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On the Generalized Hermite-Hadamard Inequalities Involving Beta Function

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Abstract

In this paper, we establish new generalized fractional integral inequalities of Hermite-Hadamard type which cover the previously published result such as Riemann integral, Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [7]:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities. It gives an estimate from both sides of the mean value of a convex function and also ensure the integrability of convex function. It is also a matter of great interest and one has to note that some of the classical inequalities for means can be obtained from Hadamard's inequality under the utility of peculiar convex functions f: These inequalities for convex functions play a crucial role in analysis and as well as in other areas of pure and applied mathematics.
In [3], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

Theorem 1.2. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.3}$$

In [5], Kirmaci proved the following results connected with the left part of (1.1). In [5] some inequalities of Hermite-Hadamard type for differentiable convex mappings were proved using the following lemma.

Lemma 1.3. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a,b \in I^\circ$ with $a < b$. If $f' \in L([a,b])$, then we have*

$$\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)f'(ta+(1-t)b)dt\right]. \tag{1.4}$$

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Theorem 1.4. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a,b \in I^\circ$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then we have*

$$\left| \frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.5}$$

The subject of the fractional calculus (integrals and derivatives) has gained considerable popularity and importance during the past there decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. The fractional integral does indeed provide several potentially useful tools for various problems involving special functions of mathematical science as well as their extensions and generalizations in one and more variables. This subject is still being studied extensively by many authors, see for instance ([1], [2], [4], [6], [8]-[23]). One of the important applications of fractional integrals is Hermite-Hadamard integral inequality, see [10], [19]-[21]. First, let's recall the above definition of the Riemann-Liouville fractional integral are defined by follows [4] and [8]:

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ \ x>a,$$

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x<b.$$

The k-Riemann-Liouville fractional integral are defined by follows:

$$J^{\alpha}_{a^+,k} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \ \ x>a,$$

$$J^{\alpha}_{b^-,k} f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \ \ x<b$$

where

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt, \ \ \mathscr{R}(\alpha)>0$$

and

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right), \ \ \mathscr{R}(\alpha)>0; k>0$$

are given by Mubeen and Habibullah in [20].

Now, let's recall the basic expressions of Hermite-Hadamard inequality for fractional integrals is proved by Sarikaya et al. in [10] as follows:

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a function with $a<b$ and $f \in L_1([a,b])$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.6}$$

with $\alpha>0$.

Meanwhile, in [10], Sarikaya et al. gave the following interesting Trapeozid identity for Riemann-Liouville integral:

Lemma 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f' \in L[a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right] = \frac{b-a}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(ta+(1-t)b)\,dt. \tag{1.7}$$

Theorem 1.7. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right]\right| \leq \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[f'(a)+f'(b)\right]. \tag{1.8}$$

On the other hand, in [19] Iqbal et al. gave the following results connected with the left part of Riemann-Liouville integral inequalities of Hermite-Hadamard type (1.6) by using the following Midpoint identity as follows.

Lemma 1.8. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $f' \in L^1[a,b]$, then the following identity for Riemann-Liouville fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)\right] = \frac{b-a}{2}\sum_{k=1}^{4} I_k, \tag{1.9}$$

where

$$I_1 = \int_0^{\frac{1}{2}} t^\alpha f'(tb+(1-t)a)\,dt, \qquad I_2 = \int_0^{\frac{1}{2}} (-t^\alpha) f'(ta+(1-t)b)\,dt,$$

$$I_3 = \int_{\frac{1}{2}}^{1} (t^\alpha - 1) f'(tb+(1-t)a)\,dt, \quad I_4 = \int_{\frac{1}{2}}^{1} (1-t^\alpha) f'(ta+(1-t)b)\,dt.$$

Many paper have been studied the Riemann-Liouville fractionals integrals and give new and interesting generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for instance see ([10]-[13]).

