Commun. Fac. Sci. Univ. Ank. Series A1
V. 52. (No.2) pp. 13-19 (2003)

A NOTE ON STONE-CECH COMPACTIFICATION OF A DISCRETE SEMIGROUP

R.KESKİN[1] and Y.ÇELİKER[2]

[1] *Department of Mathematics, Faculty of Science and Art, University of Sakarya*
[2] *Şehit Yüzbaşı Halil İbrahim Sert High School, Sakarya*

(Received: Dec.25,2002; Revised: Nov.08,2003; Accepted: Nov.12,2003)

ABSTRACT

In this study we present some theorems about βD, the Stone-Cech compactification of the discrete space D and some applications of the theorems to the semigroup βS are given.

1. INTRODUCTION

Let D be an infinite discrete space and let βD be its Stone-Cech compactification. As known (see[1]) $\{\overline{A} : A \subset D\}$ forms a basis for the topology on βD where $\overline{A} = \{p \in \beta D : A \in p\}$. Moreover, the points of βD can be regarded as ultrafilters on D with the points of D itself corresponding to the fixed ultrafilters. βD has the following properties. If Y is a compact Hausdorff space and $f : D \to Y$ is a function, then there exists a continuous function $\widetilde{f} : \beta D \to Y$ such that $\widetilde{f} \mid D = f$. Then $\widetilde{f}$ is said to be Stone-Cech extension of f. Also, $A \cap B = \varnothing$ if and only if $\overline{A} \cap \overline{B} = \varnothing$ where $A, B \subset D$ and $\overline{A} = Cl_{\beta D} A$.

Let S be an infinite discrete semigroup. Then the operation . on S extends naturally to an operation on βS making $(\beta S, .)$ into a compact right topological semigroup. By the right topological semigroup, we mean that for each $q \in \beta S$, and for each $s \in S$, the functions $\lambda_s : \beta S \to \beta S$ $\lambda_s(p) = sp$ and $\rho_q : \beta S \to \beta S$, $\rho_q(p) = pq$ are continuous.

In this study, we give a theorem which is a generalization of Lemma 8.4 in [1]. After that, we give a theorem which states that if $\widetilde{f}(p) = \widetilde{g}(p)$ and $\widetilde{f}$ is injective, then $\{t \in D \mid f(t) = g(t)\}$ is in p. Lastly, we give some applications of the theorems.

2. MAIN THEOREMS

The following theorem is well-known [1].

Theorem 1. Let D be a discrete space and let $f:D \to D$. If f has no fixed points, neither does $\widetilde{f}:\beta D \to \beta D$.

In the following we will give a theorem whose proof is a modification of that of a lemma given in [1]. Before giving our theorem, we state the lemma.

Lemma 1. Let S be a left cancellative discrete semigroup and let s and t be distinct elements of S such that $st=ts$. Then, for every $p \in \beta S$, $ps \neq pt$.

Theorem 2. Let D be an infinite discrete space and suppose that $f,g:D \to D$ are two commuting functions and that $f(u) \neq g(u)$ for every u in D. Then, for every $p \in \beta D \backslash D$, we have $\widetilde{f}(p) \neq \widetilde{g}(p)$.

Proof. Suppose that $\widetilde{f}(p) = \widetilde{g}(p)$. Since $fog = gof$, it can be shown by induction that $f^n og = gof^n$ for every $n \in N$, where f^n is the n times composition of f by itself. We can define an equivalence relation on D by stating that $u \equiv v$ if and only if $f^n(u) = f^n(v)$ for some natural number n. We now show that this is a transitive relation. Suppose that $u \equiv v$ and $v \equiv w$. Then $f^n(u) = f^n(v)$ and $f^m(v) = f^m(w)$ for some natural numbers n and m. Then

$$f^{m+n}(u) = f^m(f^n(u)) = f^m(f^n(v)) = f^{m+n}(v)$$

and

$$f^{n+m}(v) = f^n(f^m(v)) = f^n(f^m(w)) = f^{n+m}(w).$$ Thus

$f^{m+n}(u) = f^{m+n}(w)$. This implies that $u \equiv w$. Let $\theta:D \to D/\equiv$ denote the canonical projection. We define a mapping h from $\theta(D)=D/\equiv$ *into* $\theta(D)$ as follows.
$h(\theta(f(w)))= \theta(g(w))$ if $f(w) \in f(D)$ and $h(\theta(w))= \theta(f(w))$
if $\theta(w) \in \theta(D) \setminus \theta(f(D))$.

