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QUARTER - SYMMETRIC METRIC CONNECTION ON A SASAKIAN MANIFOLD

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ABSTRACT

The object of this paper is to prove the existence of a quarter-symmetric metric connection on a Riemannian manifold and to study some properties of a curvature tensor of a quarter-symmetric metric connection on a Sasakian manifold.

1. INTRODUCTION

Let (M^n, g) be a contact Riemannian manifold with a contact form η, the associated vector field ξ, (1,1) tensor field ϕ and the associated Riemannian metric g. If ξ is a Killing vector field, then (M^n, g) is called a K-contact Riemannian manifold [2],[9]. A K-contact Riemannian manifold is called Sasakian [2] if

$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X \tag{1.1}$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is called a quarter-symmetric connection [4] if its torsion tensor T satisfies

$$T(X,Y) = \pi(Y)F(X) - \pi(X)F(Y) \tag{1.2}$$

where π is a differentiable 1-form and F is a (1, 1) tensor field. If, moreover, the connection $\tilde{\nabla}$ satisfies

$$(\tilde{\nabla}_X g)(Y,Z) = 0 \tag{1.3}$$

for all vector fields X,Y,Z on (M^n, g) then it is called a quarter-symmetric metric connection.

Quarter-symmetric metric connection have been studied by K.Yano and T. Imai [10]. In this connection we can also mention the works of S.C. Rastogi [7],[8],

D. Kamilya and U.C. De [5], S.C. Biswas and U.C. De [1], R.S. Mishra and S.N. Pandey [6], S. Golab [4] and others.

If F(X)=X, then the connection is called a semi-symmetric metric connection [11]. In the present paper we have studied a Sasakian manifold with a quarter-symmetric metric connection $\tilde{\nabla}$ satisfying (1.2) and (1.3) in which the 1-form π and the (1, 1) tensor field F are respectively identical with the contact form η and the (1, 1) tensor field ϕ of the contact structure (ϕ,ξ,η,g) so that the relation (1.2) takes the form

$$T(X,Y)=\eta(Y)\phi(X)-\eta(X)\phi(Y). \tag{1.4}$$

At first we prove the existence of a quarter-symmetric metric connection in a Riemannian manifold (M^n,g) In section 3 we deduce the expressions for the curvature tensor and the Ricci tensor of (M^n,g) with respect to the quarter-symmetric metric connection. In general, the Ricci tensor of the quarter-symmetric metric connection is not symmetric. Here it is proved that in a Sasakian manifold the Ricci tensor of a quarter-symmetric metric connection is symmetric. Also, in general, the conformal curvature tensors of the quarter-symmetric metric connection and the Riemannian connection are not equal. Here we obtain a necessary and sufficient condition for the conformal curvature tensor to be equal. Finally, we obtain an expression of the projective curvature tensor of the quarter-symmetric metric connection.

2. PRELIMINARIES

Let R and S denote respectively the curvature tensor and Ricci tensor of type (1.2) of (M^n,g) It is known that in a Sasakian manifold (M^n,g) besides the relation (1.1), the following relations hold [2],[9]

$$\phi(\xi)=0. \tag{2.1}$$

$$\eta(\xi)=1 \tag{2.2}$$

$$\phi^2X=-X+\eta(X)\xi \tag{2.3}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y) \tag{2.4}$$

$$g(\xi,X)=\eta(X) \tag{2.5}$$

$$\nabla_X\xi=-\phi X \tag{2.6}$$

$$S(X,\xi)=(n-1)\eta(X) \tag{2.7}$$

$$g(R(\xi,X)Y,\xi)=g(X,Y)-\eta(X)\eta(Y) \tag{2.8}$$

$$R(\xi,X)\xi=-X+\eta(X)\xi \tag{2.9}$$

$$(\nabla_X\phi)(Y)=R(\xi,X)Y \tag{2.10}$$

for any vector fields X,Y.