The purpose of this paper is establish new Hermite-Hadamard type inequalities involving Beta functions. Using functions whose first derivatives absolute values are convex, we obtain new trapezoid and midpoint inequalities that are connected with the celebrated Hermite-Hadamard type which cover the previously puplished results.

2. Hermite-Hadamard inequalities involving beta function

In this section, using Beta function, we begin by the following theorem:

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function on $[a,b]$ with $a<b$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\beta(m,n)\le\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)\,dx\le\beta(m,n)\frac{f(a)+f(b)}{2} \tag{2.1}$$

where $\beta(m,n)$ is a beta function and

$$\Omega(x)=(b-x)^{m-1}(x-a)^{n-1}+(b-x)^{n-1}(x-a)^{m-1}$$

for $m,n>0$.

Proof. For $t\in[0,1]$, let $x=ta+(1-t)b$, $y=(1-t)a+tb$. The convexity of f yields

$$f\left(\frac{a+b}{2}\right)=f\left(\frac{x+y}{2}\right)\le\frac{f(x)+f(y)}{2} \tag{2.2}$$

i.e.,

$$2f\left(\frac{a+b}{2}\right)\le f(ta+(1-t)b)+f((1-t)a+tb). \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{m-1}(1-t)^{n-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$2f\left(\frac{a+b}{2}\right)\int_0^1 t^{m-1}(1-t)^{n-1}dt\le\int_0^1 t^{m-1}(1-t)^{n-1}f(ta+(1-t)b)\,dt+\int_0^1 t^{m-1}(1-t)^{n-1}f((1-t)a+tb)\,dt.$$

As consequence, we obtain

$$f\left(\frac{a+b}{2}\right)\beta(m,n)\le\frac{1}{2(b-a)^{m+n-1}}\int_a^b\left[(b-x)^{m-1}(x-a)^{n-1}+(b-x)^{n-1}(x-a)^{m-1}\right]f(x)\,dx$$

and the first inequality is proved.
To prove the other half of the inequality in (2.1), since f is convex, for every $t\in[0,1]$, we have,

$$f(ta+(1-t)b)+f((1-t)a+tb)\le f(a)+f(b). \tag{2.4}$$

Then multiplying both sides of (2.4) by $t^{m-1}(1-t)^{n-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\frac{1}{2(b-a)^{m+n-1}}\int_a^b\left[(b-x)^{m-1}(x-a)^{n-1}+(b-x)^{n-1}(x-a)^{m-1}\right]f(x)\,dx\le\frac{f(a)+f(b)}{2}\beta(m,n)$$

and the second inequality is proved. □

Remark 2.2. *If in Theorem 2.1, we get $n=m=1$, then the inequalities (2.1) become the inequalities (1.1).*

Remark 2.3. *If in Theorem 2.1, we get $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$) then the inequalities (2.1) become the inequalities (1.6) of Theorem 1.5.*

Remark 2.4. *If in Theorem 2.1, we get $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$) then the inequalities (2.1) become the inequalities*

$$f\left(\frac{a+b}{2}\right)\le\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f(b)+J^{\alpha}_{b^-,k}f(a)\right]\le\frac{f(a)+f(b)}{2}$$

which are proved by Hussain et. al. in [22].

3. Trapezoid inequalities involving beta function

In this section, we give an identity which use to assist us is proving our results as follows:

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\beta(m,n)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)\,dx=\frac{(b-a)}{2}\int_0^1\beta_t(m,n)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt \tag{3.1}$$

where $\beta_t(m,n)$ is incomplete beta function defiend by

$$\beta_t(m,n)=\int_0^t s^{m-1}(1-s)^{n-1}\,ds,\ \ 0<t\le 1$$

for $m,n>0$.

