Commun. Fac. Sci. Univ. Ank. Series A1
Vol. 51. (No.2) 25-36 (2002)

ON GCO-MODULES AND M-SMALL MODULES

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(Received July 10, 2002; Accepted Nov. 7, 2002)

ABSTRACT

Let M be a right R-module. Define Z^* (N) (δ^*_M (N)) to be the set of elements $n \in N$ for any R-module N in $\sigma[M]$ such that nR is an M-small (respectively δ-M-small) module. In this note it is proved that M is a GCO-module if and only if every M-small module in $\sigma[M]$ is M-projective if and only if every δ-M-small module in $\sigma[M]$ is M-projective. Also, if M/δ^*_M (M) is semisimple then M is a GCO-module if and only if M is an SI-module.

1991 Mathematics Subject Classification: 16D50, 16D60
Key words: V-module, SI-module, GCO-module, M-small module.

For a right R-module M, the submodule Z^* (M) is defined to be the set of elements $m \in M$ such that mR is a small module (see [4]). Some further properties of Z^* (.) were studied in [4, 8, 9, 10]. In this paper we think this submodule in the category $\sigma[M]$, and therefore the corresponding definition of Z^* (.) in $\sigma[M]$ is defined by Z^*_M (N) to be the set of elements $n \in N$ for a module $N \in \sigma[M]$ such that nR is M-small. In Section 1 we prove that M is a GCO-module if and only if every M-small module in $\sigma[M]$ is M-projective (Theorem 1.5). Also if M/Z^*_M (M) is semisimple, then M is a GCO-module if and only if M is an SI-module if and only if Z^*_M (M) is semisimple M-projective (Theorem 1.12). In Section 2, we define δ-M-small modules and δ^*_M (N) as a generalization of M-small modules and Z^*_M (N) in $\sigma[M]$ being inspired from [14]. Most of the results in Section 1 hold for δ-M-small modules and δ^*_M (N) but the characterization of V-modules (Example 2.6).

Throughout this paper, R will be an associative ring with unit and all modules be unitary right R-modules.

Let M be an R-module. For a direct summand N of M we write $N \leq_d M$ and for essential submodule N of M, $N \leq_e M$.

An R-module N is *subgenerated* by M if N is isomorphic to a submodule of an M-generated module. σ[M] is denoted by the full subcategory of Mod-R whose objects are all R-modules subgenerated by M [12].

Let $\hat{N}$ be the M-injective hull of N in σ[M] and let E(M) be an R-injective hull of M.

A module N in σ[M] is called *M-singular* (or *singular in* σ[M]) if $N \cong L/K$ for an $L \in \sigma[M]$ and $K \leq_e L$ (see [3]). In case M=R, instead of R-singular, we just say *singular*. Every module $N \in \sigma[M]$ contains a largest M-singular submodule which is denoted by $Z_M(N)$.

Let **G**(M) be the singular torsion theory in σ[M], that is, **G**(M) is the smallest torsion class in σ[M] which contains all M-singular modules (see [11]). **G**(M) is closed under M-injective hulls by [11, 2.4(3)], and hence **G**(M)={ $N \in \sigma[M]$: $Z_M(N) \leq_e N$ }.

Following Hirano a module M is called a *V-module* (or *co-semisimple*) if every simple module (in σ[M]) is M-injective. A module M is called a *GV-module* if every singular simple module is M-injective. M is a GV-module if and only if every simple module is projective or M-injective [5]. As a generalization of GV-modules a module M is called a *GCO-module* if every singular simple module is M-projective or M-injective [3]. M is a GCO-module if and only if every M-singular simple module is M-injective [3, 16.4]. Obviously any V-module is a GV-module and any GV-module is a GCO-module. M is called an *SI-module* if every M-singular module is M-injective [3]. Clearly SI-modules are GCO-modules. Note that a right GCO-ring coincides with a right GV-ring.

1. M-SMALL MODULES

Let K be a submodule of a module M. K is called *small* in M if $K+L \neq M$ holds for every proper submodule L of M and denoted by $K \ll M$. We write Rad(M), which is the sum of all small submodules in M, for the radical of M (see [1]).