We can define a 2 form $`\phi$ in a Sasakian manifold (M^n,g) by

$$`\phi(X,Y)=g(X,\phi Y) \tag{2.11}$$

such that

$$`\phi = d\eta. \tag{2.12}$$

From (2.3),(2.4),(2.11) and (2.12) by using $\eta \cdot \phi = 0$ we get

$$g(\phi X, Y) + g(X, \phi Y) = 0 \tag{2.13}$$

$$d\eta(\phi X, Y) + d\eta(X, \phi Y) = 0 \tag{2.14}$$

$$d\eta(\phi X, \phi Y) = d\eta(X, Y) \tag{2.15}$$

and

$$d\eta(\xi, X) = 0 \tag{2.16}$$

3. EXISTENCE OF A QUARTER-SYMMETRIC METRIC CONNECTION

Let X,Y be two vector fields on (M^n, g) We define $\tilde{\nabla}_X Y$ by the following equation

$$\begin{aligned} 2g(\tilde{\nabla}_X Y, Z) = {} & Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & -g([Y,X],Z) - g([X,Z],Y) + g([Z,Y],X) \\ & +g(\pi(X)\phi Z - \pi(Z)\phi X, Y) \\ & +g(\pi(Y)\phi Z - \pi(Z)\phi Y, Z) \\ & +g(\pi(Y)\phi X - \pi(X)\phi Y, Z) \end{aligned} \tag{3.1}$$

which should hold for all vector fields Z on (M^n, g).

It can be verified that the mapping $(X, Y) \to \tilde{\nabla}_X Y$ satisfies the following equalities:

(i) $\tilde{\nabla}_X (Y + Z) = \tilde{\nabla}_X Y + \tilde{\nabla}_X Z$

(ii) $\tilde{\nabla}_{X+Y} Z = \tilde{\nabla}_X Z + \tilde{\nabla}_Y Z$

(iii) $\tilde{\nabla}_{fX} Y = f\tilde{\nabla}_X Y, \forall f \in F(M^n)$

(iv) $\tilde{\nabla}_X f(Y) = f\tilde{\nabla}_X Y + (Xf)Y, \forall f \in F(M^n)$

where $F(M^n)$ denotes the set of all differentiable mappings over M^n.. Therefore $\tilde{\nabla}$ determines a linear connection on (M^n, g) Now we have,

$$2g(\tilde{\nabla}_X Y, Z) - 2g(\tilde{\nabla}_Y Y, Z) = 2g([X, Y]Z) + 2g(\pi(Y)\phi(X) - \pi(X)\phi(Y), Z). \tag{3.2}$$

Hence,

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \pi(Y)\phi(X) - \pi(Y)\phi(Y)$$

or,

$$T(X, Y) = \pi(Y)\phi(X) - \pi(X)\phi(Y). \tag{3.3}$$

Also we have,

$$2g(\tilde{\nabla}_X Y, Z) + 2g(\tilde{\nabla}_X Y, Z) = 2Xg(Y, Z)$$

or,

$$(\tilde{\nabla}_X g)(Y, Z) = 0$$

that is,

$$\tilde{\nabla}_g = 0. \tag{3.4}$$

From (3.3) and (3.4) it follows that $\tilde{\nabla}$ determines a quarter-symmetric metric connection on (M^n, g) It can be easily shown that $\tilde{\nabla}$ determines a unique quarter-symmetric metric connection on (M^n, g) Thus we have the following theorem:

Theorem 2.1. Let (M^n, g) be a Riemannian manifold and π be a 1-form on M^n. Then there exists a unique linear connection $\tilde{\nabla}$ satisfying (3.3) and (3.4).

Remark. Theorem (2.1) proves the existence of a quarter-symmetric metric connection on (M^n, g).

4. CURVATURE TENSOR

Let us write

$$\tilde{\nabla}_X Y = \nabla_X Y + H(X, Y) \tag{4.1}$$

For a quarter-symmetric metric connection $\tilde{\nabla}$ and a Levi-Civita connection ∇ on (M^n, g).