Suppose that $\theta(f(w))= \theta(f(v))$. Then $f^n(f(w)) = f^n(f(v)$ for some natural number n. Then $g(f^{n+1}(w)) = g(f^{n+1}(v))$. Thus $f^{n+1}(g(w)) = f^{n+1}(g(v))$. That is, $g(w)\equiv g(v)$. This implies that $\theta(g(w))=\theta(g(v))$. It follows from the definition that if $v\equiv w$, then $f(v)\equiv f(w)$. In order to show that h is well-defined on $\theta(D)\setminus\theta(f(D))$, suppose that $\theta(w)=\theta(v)$. Then $w\equiv v$, so $f(w)\equiv f(v)$. Thus $\theta(f(w))=\theta(f(v))$. That is, $h(\theta(w))=h(\theta(v))$. This shows that h is well-defined on $\theta(D)$. On the other hand, we see that $\widetilde{h} o \widetilde{\theta} o \widetilde{f}$ and $\widetilde{\theta} o \widetilde{g}$ are continuous functions agreeing on D, hence on βD. Thus $\widetilde{h}(\widetilde{\theta}(\widetilde{f}(p))) = \widetilde{\theta}(\widetilde{g}(p)) = \widetilde{\theta}(\widetilde{f}(p))$. We next show that h has no fixed points. Suppose that $h(\theta(f(v)))=\theta(f(v))$. Then since $h(\theta(f(v)))=\theta(g(v))$, we have $\theta(f(v))=\theta(g(v))$. Thus $f^n(g(v)) = f^n(f(v))$ for some natural number n. This shows that $g(f^n(v)) = f(f^n(v))$. This contradicts to the fact that $g(u)\neq f(u)$ for every $u\in D$. If $\theta(w)\in\theta(D)\setminus\theta(f(D))$, then $h(\theta(w))=\theta(f(w))\neq\theta(w)$. Therefore h has no fixed points. Since $\widetilde{h}$ has a fixed point $\widetilde{\theta}(\widetilde{f}(p))$, this contradicts to Theorem1. Thus the proof is complete.

An element s of a semigroup S is said to be a left(right) cancellable if, for every x and y in S, $sx=sy(xs=ys)$ implies $x=y$. A semigroup S is called left(right) cancellative if every element of S is left(right) cancellable. In view of the above theorem, we can give the following corollary. The corollary appears in [1] as lemma 8.4.

Corollary 1. Let S be a left cancellative infinite discrete semigroup and let s and t be distinct elements of S such that $st=ts$. Then, for every $p\in\beta S$, we have $ps\neq pt$.

Proof. Let $\rho_s : S \to S \quad \rho_s(u) = us$ and $\rho_t : S \to S \quad \rho_t(u) = ut$. Since S is left cancellative, it follows that $\rho_t(u) \neq \rho_s(u)$ for every u in S. Moreover, $\rho_t o \rho_s = \rho_s o \rho_t$. Therefore, by Theorem1, we have $\rho_t(p) \neq \rho_s(p)$. That is, $pt\neq ps$.

Recall that we represent the Stone-Cech extension of ρ_t by the same symbol ρ_t.

We also give the following corollary easily.

Corollary 2. Let S be a right cancellative infinite discrete semigroup and let s and t be distinct elements of S such that $st=ts$. Then $sp\neq tp$ for every $p\in\beta S$.

Proof. Let λ_t , $\lambda_s : S \to S$, $\lambda_t(u) = tu$ and $\lambda_s(u) = su$. Then $\lambda_t(u) \neq \lambda_s(u)$ for every $u \in S$, and $\lambda_s o \lambda_t = \lambda_t o \lambda_s$. Thus $\lambda_s(p) \neq \lambda_t(p)$. That is, $sp \neq tp$.

