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ON THE SOLUTIONS OF KLEIN-GORDON EQUATION

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ABSTRACT

In this paper we find the solutions of the equation

$$y''+(\lambda-Q(x))^2y-\frac{n(n+1)}{x^2}y=0\ , \qquad x\in R_+\ ,$$

using the solutions of the Klein-Gordon equation

$$y''-(\lambda-Q(x))^2y=0\ , \quad x\in R_+=[0,\infty),$$

where Q is a real valued function, λ is a spectral parameter and n is a natural number.

1. INTRODUCTION

Let us consider the following boundary value problems

$$\begin{aligned}&y''-(\lambda-Q(x))^2y=0\ , \quad x\in R_+\\&y(0)=0\end{aligned} \tag{1.1}$$

and

$$y''+(\lambda-Q(x))^2y-\frac{n(n+1)}{x^2}y=0\ , \quad x\in R_+\ , \tag{1.2}$$

where Q is an absolutely continuous real valued function in each finite subinterval of R_+ and satisfying

$$\int_0^\infty x\{|Q(x)|+|Q'(x)|\}dx<\infty\ . \tag{1.3}$$

Under the condition (1.3) the equation (1.1) has the solutions

$$e^{+}(x,\lambda)=e^{i\lambda x+i\alpha(x)}+\int_{x}^{\infty}K^{+}(x,t)e^{i\lambda t}dt, \tag{1.4}$$

and

$$e^{-}(x,\lambda)=e^{-i\lambda x-i\alpha(x)}+\int_{x}^{\infty}K^{-}(x,t)e^{-i\lambda t}dt, \tag{1.5}$$

for λ in the closed upper and lower half-planes, respectively, where

$$\alpha(x)=\int_{x}^{\infty}Q(t)dt,$$

and the kernels $K^{\pm}(x,t)$ are expressed in terms of Q and $K^{\pm}(x,t)$ are the solutions of Volterra type integral equations([6]).

As it is known, the solutions of $e^{+}(x,\lambda)$ and $e^{-}(x,\lambda)$ given (1.4) and (1.5) are important in the investigation of spectral analysis and scattering theories of the boundary value problems (1.1) ([2]-[5]). But the equation (1.2) has no solution represented as the solutions (1.4) and (1.5) due to the factor $\frac{n(n+1)}{x^2}$.

In this study, our purpose is to find that the equation (1.2) have the similar solutions to (1.4) and (1.5) using the solution of the equation (1.1).

The similar problem has been studied for Sturm-Lioville equation in [1].

2. The solution of (1.2)

Let us consider the following equation

$$y''+\left(Q^{2}(x)-2\lambda Q(x)\right)y=0. \tag{2.1}$$

Then we get

Theorem2.1. For all λ, equation (2.1) has the solution $f(x,\lambda)$ which satisfies the initial conditions $f(0,\lambda)=0, f'(0,\lambda)=1$ and $f(x,\lambda)$ has the representation

$$f(x,\lambda)=x-\int_{0}^{x}(x-t)\left\{Q^{2}(t)-2\lambda Q(t)\right\}f(t,\lambda)dt. \tag{2.2}$$

Moreover the asymptotic equalities

$$f(x,\lambda)\cong x(1+o(1)) \ , \ f'(x,\lambda)\cong(1+o(1)) \tag{2.3}$$

are valid.

Proof. If we integrate equation (2.1) twice and use the initial conditions, we get equation (2.2). From equation (2.2), we get asymptotic equalities (2.3) by means of standard technique ([7], p.145).

Let $h(x, \lambda)$ be the normalized eigen-function of the boundary value problem (1.1). The solution $h(x, \lambda)$ has the following asymptotic behavior

$$h(x,\lambda) = x(h'(x,\lambda) + o(1)), \tag{2.4}$$

for $x \to 0$.

If we consider the function

$$y(x,\lambda) = \frac{f(x,\lambda)h'(x,\lambda) - f'(x,\lambda)h(x,\lambda)}{\lambda f(x,\lambda)}. \tag{2.5}$$

We can give following

Theorem2.2. If the function $f(x, \lambda)$ is not vanished in the interval $(0,\infty)$, then the function $y(x, \lambda)$ defined by (2.5) satisfies the equation

$$y'' + V(x,\lambda)y + \lambda^2 y = 0 \tag{2.6}$$

where

$$V(x,\lambda) = Q^2(x) - 2\lambda Q(x) + 2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]'. \tag{2.7}$$