An R-module N is called *M-small* (or *small in* σ[M]) if $N \cong K \ll L$ for $K, L \in \sigma[M]$. Note that M-small modules are dual notion to that of M-singular modules. In case M=R, instead of R-small, we just say *small*. M-small modules are small, since the class of small modules is closed under isomorphism. An R-module N is M-small if and only if $N \ll \hat{N}$. Every simple R-module is M-injective or M-small.The class of M-small modules is closed under submodules, homomorphic images and finite direct sums. (see [6])

Let M be an R-module. Denote

$$Z^*_M(N) = \{ n \in N : nR \text{ is M-small} \}$$

for an R-module $N \in \sigma[M]$. In case M=R, we write $Z^*(N)$ instead of $Z^*_R(N)$. Let $N \in \sigma[M]$. Then it can be easily seen that

$$\mathrm{Rad}(N) \leq Z^*_M(N) \leq Z^*(N).$$

If N is M-small, then $Z^*_M(N)=N$. Since $\sigma[N] \subseteq \sigma[M]$, we also have $Z^*_N(X) \leq Z^*_M(X)$ for any module $X \in \sigma[M]$.

Lemma 1.1. Let M be a module. Then

a) $Z^*_M(N) = \mathrm{Rad}\,\hat{N} \cap N$ for any $N \in \sigma[M]$.

b) Let $N \in \sigma[M]$. For any submodule K of N, $Z^*_M(K) = K \cap Z^*_M(N)$.

c) Let $f: N \to K$ be a homomorphism of modules N, K where $N, K \in \sigma[M]$. Then $f(Z^*_M(N)) \leq Z^*_M(K)$.

d) Let N_i $(i \in I)$ be any collection of modules in $\sigma[M]$ and let $N = \oplus_{i \in I} N_i$.

Then $Z^*_M(N) = \oplus_{i \in I} Z^*_M(N_i)$.

Proof. (a) and (b) are clear. (c) and (d) can be obtained by the similar techniques of [10, Lemma 2.1 and 2.3].

Now we give a lemma showing some properties of $Z^*_M(.)$ in case it is zero.

Lemma 1.2. Let $N \in \sigma[M]$. Then

a) $Z^*_M(N)=0$ if and only if $\mathrm{Rad}(\hat{N})=0$.

b) $Z^*_M(N)=0$ if and only if $Z^*_K(N)=0$ for every $K \in \sigma[M]$ with $N \in \sigma[K]$.

Proof. a) By Lemma 1.1 and, since $N \leq_e \hat{N}$.

b) Suppose that $Z^*_M(N)=0$, and let $K \in \sigma[M]$ with $N \in \sigma[K]$ and $x \in Z^*_K(N)$. Then xR is K-small, i.e. $xR \cong L \ll T$ for some $L, T \in \sigma[K]$. Since $K \in \sigma[M]$, $L, T \in \sigma[M]$.This implies that xR is M-small. Thus $x \in Z^*_M(N)=0$. Converse is open.

Since $Z^*_M(.)$ is related with the radical of a module then one may think whether the results hold for radicals of modules are true for $Z^*_M(.)$. Therefore here we consider V-modules and GCO-modules by being encouraged from [12, 23.1] and [3, 16.4].

Theorem 1.3. The following are equivalent for a module M.
a) M is a V-module,
b) Z^*_M (N)=0 for every module N∈σ[M],
c) Z^*_M (N)=0 for every factor module N of M.

Proof. Since Z^*_M (N)=Rad($\hat{N}$) ∩ N for N∈σ[M], it is clear from [12, 23.1]. □

Let N∈σ[M]. N is called *cogenerator* in σ[M] if there exists a monomorphism N→ $\prod_\Lambda M_\lambda$ with modules M_λ∈σ[M] [12]. A module M is called *locally noetherian* if every finitely generated submodule of M is noetherian.

Theorem 1.4. Let M be a locally noetherian module. The following are equivalent.
a) M is a V-module,
b) σ[M] has a semisimple M-injective cogenerator,
c) σ[M] has a cogenerator Q with Z^*_M (Q)=0.

Proof. It is clear from [12, 23.1].