The following lemma is useful for the semigroup βS.

Lemma 2. Let D and T be two infinite discrete spaces and let $f,g: D \to T$ with $f \mid A$ is injective for some $A \in p$. Then $\widetilde{f}\,(p) = \widetilde{g}\,(p)$ implies $E = \{t \in D : f(t) = g(t)\} \in p$.

Proof. Assume that $E \notin p$. Then $D \setminus E \in p$ and thus $(D \setminus E) \cap A \in p$. Let $X = (D \setminus E) \cap A$. Then $X \in p$. We define $h: T \to T$ by putting $h(f(t)) = g(t)$ for every $t \in X$, defining $h(T \setminus f(X)) = f(b)$ if $T \setminus f(X) \neq \varnothing$ where $b \in X$ is fixed. Since f is injective, h has no fixed points. Since hof and g agree on X, it follows that $\widetilde{h}\,(\widetilde{f}\,(p)) = \widetilde{g}(p) = \widetilde{f}(p)$. This is a contradiction, since h has no fixed points. Thus $E \in p$.

Corollary 3. Let D be an infinite discrete space and let $f,g: D \to D$ be two functions. Suppose that f is injective. Then $\widetilde{f}(p) = \widetilde{g}(p)$ if and only if $E = \{t \in D : f(t) = g(t)\} \in p$.

Proof. If $\widetilde{f}(p) = \widetilde{g}(p)$, then $E \in p$ follows from Lemma 2. Conversely, if $E \in p$ then $p \in \overline{E}$ so that $\widetilde{f}\,(p) = \widetilde{g}\,(p)$.

The corollary has many applications to the semigroup βS. The following two lemmas are proved in [1], page 115 and page 160. See also [2] for the first lemma.

Lemma 3. Let S be an infinite discrete semigroup and let $x \in \beta S \setminus S$. Let s and t be distinct elements of S.

i) If s is left cancellable and S is right cancellative, then $sx \neq tx$,

ii) If s is right cancellable and S is left cancellative, then $xs \neq xt$.

Proof. i) Since s is left cancelleble, the translation $s \to su$ is injective. Suppose that $sx=tx$. Then $\lambda_s(x) = \lambda_t(x)$. Thus $\{u \in S : su = tu\} \in x$. Therefore there exists $u \in S$ such that $su = tu$. Since S is right cancellative, we have $s = t$.

ii) Since s is right cancellable, the translation $u \to us$ is injective. Suppose that $\rho_t(x) = \rho_s(x)$. Thus there exists an element $u \in S$ such that $\rho_t(u) = \rho_s(u)$, i.e., $ut = us$. This implies that $t = s$, since S is left cancellable.

Lemma 4. Let S be a discrete left cancellative semigroup and let s and t be distinct elements of S. Let $p \in \beta S$. Then $sp = tp$ if and only if $\{u \in S : su = tu\} \in p$.

Proof. Let $\lambda_t, \lambda_s : S \to S$, $\lambda_t(u) = tu$ and $\lambda_s(u) = su$. Since λ_t is injective, the proof then follows from Corollary 3.

Let X be a topological space. A point x is said to be a weak p-point if x is not a limit point of any countable subset of $X \setminus \{x\}$.

Lemma 5. Let D and E be two discrete spaces and let $f : D \to E$ be an injective function. Then

a) $\tilde{f} : \beta D \to \beta E$ is injective.

x is a weak p-point in $\beta D \setminus D$ if and only if $\tilde{f}(x)$ is a weak p-point in $\beta E \setminus E$.

Proof. a) Let $x,y \in \beta D$ such that $x \neq y$. Then, since βD is a Hausdorff space there exists two subsets A and B of D such that $x \in \overline{A}$, $y \in \overline{B}$ and $\overline{A} \cap \overline{B} = \varnothing$. Then $A \cap B = \varnothing$ and so

$f(A) \cap f(B) = \varnothing$. Therefore $\overline{f(A)} \cap \overline{f(B)} = \varnothing$. Moreover, for any subset $U \subset D$, we see that $\tilde{f}(\overline{U}) = \overline{f(U)}$.