Proof. Here, we apply integration by parts in integrals of right part of (3.1), then we have

$$\begin{aligned}F_1&=\int_0^1\beta_t(m,n)f'(tb+(1-t)a)\,dt\\&=\frac{f(b)}{b-a}\beta(m,n)-\frac{1}{b-a}\int_0^1 t^{m-1}(1-t)^{n-1}f(tb+(1-t)a)dt\\&=\frac{f(b)}{b-a}\beta(m,n)-\frac{1}{(b-a)^{m+n}}\int_a^b(x-a)^{m-1}(b-x)^{n-1}f(x)dx.\end{aligned}$$

And similarly, we obtain

$$\begin{aligned}F_2=\int_0^1\beta_t(m,n)f'(ta+(1-t)b)\,dt&=-\frac{f(a)}{b-a}\beta(m,n)+\frac{1}{b-a}\int_0^1 t^{m-1}(1-t)^{n-1}f(ta+(1-t)b)dt\\&=-\frac{f(a)}{b-a}\beta(m,n)+\frac{1}{(b-a)^{m+n}}\int_a^b(x-a)^{n-1}(b-x)^{m-1}f(x)dx.\end{aligned}$$

If we subtract F_1 from F_2 and multiply by $\frac{(b-a)}{2}$, we obtain proof of the (3.1). □

Remark 3.2. *If in Lemma 3.1, we get $m=n=1$, then the identity (3.1) becomes the identity (1.2) of Lemma 1.1.*

Remark 3.3. *If in Lemma 3.1, we get $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$), then the identity (3.1) becomes the identity (1.7) of Lemma 1.6.*

Remark 3.4. *If in Lemma 3.1, we get $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$), then the identity (3.1) reduces to*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f(b)+J^{\alpha}_{b^-,k}f(a)\right]=\frac{b-a}{2}\int_0^1\left[(1-t)^{\frac{\alpha}{k}}-t^{\frac{\alpha}{k}}\right]f'(ta+(1-t)b)\,dt$$

which are proved by Hussain et. al. in [22].

Remark 3.5. *By the change of variable in Lemma 3.1, the identity (3.1) reduces to*

$$\beta(m,n)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)\,dx=\frac{(b-a)}{2}\int_0^1[\beta_t(m,n)-\beta_{1-t}(m,n)]f'(tb+(1-t)a)\,dt.$$

Now, we extend some estimates of the right hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex as follows:

Theorem 3.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\beta(m,n)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)\,dx\right|\le(b-a)\left(\frac{|f'(a)|+|f'(b)|}{2}\right)\int_0^{\frac{1}{2}}[\beta_{1-t}(m,n)-\beta_t(m,n)]\,dt \tag{3.2}$$

for $m,n>0$.

Proof. Using Lemma 3.1 and the convexity of $|f'|$, we find that

$$\begin{aligned}
\left|\beta(m,n)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x)f(x)\,dx\right| &\leq \frac{(b-a)}{2}\int_0^1 |\beta_t(m,n)-\beta_{1-t}(m,n)|\left|f'((1-t)a+tb)\right|dt \\
&\leq \frac{(b-a)}{2}\int_0^{\frac{1}{2}} [\beta_{1-t}(m,n)-\beta_t(m,n)]\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dt \\
&\quad +\frac{(b-a)}{2}\int_{\frac{1}{2}}^1 [\beta_t(m,n)-\beta_{1-t}(m,n)]\left[(1-t)\left|f'(a)\right|+t\left|f'(b)\right|\right]dt \\
&= \frac{(b-a)}{2}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\int_0^{\frac{1}{2}} [\beta_{1-t}(m,n)-\beta_t(m,n)]\,dt
\end{aligned}$$

which this completes the proof of the (3.2). □

Remark 3.7. *If in Theorem 3.6, we get $n=m=1$, then, the inequality (3.2) becomes the inequality (1.3) of Theorem 1.2.*

Remark 3.8. *If in Theorem 3.6, we get $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$), then the inequality (3.2) becomes the inequality (1.8) of Theorem 1.7.*

Remark 3.9. *If in Theorem 3.6, we get $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$), then the inequality (3.2) becomes*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f(b)+J^{\alpha}_{b^-,k}f(a)\right]\right| \leq \frac{(b-a)}{\left(\frac{\alpha}{k}+1\right)}\left(1-\frac{1}{2^{\frac{\alpha}{k}}}\right)\left(\frac{|f'(a)|+|f'(b)|}{2}\right)$$

which are proved by Hussain et. al. in [22].

