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QUARTER - SYMMETRIC METRIC CONNECTION ON A (k,μ) - CONTACT METRIC MANIFOLD

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ABSTRACT. The object of the present paper is to prove the existence of a quarter-symmetric metric connection on a Riemannian manifold and to study some properties of a quarter-symmetric metric connection on a non-Sasakian (k,μ)-contact metric manifold.

1. INTRODUCTION AND RESULTS

A. (k,μ) - contact metric manifolds

An odd dimensional differentiable manifold $M^m(m=2n+1)$ is called a contact manifold if it carries a global differentiable 1- form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on M^m . This 1 - form η is called the contact form of M^{2n+1}. A Riemannian metric g is said to be associated with a contact manifold if there exists a (1, 1) tensor field ϕ and a contravariant global vector field ξ, called the characteristic vector field of the manifold such that

(1.1) $\quad (a)\ \phi^2X=-X+\eta(X)\xi,\ (b)\ \eta(\xi)=1,\ (c)\ g(X,\xi)=\eta(X)$
$\quad (d)\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y),\ (e)\ d\eta(X,Y)=g(X,\phi Y)$

for all vector fields X,Y on M. Then the structure (ϕ,ξ,η,g) is said to be a contact metric structure and the manifold M^m equipped with such a structure is said to be a contact metric manifold [1]. In a contact metric manifold the following relations hold:

(1.2) $\quad (a)\ \phi\xi=0,\ (b)\ \eta\circ\phi=0,\ (c)\ d\eta(\xi,X)=0,\ (d)\ g(X,\phi Y)+g(\phi X,Y)=0.$

A contact metric manifold is said to be η - Einstein if its Ricci tensor S is of the form

$$S=ag+b\eta\otimes\eta,$$

where a and b are smooth functions on the manifold.

In a contact metric manifold we define a (1, 1) tensor field h by $h=\frac{1}{2}\pounds_\xi\phi$ where

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$\pounds$ denotes the Lie differentiation. Then h is self-adjoint and satisfies

(1.3) $(a)\ h\xi = 0,\ (b)\ h\phi = -\phi h,\ (c)\ Tr.h = Tr.\phi h = 0.$

A contact metric manifold is said to be a (k, μ) - contact metric manifold [2] if it satisfies the relation

(1.4) $R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]$

for all vector fields X and Y on M where k, μ are real constants and R is the Riemann curvature tensor of the manifold of type (1, 3).

The class of (k, μ) - contact metric manifolds contains both the class of Sasakian ($k = 1$ and $h = 0$) and non-Sasakian ($k \neq 1$ and $h \neq 0$) manifolds. For example, the unit tangent sphere bundle of a flat Riemannian manifold with the usual contact metric structure is a non- Sasakian (k, μ) - contact metric manifold. Throughout the present paper we confined ourselves with the study of non-Sasakian cases and hence $k \neq 1$ and $h \neq 0$.

B. Quarter - symmetric metric connections.

A linear connection $\tilde{\nabla}$ on an m - dimensional Riemannian manifold (M^m, g) is said to be a quarter-symmetric metric connection [7] if its torsion tensor T satisfies

(1.5) $(a)\ T(X,Y) = \pi(Y)F(X) - \pi(X)F(Y)$ and $(b)\ \tilde{\nabla}g = 0$

where π is a differentiable 1 - form and F is a (1, 1) tensor field.

Especially, if $F(X) = X$, then the quarter-symmetric metric connection reduces to a semi-symmetric metric connection [11].

Quarter-symmetric metric connection studied by many authors in several ways to a different extent such as [4], [5], [7], [8], [12].

If the contact form η and the (1, 1) - tensor field h of the contact metric structure are respectively taken in lieu of the 1 - form π and the (1, 1) - tensor field F of the quarter - symmetric metric connection, then (1.5) reduces to the following form

(1.6) $(a)\ T(X,Y) = \eta(Y)hX - \eta(X)hY$ and $(b)\ \tilde{\nabla}g = 0.$

C. Results of the Paper.

The present paper deals with a study of non - Sasakian (k, μ) - contact metric manifold with a quarter - symmetric metric connection $\tilde{\nabla}$ satisfying (1.6) and obtained the following results:

Theorem 1. *On a Riemannian manifold (M, g) there exists a unique quarter - symmetric metric connection.*

Theorem 2. *In a non-Sasakian (k, μ) - contact metric manifold (M, g), a linear connection $\tilde{\nabla}$ is a quarter-symmetric metric connection if and only if*

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)hX - g(hX, Y)\xi \text{ for all } X,\ Y \in \chi(M).$$

Theorem 3. *The curvature tensor $\tilde{R}$ of a non-Sasakian (k, μ) - contact metric manifold with respect to the quarter-symmetric metric connection satisfies*

(i) $\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z,$

(ii) $g(\tilde{R}(X,Y)Z,W) = -g(\tilde{R}(X,Y)W,Z),$

(iii) $g(\tilde{R}(X,Y)Z,W) = g(\tilde{R}(Z,W)X,Y),$

(iv)

$$\tilde{R}(X,Y)Z+\tilde{R}(Y,Z)X+\tilde{R}(Z,X)Y = 2[d\eta(X,Z)hY - d\eta(Y,Z)hX - d\eta(X,Y)hZ +(1-k)\{d\eta(Y,Z)\eta(X)\xi + d\eta(X,Y)\eta(Z)\xi +d\eta(X,Z)\eta(Y)\xi\}]$$

for all vector fields X, Y, $Z \in \chi(M)$.