Proof. Let us write the first and second derivatives of $y(x, \lambda)$:

$$y'(x,\lambda) = -\lambda h(x,\lambda) - \frac{f'(x,\lambda)}{f(x,\lambda)}\left\{\frac{f(x,\lambda)h'(x,\lambda) - f'(x,\lambda)h(x,\lambda)}{\lambda f(x,\lambda)}\right\},$$

and

$$y''(x,\lambda) = -\lambda^2 \frac{f(x,\lambda)h'(x,\lambda)}{\lambda f(x,\lambda)} - \left(\frac{f'(x,\lambda)}{f(x,\lambda)}\right)' y(x,\lambda) + \lambda \frac{f'(x,\lambda)h(x,\lambda)}{f(x,\lambda)} + \frac{f'^2(x,\lambda)}{f^2(x,\lambda)} y(x,\lambda)$$

$$= -\lambda^2 \left\{\frac{f(x,\lambda)h'(x,\lambda) - f'(x,\lambda)h(x,\lambda)}{\lambda f(x,\lambda)}\right\} + \left\{\frac{f'^2(x,\lambda)}{f^2(x,\lambda)} - \left(\frac{f'(x,\lambda)}{f(x,\lambda)}\right)'\right\} y(x,\lambda)$$

$$= -\lambda^2 y(x,\lambda) + \left\{ \frac{f'^2(x,\lambda)}{f^2(x,\lambda)} - \frac{f''(x,\lambda)f(x,\lambda) - f'^2(x,\lambda)}{f^2(x,\lambda)} \right\} y(x,\lambda)$$

$$= -\lambda^2 y(x,\lambda) - \left(Q^2(x) - 2\lambda Q(x)\right) y(x,\lambda) + \left\{ 2\frac{f'^2(x,\lambda)}{f^2(x,\lambda)} - 2\frac{f''(x,\lambda)}{f(x,\lambda)} \right\} y(x,\lambda) .$$

Hence, we find that

$$y''(x,\lambda) + \left\{Q^2(x) - 2\lambda Q(x)\right\} y(x,\lambda) + 2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right] y(x,\lambda) + \lambda^2 y(x,\lambda) = 0$$

If we use the asymptotic equalities (2.3) and (2.4) as $x \to 0$, we have

$$\lim_{x\to 0} y(x,\lambda) = \lim_{x\to 0} \frac{f(x,\lambda)h'(x,\lambda) - f'(x,\lambda)h(x,\lambda)}{\lambda f(x,\lambda)}$$

$$= \lim_{x\to 0} \frac{f(x,\lambda)h'(x,\lambda)}{\lambda f(x,\lambda)} - \lim_{x\to 0} \frac{f'(x,\lambda)h(x,\lambda)}{\lambda f(x,\lambda)}$$

$$= \lim_{x\to 0} \lambda^{-1} h'(x,\lambda) - \lim_{x\to 0} \lambda^{-1} f'(x,\lambda) \frac{h(x,\lambda)}{f(x,\lambda)}$$

$$= \lambda^{-1} h'(0,\lambda) - \lim_{x\to 0} \lambda^{-1} f'(x,\lambda) \frac{h'(x,\lambda)}{f'(x,\lambda)}$$

$$= 0$$

$$2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]' = 2\left\{(1+o(1))\frac{1}{x(1+o(1))}\right\}'$$

$$= 2\left\{\frac{1}{x} + o\left(\tfrac{1}{x}\right)\right\}'$$

$$= -\frac{2}{x^2} + o\left(\tfrac{1}{x^2}\right), \quad x \to 0$$

Hence, by (2.7) it follows that the potential $V(x,\lambda)$ behaves like $-\tfrac{2}{x^2}$ in the neighborhood of zero. In this way we use the function (2.5) in the non-singular boundary value problem (1.1) and find the singular boundary value problem (2.6)

Now we find the inverse transformation of the transformation (2.5). Since

$$y(x,\lambda) = \frac{f(x,\lambda)h'(x,\lambda) - f'(x,\lambda)h(x,\lambda)}{\lambda f(x,\lambda)}$$

then we get

$$\lambda\frac{y(x,\lambda)}{f(x,\lambda)}=\left\{\frac{h(x,\lambda)}{f(x,\lambda)}\right\}'$$

and hence , we also find

$$h(x,\lambda)=\lambda f(x,\lambda)\int_0^x\frac{y(t,\lambda)}{f(t,\lambda)}dt\,. \tag{2.8}$$

Now in a similar way, we find the differential equation

$$h''(x,\lambda)+\left\{V(x,\lambda)-2\left[f'(x,\lambda)f^{-1}(x,\lambda)\right]'\right\}h(x,\lambda)+\lambda^2h(x,\lambda)=0\,. \tag{2.9}$$

If we substitute the potential $V(x,\lambda)$ defined in (2.7) in the last statement then we find the non-singular equation

$$y''-(\lambda-Q(x))^2y=0\ .$$

So we get the following theorem .

Theorem 2.3. If the function $h(x,\lambda)$ defined by (2.8) is the solution of (2.9) then the function $y(x,\lambda)$ defined by (2.5) is the solution of (2.6).

Remark: Now if the boundary value problem defined by (1.1) and the condition y(0)=0 has no negative spectrum then the solution $f(x,\lambda)$ of (1.1) for λ=0 which satisfies the initial conditions $f(0,\lambda)=0, f'(0,\lambda)=1$ is not vanished in the interval $0<x<\infty$. The normalized eigen-functions of this boundary value problem and their derivatives will have the following asymptotic behavior at infinity

$$h(x,\lambda)\cong\sin(\lambda x+\delta(x)) \qquad h'(x,\lambda)\cong\lambda\cos(\lambda x+\delta(x)), \qquad x\to\infty\,.$$

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