Thus $\tilde{f}(\overline{A}) \cap \tilde{f}(\overline{B}) = \varnothing$. It follows that $\tilde{f}(x) \neq \tilde{f}(y)$.

b) Suppose x is not a weak p-point in $\beta D \setminus D$. Let $C \subset \beta D \setminus D$ such that $x \in \overline{C}$ where $x \notin C$ and C is countable. Since $\tilde{f}$ is injective and C is countable, it follows that $\tilde{f}$ (C) is countable. Now we show that if $x \in \beta D \setminus D$, then $\tilde{f}(x) \in \beta E \setminus E$.

Assume that $\tilde{f}(x) \in E$. Since $x\in \overline{D}$, it follows that $\tilde{f}(x) \in \tilde{f}(\overline{D}) = \overline{f(D)}$. Since $\tilde{f}(x)\in E$, it is seen that $\tilde{f}(x) \in f(D)$, which implies $\tilde{f}(x) = f(t)$ for some $t\in D$. This shows that $x=t$. That is, $x\in D$, which is a contradiction. Since $x\notin C$, it is seen that $\tilde{f}(x) \notin \tilde{f}(C)$.

On the other hand, since $x\in \overline{C}$, we see that $\tilde{f}(x) \in \tilde{f}(\overline{C}) = \overline{f(C)}$. Therefore $\tilde{f}(x)$ is not a weak p-point in $\beta E\backslash E$.

Suppose that $\tilde{f}(x)$ is not a weak p-point of $\beta E\backslash E$. Let C be a countable subset of $\beta E\backslash E$ such that $\tilde{f}(x) \in \overline{C}\backslash C$.

Let $U=\{w\in\beta D\backslash D$: $\tilde{f}(w) \in C\}$. Then the set U does not contain x by the assumption. Since $\tilde{f}$ is injective, U is countable. Let $x\in \overline{A}$. Then $\tilde{f}(x) \in \overline{f(A)}$. Therefore, since $\tilde{f}(x) \in \overline{C}$, it is seen that $\overline{f(A)} \cap C \neq \varnothing$. Thus $\tilde{f}(w) \in C$ for some $w\in \overline{A}$. Since $C\subset\beta E\backslash E$, it follows that $w\in\beta D\backslash D$. Thus $w\in U$. Since $U \cap \overline{A} \neq \varnothing$, we see that $x\in \overline{U}$. This implies that x is not weak p-point in $\beta E\backslash E$. Thus we can give the following corollary(see also[7])

Corollary 4. Let S be an infinite, discrete cancellative semigroup, $x\in\beta S\backslash S$ and $s\in S$. Then the following statements are equivalent.

(1) xs is a weak p-point in $\beta S\backslash S$.

(2) sx is a weak p-point in $\beta S\backslash S$.

(3) x is a weak p-point in $\beta S\backslash S$.

Proof. Take the translations λ_s, $\rho_s : S\to S$ $\lambda_s(u) = su$ and $\rho_s(u) = us$. It follows that λ_s and ρ_s are injective. The proof then follows.

Acknowledgements. The first author is very grateful to Dr. Dona Strauss for private communications.

REFERENCES

[1] Neil Hindman, Dona Strauss, Algebra in the Stone-Cech compactification, Walter de Gruyter, Berlin-New York, 1999.

[2] M. Filali, Right Cancellation in βS and UG, Semigroup Forum 52(1996) 381-388.

[3] M. Filali, On the semigroup βS, Semigroup Forum 58(1999) 241-247.

[4] Neil Hindman, Dona Strauss, Cancellation in the Stone-Cech compactification of a discrete semigroup Proceedings of the Edinburg Math. Soc.(1994)37, 379-397.

[5] Neil Hindman, Algebra in βS and its Applications to Ramsey Theory, 44(3) (1996) 581-625.

[6] Dona Strauss, Semigroup Structure on βN, Semigroup Forum, 44(1992) 238-244.

[7] M. Filali, Weak P-points and Cancellation in βS.
Annals of the New York Academy of Sciences 806(1996) 130-139.