{s}_j(V_j).$$

Hence $U_1 = \bar{s}_1(W)$, $U_2 = \bar{s}_2(V)$ and $U = \bar{s}_1(W) \times \bar{s}_2(V)$, where $\bar{s}_1 \in \Gamma(W, \overline{H}_1), \bar{s}_2 \in \Gamma(V, \overline{H}_2)$ and $W = \bigcup_{i \in J} W_i, V = \bigcup_{j \in J} V_j$. So, $\theta(U) = W \times V$.

Show that, $F^{-1}(U) \subset (\overline{H})^*$ is an open set. In fact, if $\sigma^* \in F^{-1}(U)$, then there exists an element $\bar{\sigma} \in U$ such that $F(\sigma^*) = \bar{\sigma}$. Therefore, if

$$\sigma^* = (\theta(U), (\bar{s}_1, \bar{s}_2))_{(x_1,x_2)} ,$$

then

$$F(\sigma^*) = \left(\overline{s_1(x_1)}, \overline{s_2(x_2)}\right) \in U$$

and

$$\theta(F(\sigma^*)) = (x_1, x_2) \in \partial(U).$$

Thus, there exists a section $\gamma\bar{s} = \gamma(\bar{s}_1, \bar{s}_2) : \theta(U) \to (\overline{H})^*$ such that

$$\begin{aligned}\gamma\bar{s}((x_1, x_2)) &= (\theta(U), (\bar{s}_1, \bar{s}_2))_{(x_1,x_2)} \\ &= \sigma^* \in \gamma\bar{s}(\theta(U)).\end{aligned}$$

Therefore, since σ^* is an arbitrary element, $F^{-1}(U) \subset \gamma\bar{s}(\theta(U))$.

On the other hand, if $(\sigma^*)' \in \gamma\bar{s}(\theta(U))$ it is similarly shown that $(\sigma^*)' \in F^{-1}(U)$. Thus,

$$F^{-1}(U) = \gamma\bar{s}(\theta(U)).$$

Since $\gamma\bar{s}(\theta(U))$ is open, F is continuous.

By Theorem 3.1, the mapping F is a bijection since $F/(\overline{H})^*_{(x_1,x_2)} = f$ for each stalk $(\overline{H})^*_{(x_1,x_2)} \subset (\overline{H})^*$. Furthermore, $(\theta \circ F)(\sigma^*) = \psi^*(\sigma^*)$ for every $\sigma^* \in (\overline{H})^*$. Therefore F is a stalk preserving mapping. Moreover F^{-1} is continuous, since F is an open mapping.

Thus, the mapping F is a sheaf isomorphism.

We can now state the following theorem.

Theorem 3.3. $\Gamma(W, (\overline{H})^*)$ is isomorphic to $\Gamma(W, \overline{H}_1 \times \overline{H}_2)$, for each open set $W \subset X_1 \times X_2$.

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