From (3.4) we get

$$Xg(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) = 0.$$

By virtue of (4.1) we have,

$$Xg(Y,Z) - g(\nabla_X Y + H(X,Y), Z) - g(Y, \nabla_X Z + H(X,Z)) = 0.$$

From here

$$Xg(Y,Z) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) - g(H(X,Y),Z) - g(H(X,Z),Y) = 0$$

that is,

$$(\nabla_X g)(Y,Z) - g(H(X,Y),Z) - g(H(X,Z),Y) = 0.$$

Since ∇ is the Levi-Civita connection, $(\nabla_X g)(Y,Z) = 0$ and hence we have

$$g(H(X,Y),Z) - g(H(X,Z),Y) = 0. \tag{4.2}$$

Also, from (4.1) it follows that

$$\begin{aligned} H(X,Y) - H(Y,X) &= \tilde{\nabla}_X Y - \nabla_X Y - \tilde{\nabla}_Y X + \nabla_Y X \\ &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \\ &= T(X,Y) \cdot \end{aligned}$$

Hence by using (1.4) we obtain,

$$H(X,Y) - H(Y,X) = \eta(Y)\phi(X) - \eta(Y)\phi(X) \cdot \tag{4.3}$$

Again from (4.3) we get

$$g(H(X,Y),Z) - g(H(Y,X),Z) = \eta(Y)g(\phi(X),Z) - \eta(X)g(\phi(Y),Z) \tag{4.4}$$

$$g(H(X,Z),Y) - g(H(Z,X),Y) = \eta(Z)g(\phi(X),Y) - \eta(X)g(\phi(Z),Y) \tag{4.5}$$

and

$$g(H(Y,Z),X) - g(H(Z,Y),X) = \eta(Z)g(\phi(Y),X) - \eta(Y)g(\phi(Z),X) \tag{4.6}$$

Adding (4.4) and (4.5) and then subtracting (4.6) from the result we get by applying (2.13) and (4.2)

$$H(Z,Y) = -\eta(Z)\phi(Y)\cdot \tag{4.7}$$

So that from (4.1) we can write

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi(Y)\cdot \tag{4.8}$$

If

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$$

denote the curvature tensor of the connection $\tilde{\nabla}$ then from (4.8) by using the relation (1.1) we obtain,

$$\begin{aligned}\tilde{R}(X,Y)Z = R(X,Y)Z - 2d\eta(X,Y)\phi(Z) + \eta(X)g(Y,Z)\xi \\ +\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - \eta(Y)g(X,Z)\xi\end{aligned} \tag{4.9}$$

where R(X,Y)Z is the Riemannian curvature tensor of the manifold.
Therefore from (4.9) we obtain,

$$\tilde{S}(Y,Z) = S(Y,Z) - 2d\eta(\phi Z,Y) + g(Y,Z) + (n-2)\eta(Y)\eta(Z) \tag{4.10}$$

where S and $\tilde{S}$ denote the Ricci tensors of ∇ and $\tilde{\nabla}$ respectively.
Further, since $d\eta$ is skew-symmetric, we get from (4.10)

$$\tilde{r} = r + 2(n-1)\cdot \tag{4.11}$$

It is seen from (4.10) by using (2.14) that

$$\tilde{S}(X,Y) = \tilde{S}(Y,X)$$

that is, the Ricci tensor of the connection $\tilde{\nabla}$ is symmetric.

The conformal curvature tensor $\tilde{C}(X,Y)Z$ of the quarter-symmetric metric connection $\tilde{\nabla}$ is given by

$$\begin{aligned}\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-2}[g(Y,Z)\tilde{L}X - g(X,Z)\tilde{L}Y + \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] \\ + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{4.12}$$