Theorem 3.10. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some $q>1$, then the following inequality holds: for $m,n>0$*

$$\left|\beta(m,n)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x)f(x)\,dx\right| \leq \frac{(b-a)}{2}\left(\int_0^1 |\beta_t(m,n)-\beta_{1-t}(m,n)|^p\,dt\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \tag{3.3}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Lemma 3.1, Hölder's inequality and the convexity of $|f'|^q$, we find that

$$\begin{aligned}
&\left|\beta(m,n)\frac{f(a)+f(b)}{2}-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x)f(x)\,dx\right| \\
\leq\ & \frac{(b-a)}{2}\left(\int_0^1 |\beta_t(m,n)-\beta_{1-t}(m,n)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'((1-t)a+tb)\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{(b-a)}{2}\left(\int_0^1 |\beta_t(m,n)-\beta_{1-t}(m,n)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[(1-t)\left|f'(a)\right|^q+t\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}}
\end{aligned}$$

which this completes the proof of the (3.3). □

Remark 3.11. *If in Theorem 3.10, we get $m=n=1$, then, the inequality (3.3) becomes the inequality (2.4) of Theorem 2.3 in [3].*

Remark 3.12. *If in Theorem 3.10, we get $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$), then the inequality (3.3) becomes the inequality (2.7) of Theorem 8 in [14].*

Remark 3.13. *If in Theorem 3.10 we get $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$), then the inequality (3.3) becomes*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma_k(\alpha+k)}{(b-a)^{\frac{\alpha}{k}}}\left[J^{\alpha}_{a^+,k}f(b)+J^{\alpha}_{b^-,k}f(a)\right]\right| \leq \frac{(b-a)}{2\left(\frac{\alpha}{k}p+1\right)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1,\ \frac{\alpha}{k}\in[0,1]$, which are proved Hussain et. al. in [22].

4. Midpoint inequalities involving beta function

Before starting and proving our next result, we need the following lemma.

Lemma 4.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$f\left(\frac{a+b}{2}\right)\beta(m,n)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x)f(x)\,dx=\frac{b-a}{2}\sum_{k=1}^{4}T_k \tag{4.1}$$

where

$$T_1=\int_0^{\frac{1}{2}}\beta_t(m,n)f'(tb+(1-t)a)\,dt,\qquad T_2=\int_0^{\frac{1}{2}}(-\beta_t(m,n))f'(ta+(1-t)b)\,dt,$$
$$T_3=\int_{\frac{1}{2}}^{1}(-\beta_{1-t}(n,m))f'(tb+(1-t)a)\,dt,\qquad T_4=\int_{\frac{1}{2}}^{1}\beta_{1-t}(n,m)f'(ta+(1-t)b)\,dt$$

for $m,n>0$.

Proof. In the proof of (4.1), we apply integration by parts, then we have

$$\begin{aligned}T_1&=\int_0^{\frac{1}{2}}\beta_t(m,n)f'(tb+(1-t)a)\,dt\\&=\frac{1}{b-a}\beta_{\frac{1}{2}}(m,n)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_0^{\frac{1}{2}}t^{m-1}(1-t)^{n-1}f(tb+(1-t)a)\,dt,\end{aligned}$$

$$\begin{aligned}T_2&=\int_0^{\frac{1}{2}}(-\beta_t(m,n))f'(ta+(1-t)b)\,dt\\&=\frac{1}{b-a}\beta_{\frac{1}{2}}(m,n)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_0^{\frac{1}{2}}t^{m-1}(1-t)^{n-1}f(ta+(1-t)b)\,dt,\end{aligned}$$