Theorem 1.5. The following are equivalent for a module M.
a) M is a GCO-module,
b) For every module N∈σ[M], Z^*_M (N) is M-projective,
c) Every M-small module in σ[M] is M-projective,
d) For every module N∈σ[M], Z_M (N) ∩ Z^*_M (N)=0,
e) For every simple module E∈σ[M], Z_M ($\hat{E}$) ∩ Z^*_M ($\hat{E}$)=0,
f) M/Soc(M) is a V-module and Z_M (M) ∩ Z^*_M (M)=0,
g) Z^*_M (M/K)=0 for every $K \leq_e M$ and Z_M (M) ∩ Z^*_M (M)=0,
h) Every non-zero module N with Z^*_M (N)=N contains a non-zero M-projective submodule,
i) For every module N∈σ[M] with $Z_M (N) \leq_e N$ (i.e. N∈ ***G***(M)), Z^*_M (N)=0.

Proof. (a) ⇒ (b) Since simple modules in σ[M] splits into four disjoint classes by combining the exclusive choices [M-projective or M-singular] and [M-injective or M-small], one deduces that M is a GCO-module if and only if every M-small simple module is M-projective. So, let $n \in Z^*_M(N)$ for $N \in \sigma[M]$ and K be a maximal submodule of nR. Then nR/K is simple and M-projective. This implies that $K \leq_d nR$. Hence nR and then $Z^*_M(N)$ is semisimple. By [7, Proposition 4.32], $Z^*_M(N)$ is M-projective.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) It is clear.

(e) $\Rightarrow$ (a) It follows from [3, 16.4 (d) $\Rightarrow$ (a)].

(d) $\Rightarrow$ (g) Let $K \leq_e M$. Then M/K is M-singular. This implies that $Z_M(M/K)=M/K$. By hypothesis, $Z^*_M(M/K)=0$.

(g) $\Leftrightarrow$ (f) It follows from [3, 16.1 (a) $\Leftrightarrow$ (d)].

(f) $\Rightarrow$ (a) It follows from [3, 16.4 (e) $\Rightarrow$ (a)].

(b) $\Rightarrow$ (h) It is clear.

(h) $\Rightarrow$ (a) Let N be an M-singular simple module in $\sigma[M]$. If N is M-small then N contains a non-zero M-projective module P in $\sigma[M]$. Since N is simple N=P and then N is projective and M-singular in $\sigma[M]$, a contradiction. Hence N is M-injective.

(d) $\Rightarrow$ (i) It is clear.

(i) $\Rightarrow$ (d) Let $0 \neq n \in Z_M(N) \cap Z^*_M(N)$. Then nR is M-singular and M-small. Since $nR=Z_M(nR) \leq_e nR$, $Z^*_M(nR)=0$ by hypothesis, a contradiction. □

If we consider the GCO-modules with ascending (descending) chain condition on essential submodules we have the following corollaries. First one is a generalization of [3, 16.13 (1)].

Corollary 1.6. The following are equivalent for a module M.

a) M is a GCO-module with ascending chain condition on essential submodules,

b) M/SocM is a V-module and Noetherian, $Z_M(M) \cap Z^*_M(M)=0$.

Proof. By Theorem 1.5 and [3, 5.15].

Corollary 1.7. For a module M with M/SocM finitely generated, the following are equivalent.

a) M is a GCO-module with descending chain condition on essential submodules,

b) M/SocM is semisimple, $Z_M(M) \cap Z^*_M(M)=0$.

Proof. By Theorem 1.5, [3, 5.15] and [1, Proposition 10.15].

GV-modules can be characterized by replacing $Z_M(N)$ by the singular submodule Z(N) and M-projectivity by projectivity in Theorem 1.5.

Theorem 1.8. The following are equivalent for a module M.
a) M is a GV-module,
b) For every module $N \in \sigma[M]$, $Z^*_M(N)$ is projective,
c) Every M-small module in $\sigma[M]$ is projective,
d) For every module $N \in \sigma[M]$, $Z(N) \cap Z^*_M(N)=0$,
e) For every simple module $E \in \sigma[M]$, $Z(\hat{E}) \cap Z^*_M(\hat{E})=0$,
f) M/ Soc(M) is a V-module and $Z(M) \cap Z^*_M(M)=0$,
g) $Z^*_M(M/K)=0$ for every $K \leq_e M$ and $Z(M) \cap Z^*_M(M)=0$,
h) Every non-zero module N with $Z^*_M(N)=N$ contains a non-zero projective submodule,
i) For every module $N \in \sigma[M]$ with $Z(N) \leq_e N$, $Z^*_M(N)=0$.

Example 1.9. If M is a GV-module, $Z(M) \cap Rad(M)=0$ but $Z(M) \cap Z^*(M)$ need not be zero in general.