From which we get

$$\begin{aligned}\tilde{C}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) - \frac{1}{n-2}[g(Y,Z)\tilde{S}(X,W) - g(X,Z)\tilde{S}(Y,W) \\ + \tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W)] \\ + \frac{\tilde{r}}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\end{aligned} \tag{4.13}$$

where

$$\tilde{C}(X,Y,Z,W) = g(\tilde{C}(X,Y),Z,W)$$

$$\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$$

$$\tilde{S}(X,Y) = g(\tilde{L}X,Y),$$

$\tilde{L}$ being the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor $\tilde{S}$.
By using (4.9), (4.10) and (4.11) we get from (4.13)

$$\tilde{C}(X,Y,Z,W) = C(X,Y,Z,W) - 2d\eta(X,Y)g(\phi Z,W) + \frac{2}{n-2}[g(Y,Z)d\eta(\phi W,X) - g(X,Z)d\eta(\phi W,Y) + g(X,W)d\eta(\phi Z,Y) - g(Y,W)d\eta(\phi Z,X)] = 0 \tag{4.14}$$

where $C(X,Y,Z,W) = g(C(X,Y)Z,W)$ and $C(X,Y)Z$ is the conformal curvature tensor of the Levi-Civita connection ∇.

Clearly $d\eta = 0$ is a sufficient condition for

$$\tilde{C}(X,Y,Z,W) = C(X,Y,Z,W). \tag{4.15}$$

If one considers the relation (4.15) to be true one can get from (4.14)

$$d\eta(X,Y)g(\phi Z,W) - \frac{1}{n-2}[g(Y,Z)d\eta(\phi W,X) - g(X,Z)d\eta(\phi W,Y) + g(X,W)d\eta(\phi Z,Y) - g(Y,W)d\eta(\phi Z,X)] = 0. \tag{4.16}$$

Putting $Z = \xi$ in (4.16) we get

$$\eta(Y)d\eta(\phi W,X) - \eta(X)d\eta(\phi W,Y) = 0\cdot \tag{4.17}$$

Again putting $Y = \xi$ and $W = \phi W$ in (4.17) we obtain

$$d\eta(X,W) = 0\cdot$$

The following theorem can now be stated:

Theorem 4.1. A necessary and sufficient condition for the conformal curvature tensor of the quarter-symmetric metric connection $\tilde{\nabla}$ given by (4.1) to be equal to the conformal curvature tensor of a Sasakian manifold M^n is that the contact form η is closed.

Next we consider the projective curvatue tensor of $\tilde{\nabla}$. Let

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n+1}\left[\tilde{S}(X,Y)Z - \tilde{S}(Y,X)Z\right] + \frac{1}{n^2-1}\left[\left\{n\tilde{S}(X,Z) + \tilde{S}(Z,X)\right\}Y - \left\{n\tilde{S}(Y,Z) + \tilde{S}(Z,Y)\right\}X\right] \tag{4.18}$$

be the generalized projective curvature tensor [3] of the quarter-symmetric metric connection $\tilde{\nabla}$. But here the Ricci tensor of the quarter-symmetric metric connection is symmetric. Therefore the expression of the generalized projective curvature tensor of the connection $\tilde{\nabla}$ reduces to

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\left[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\right] \tag{4.19}$$

From which we get

$$\tilde{P}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) - \frac{1}{n-1}\left[\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W)\right] \tag{4.20}$$

where

$$\tilde{P}(X,Y,Z,W) = g(\tilde{P}(X,Y)Z,W)$$
$$\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)\cdot$$

By virtue of (4.9) and (4.10) we get from (4.20)

$$\begin{aligned}\tilde{P}(X,Y,Z,W) = {} & P(X,Y,Z,W) - 2d\eta(X,Y)g(\phi Z,W) \\ & + g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ & + \frac{2}{n-1}\left[g(X,W)d\eta(\phi Z,Y) - g(Y,W)d\eta(\phi Z,X)\right] \\ & - \frac{1}{n-1}\left[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\right] \\ & + \frac{1}{n-1}\left[g(X,X)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z)\right]\end{aligned} \tag{4.21}$$

where $P(X,Y,Z,W) = g(P(X,Y)Z,W)$ and $P(X,Y)Z$ is the projective curvature tensor of the Levi-Civita connection. Hence the projective curvature tensor of the quarter-symmetric metric connection given by (4.1) is not, in general, equal to the projective curvature tensor of the Levi-Civita connection.

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