$$\begin{aligned}T_3&=\int_{\frac{1}{2}}^{1}(-\beta_{1-t}(m,n))f'(tb+(1-t)a)\,dt\\&=\frac{1}{b-a}\beta_{\frac{1}{2}}(n,m)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{1}{2}}^{1}(1-t)^{m-1}t^{n-1}f(tb+(1-t)a)\,dt,\end{aligned}$$

$$\begin{aligned}T_4&=\int_{\frac{1}{2}}^{1}\beta_{1-t}(m,n)f'(ta+(1-t)b)\,dt\\&=\frac{1}{b-a}\beta_{\frac{1}{2}}(n,m)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{1}{2}}^{1}(1-t)^{m-1}t^{n-1}f(ta+(1-t)b)\,dt.\end{aligned}$$

Thus, by the above expressions, the desired identity (4.1) is obtained. □

Remark 4.2. *If in Lemma 4.1, we take $m=n=1$, then, the identity (4.1) becomes the identity (1.4) of Lemma 1.3 by Kirmaci in [5].*

Remark 4.3. *If in Lemma 4.1, we take $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$), then the identity (4.1) reduces to (1.9) of Lemma 1.8 by Iqbal et al. in [19] .*

Remark 4.4. *If in Lemma 4.1, we take $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$), then the identity (4.1) reduces to the identity of Corollary 6 by Sarikaya and Ertugral in [13] .*

Remark 4.5. *By the change of variable in Lemma 4.1, the identity (4.1) reduces to*

$$f\left(\frac{a+b}{2}\right)\beta(m,n)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b \Omega(x)f(x)\,dx=\frac{(b-a)}{2}\int_0^{\frac{1}{2}}\left[\beta_t(m,n)+\beta_t(n,m)\right]\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt. \tag{4.2}$$

Finally, we extend some estimates of the left hand side of a Hermite-Hadamard type inequality for functions whose first derivatives absolute values are convex as follows:

Theorem 4.6. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)\beta(m,n)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)dx\right|\le\frac{(b-a)}{2}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]\left(\frac{1}{2}\beta(m,n)-\beta_{\frac{1}{2}}(m+1,n)-\beta_{\frac{1}{2}}(n+1,m)\right)\tag{4.3}$$

for $m,n>0$.

Proof. From (4.2) and using the convexity of $|f'|$, then we have

$$\left|f\left(\frac{a+b}{2}\right)\beta(m,n)-\frac{1}{2(b-a)^{m+n-1}}\int_a^b\Omega(x)f(x)dx\right|\le\frac{(b-a)}{2}\left(\int_0^{\frac{1}{2}}\left[\beta_t(m,n)+\beta_t(n,m)\right]dt\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].\tag{4.4}$$

By change variable in the integrals, we get

$$\begin{aligned}\int_0^{\frac{1}{2}}\left[\beta_t(m,n)+\beta_t(n,m)\right]dt &= \int_0^{\frac{1}{2}}\int_0^t s^{m-1}(1-s)^{n-1}dsdt+\int_0^{\frac{1}{2}}\int_0^t(1-s)^{m-1}s^{n-1}dsdt \\ &= \frac{1}{2}\beta(m,n)-\beta_{\frac{1}{2}}(m+1,n)-\beta_{\frac{1}{2}}(n+1,m).\end{aligned}\tag{4.5}$$

By writing (4.5) in the (4.4), we obtain the required inequality. This completes the proof. □

Remark 4.7. *If in Theorem 4.6, we take $m=n=1$, then, the inequality (4.3) reduces to the inequality (2.3) of Theorem 2.3 by Kirmaci in [5].*

Remark 4.8. *If in Theorem 4.6, we take $m=1,\ n=\alpha$,(or $m=\alpha,\ n=1$), then the inequality (4.3) reduces to the inequality (3) of Theorem 2 by Iqbal et al. in [19] .*

Remark 4.9. *If in Theorem 4.6, we take $m=1,\ n=\frac{\alpha}{k}$,(or $m=\frac{\alpha}{k},\ n=1$), then the inequality (4.3) reduces to the inequality of Corollary 9 by Sarikaya and Ertugral in [13].*

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