Proof. Let $M=\mathbb{Z}/2\mathbb{Z}$. M is simple and hence a GV-module. Also $Z(M) \cap Rad(M)=0$. But $Z(M) \cap Z^*(M)=M$ since M is singular and small $\mathbb{Z}$-module.

Applying Theorem 1.8 to M=R, we immediately have the following corollary which is a generalization of [8, Theorem 10].

Corollary 1.10. The following are equivalent for a ring R.
a) R is a right GV-ring,
b) For every R-module M, $Z^*(M)$ is projective,
c) Every small module is projective,
d) For every R-module M, $Z(M) \cap Z^*(M)=0$,
e) For every simple module S, $Z(E(S)) \cap Z^*(E(S))=0$.
f) R/ Soc(R) is a V-module and $Z(R_R) \cap Z^*(R_R)=0$,
g) $Z^*(R/K)=0$ for every essential right ideal K of R and $Z(R_R) \cap Z^*(R_R)=0$,
h) Every non-zero R-module M with $Z^*(M)=M$ contains a non-zero projective submodule,

i) For every R-module M with $Z(M) \leq_e M$, $Z^*(M)=0$.

Theorem 1.11. Let M be a module with $M/Z^*_M(M)$ a V-module. Then the following are equivalent.
a) M is a GCO-module,
b) $Z^*_M(M)$ is semisimple M-projective.

Proof. (a) $\Rightarrow$ (b) By Theorem 1.5.
(b) $\Rightarrow$ (a) Since $Z^*_M(M)$ is semisimple, $Z^*_M(M) \leq Soc(M)$. Then by hypothesis, M/ Soc(M) is a V-module. $Z_M(M) \cap Rad(M)$ is a direct summand of $Z^*_M(M)$. Since $Z^*_M(M)$ is M-projective, we have $Z_M(M) \cap Rad(M)=0$. By [3, 16.4], M is a GCO-module.

In [3, 17.5], we do not need the condition that M is self-projective.

Theorem 1.12. Let M be a module with $M/Z^*_M(M)$ semisimple. Then the following are equivalent.
a) M is a GCO-module,
b) M is an SI-module,
c) $Z^*_M(M)$ is semisimple M-projective.

Proof. (a) $\Leftrightarrow$ (c) By Theorem 1.11.
(b) $\Rightarrow$ (a) Clear.
(c) $\Rightarrow$ (b) Since $Z^*_M(M) \leq Soc(M)$, M/ SocM is semisimple. Let $K \leq_e M$. Then $SocM \leq K$. This implies that M/K is semisimple. On the other hand, since finitely generated M-singular modules can not be M-projective, we have $Z_M(M) \cap Rad(M)=0$. Thus M is an SI-module by [3, 17.2].

2. δ-M-SMALL MODULES

In this section, we define δ-M-small modules and use them to characterize GCO-modules.

Zhou [14] introduced the concept "δ-small submodule" as a generalization of small submodule. Let N be a submodule of a module M. N is called *δ-small* in M if whenever M=N+K and M/K is singular for any $K \leq M$ we have M=K, denoted by $N \ll_\delta M$. Here we consider this definition in the category σ[M] for a module M.

Definition 2.1. Let $N \leq K \in \sigma[M]$. N is called a *δ-M-small submodule of K in $\sigma[M]$* if whenever K=N+X and K/X is M-singular for X≤K we have K=X, we denoted by $N \ll_{\delta_M} K$.

For modules N, $K \in \sigma[M]$, $N \ll_{\delta} K \Rightarrow N \ll_{\delta_M} K$. The properties of δ-small submodules that are listed in Lemma 1.3 in [14] also hold in σ[M]. We write them for convenience. Note that the class of M-singular modules is closed under submodules, homomorphic images and direct sums [3].

Lemma 2.2. Let $N \in \sigma[M]$.

a) For modules K, $L \in \sigma[M]$ with $K \leq L \leq N$ we have

$$L \ll_{\delta_M} N \text{ if and only if } K \ll_{\delta_M} N \text{ and } L/K \ll_{\delta_M} N/K.$$

b) For K, $L \in \sigma[M]$, $K+L \ll_{\delta_M} N$ if and only if $K \ll_{\delta_M} N$ and $L \ll_{\delta_M} N$.

c) If $K \ll_{\delta_M} N$ and $f: N \to L$ is a homomorphism, then $f(K) \ll_{\delta_M} L$.

In particular, if $K \ll_{\delta_M} N \leq L$, then $K \ll_{\delta_M} L$.

d) If $K \leq L \leq_d N \in \sigma[M]$ and $K \ll_{\delta_M} N$, then $K \ll_{\delta_M} L$.

As a generalization of M-small module we define δ-M-small module.

Definition 2.3. Let $N \in \sigma[M]$. N is called a *δ-M-small module in $\sigma[M]$* if

$$N \cong K \ll_{\delta_M} L \in \sigma[M].$$

The following equivalence can be seen similarly as it is for M-small modules. For M-small modules it is proved in [6].

Lemma 2.4. N is a δ-M-small module in σ[M] if and only if $N \ll_{\delta_M} \hat{N}$.

Proof. It is enough to show that if N is δ-M-small then $N \ll_{\delta_M} \hat{N}$. Let K, $L \in \sigma[M]$ be such that $N \cong K \ll_{\delta_M} L$. Since $\hat{K}$ is injective in σ[M], there exists a homomorphism $f: L \to \hat{K}$ such that $f \circ i = g$ where $i: K \to L$ and $g: K \to \hat{K}$ are inclusion maps. Since $K \ll_{\delta_M} L$, $K = f(K) \ll_{\delta_M} \hat{K}$. This implies that $N \ll_{\delta_M} \hat{N}$.

If N is an M-small module then it is δ-M-small. The class of δ-M-small modules is closed under submodules, homomorphic images and finite direct sums.

Definition 2.5. Let $N\in\sigma[M]$. We define

$\delta_M(N) := \{n\in N : nR \ll_{\delta_M} N\}$

$\delta^*_M(N) := \{n\in N : nR \ll_{\delta_M} \widehat{nR}\} = \{n\in N : nR \ll_{\delta_M} \widehat{N}\} = \delta_M(\widehat{N}) \cap N.$

In case M=R, we write $\delta_R(N)=\delta(N)$ and $\delta^*_R(N)=\delta^*(N)$. Then

$$\mathrm{Rad}(N) \le \delta_M(N) \le \delta^*_M(N)$$
$$\mathrm{Rad}(N) \le Z^*_M(N) \le \delta^*_M(N).$$

If N is a δ-M-small module then $\delta^*_M(N)=N$. Also by definition for $N\le K \in\sigma[M]$, $\delta^*_M(N)=N \cap \delta^*_M(K)$. In particular, $\delta^*_M(\delta^*_M(N))= \delta^*_M(N)$. δ(N) is defined by [14]. Note that for any ring R, $\mathrm{Soc}(R_R)\le \delta(R_R)$ by [14, Theorem 1.6].

If for every $N\in\sigma[M]$, $\delta^*_M(N)=0$, then M is a V-module. But the converse is not true in general:

Example 2.6. Let F be any field and R be the direct product of any infinite number of copies of F. Then R is a commutative V-ring and Soc(R) is the ideal of R consisting of all elements which have at most a finite number of non-zero components. Then by [14, Theorem 1.6], $\mathrm{Soc}(R) \le \delta(R) \le \delta^*(R)$ implies that $\delta^*(R)\neq 0$. Hence R is a V-ring but $\delta^*(R)\neq 0$. Actually, by Corollary 2.9 $\mathrm{Soc}(R)= \delta^*(R)$.

But Theorem 1.5 still holds when $Z^*_M(.)$ is replaced by $\delta^*_M(.)$.

Theorem 2.7. The following are equivalent for a module M.
a) M is a GCO-module,
b) For every module $N\in\sigma[M]$, $\delta^*_M(N)$ is M-projective,
c) Every δ-M-small module in σ[M] is M-projective,
d) For every module $N\in\sigma[M]$, $Z_M(N) \cap \delta^*_M(N)=0$,
e) For every simple module $E\in\sigma[M]$, $Z_M(\widehat{E}) \cap \delta^*_M(\widehat{E})=0$,
f) M/Soc(M) is a V-module and $Z_M(M) \cap \delta^*_M(M)=0$,
g) $\delta^*_M(M/K)=0$ for every $K\le_e M$ and $Z_M(M) \cap \delta^*_M(M)=0$,

h)Every non-zero module N with δ^*_M(N)=N contains a non-zero M-projective submodule,

i) For every module N∈σ[M] with Z_M(N) $\leq_e$ N, δ^*_M(N)=0.

Proof. (a) implies (b), since M-singular M-injective and δ-M-small modules are zero. Then δ^*_M(N) is semisimple and then M-projective. The others can be seen by definitions and Theorem 1.5.

Replacing Z_M(N) by the singular submodule Z(N) and M-projectivity by projectivity in Theorem 2.7 we have the following.

Theorem 2.8. The following are equivalent for a module M.
a) M is a GV-module,
b) For every module N∈σ[M], δ^*(N) is projective,
c) Every δ-M-small module in σ[M] is projective,
d) For every module N∈σ[M], Z(N) ∩ δ^*(N)=0,
e) For every simple module E∈σ[M], Z($\hat{E}$) ∩ δ^*($\hat{E}$)=0,
f) M/ Soc(M) is a V-module and Z(M) ∩ δ^*(M)=0,
g) δ^*(M/K)=0 for every K$\leq_e$M and Z(M) ∩ δ^*(M)=0,
h) Every non-zero module N with δ^*(N)=N contains a non-zero projective submodule,
i) For every module N∈σ[M] with Z(N) $\leq_e$ N, δ^*(N)=0.

Applying the above theorem to a ring we have the following corollary.

Corollary 2.9. The following are equivalent for a ring R.
a) R is a right GV-ring,
b) For every R-module M, δ^*(M) is projective,
c) Every δ-small module is projective,
d) For every R-module M, Z(M) ∩ δ^*(M)=0,
e) For every simple module S, Z(E(S)) ∩ δ^*(E(S))=0.
f) R/Soc(R) is a V-module and Z(R_R) ∩ δ^*(R_R)=0,
g) δ^*(R/K)=0 for every essential right ideal K of R and Z(R_R) ∩ δ^*(R_R)=0,
h) Every non-zero R-module M with δ^*(M)=M contains a non-zero projective submodule,

i) For every R-module M with $Z(M) \leq_e M$, $\delta^*(M)=0$.

In this case $Soc(R_R)=\delta(R_R)=\delta^*(R_R)$.

Proof. The last part is because of that $\delta^*(R_R)$ is semisimple.

If $M/Z^*_M(M)$ is a V-module (semisimple) then $M/\delta^*_M(M)$ is a V-module (respectively semisimple). Then Theorem 1.11 and 1.12 still hold for $\delta^*_M(.)$.

Theorem 2.10. Let M be a module with $M/\delta^*_M(M)$ a V-module. Then the following are equivalent.
a) M is a GCO-module,
b) $\delta^*_M(M)$ is semisimple M-projective.

Theorem 2.11. Let M be a module with $M/\delta^*_M(M)$ semisimple. Then the following are equivalent.
a) M is a GCO-module,
b) M is an SI-module,
c) $\delta^*_M(M)$ is semisimple M-projective.

Also under the assumption "$M/Z^*_M(M)$ is V-module (semisimple)" the conditions of Theorem 1.11 (respectively 1.12) are equivalent to "$\delta^*_M(M)$ is semisimple M-projective".

Consider some examples.

Examples 2.12. 1) Let R be the 2×2 upper triangular matrix over a field F. R is a right GV-ring but not a right V-ring by [2]. Then

$$Soc(R_R)=\delta(R_R)=\delta^*(R_R)=Z^*(R_R)=\begin{bmatrix} 0F \\ 0F \end{bmatrix}$$

([8, Example 11]), $J(R)=\begin{bmatrix} 0F \\ 00 \end{bmatrix}$.

2) Let $R=\mathbf{Z}/4\mathbf{Z}$. Then Soc(R)=Z(R)=2R. Since $R/Soc(R) \cong \mathbf{Z}/2\mathbf{Z}$, $Soc(R)=\delta(R)$. $\mathbf{Z}$ is a small module. This implies that for every R-module M, $Z^*(M)=M$ [8, Lemma 8] and hence for every R-module M, $\delta^*(M)=M$. On the other hand R is not an SI-ring but every singular R-module is semisimple by [13, Example 8].

If R is a right SI-ring, then Soc(R_R)=δ^*(R_R) is projective. But the second example above says that if every singular right R-module is semisimple and δ^*(R_R) is projective then R need not be a right SI-ring, compare with [3, 17.4 (a)⇔(c)].

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