Theorem 4. *The curvature tensor of a non-Sasakian* (k,μ) *- contact metric manifold with respect to the quarter-symmetric metric connection satisfies the Bianchi identity if and only if the contact form* η *is closed.*

Theorem 5. *The Ricci tensor of a non-Sasakian* (k,μ) *- contact metric manifold* (M^m, g) *with respect to the quarter-symmetric metric connection is symmetric if and only if the contact form* η *is closed.*

Theorem 6. *If the Ricci tensor of a non-Sasakian* (k,μ) *- contact metric manifold with respect to the quarter-symmetric metric connection vanishes, then the manifold is locally isometric to either an* η *- Einstein or a 3-dimensional non-Sasakian* (k,μ) *- contact metric manifold.*

Theorem 7. *If the Ricci tensor of a complete non-Sasakian* (k,μ) *- contact metric manifold with respect to the quarter-symmetric metric connection vanishes and the manifold is not* η *- Einstein, then it is locally isometric to one of the following Lie groups with a left invariant metric :*

$SU(2)$*: the group of* 2×2 *unitary matrices of determinant 1,*

$SO(3)$*: the rotation group of 3 - space,*

$SL(2,R)$*: the group of* 2×2 *real matrices of determinant 1,*

$E(2)$*: the group of rigid motions of the Euclidean 2-space,*

$E(1,1)$*: the group of rigid motions of the Minkowski 2-space,*

$O(1,2)$*: the Lorentz group consisting of linear transformations preserving the quadratic form* $t^2 - x^2 - y^2$.

Theorem 8. *If the Weyl conformal curvature tensor of a non-Sasakian* (k,μ) *- contact metric manifold with respect to the quarter-symmetric metric connection is equal to that with respect to the Riemannian connection then the manifold is either locally isometric to a 3-dimensional non-Sasakian* (k,μ) *- contact metric manifold or the contact form* η *is closed provided that* $k+\mu^2-2\mu \neq 0$.

Theorem 9. *Let* (M,g) *be a complete non-Sasakian* (k,μ) *- contact metric manifold whose contact form* η *is not closed and* $k+\mu^2-2\mu \neq 0$. *If the Weyl conformal curvature tensor of the manifold with respect to the quarter-symmetric metric connection is equal to that with respect to the Riemannian connection then the manifold is locally isometric to one of the following Lie groups with a left invariant metric :*

$SU(2)$*: the group of* 2×2 *unitary matrices of determinant 1,*

$SO(3)$*: the rotation group of 3 - space,*

$SL(2,R)$*: the group of* 2×2 *real matrices of determinant 1,*

$E(2)$*: the group of rigid motions of the Euclidean 2-space,*

$E(1,1)$*: the group of rigid motions of the Minkowski 2-space,*

$O(1,2)$*: the Lorentz group consisting of linear transformations preserving the* *quadratic form* $t^2 - x^2 - y^2$.

Theorem 10. *If the contact form* η *of a non-Sasakian* (k,μ) *- contact metric manifold is closed, then* $\tilde{C}(X,Y)\xi = C(X,Y)\xi$ *for all* $X, Y \in \chi(M)$ *where* $\tilde{C}$ *and* C *are respectively the Weyl conformal curvature tensor with respect to the quarter-symmetric metric connection and the Riemannian connection.*

Theorem 11. *The contact form* η *of a non-Sasakian* (k,μ) *- contact metric manifold is closed if and only if the Weyl projective curvature tensor with respect to the quarter-symmetric metric connection is equal to that with respect to the Riemannian connection.*

Theorem 12. *If the Weyl projective curvature tensor of a non-Sasakian* (k,μ) *- contact metric manifold with respect to the quarter-symmetric metric connection is equal to that with respect to the Riemannian connection then the characteristic vector field* ξ *is a harmonic vector field.*

2. PRELIMINARIES

This section deals with some fundamental results of (k,μ) - contact metric manifolds and quarter-symmetric metric connection, which will be frequently used later on.

Lemma 2.1. *In a* (k,μ) *- contact metric manifold* $(M^m, g)(m = 2n+1)$ *the following relations hold:*

(2.1) $\nabla_X \xi = -\phi X - \phi h X,$

(2.2) $h^2 X = (k-1)\phi^2 X,\ k \leq 1$

(2.3) $(\nabla_X h)(Y) = (1-k)[g(X,\phi Y)\xi - \eta(Y)\phi X] + g(X, h\phi Y)\xi + \eta(Y)h\phi X - \mu\eta(X)\phi h Y,$

(2.4) $S(X,Y) = [2n-2-n\mu]g(X,Y) + [2n-2+\mu]g(hX,Y) + [2-2n+2nk+n\mu]\eta(X)\eta(Y),$

(2.5) $r = 2n(2n-2-n\mu) + 2nk$

where ∇ *denotes the Riemannian connection,* S *and* r *denotes respectively the Ricci-tensor of type (0, 2) and the scalar curvature of the manifold with respect to the Riemannian connection* ∇.

Proof. The proof of this Lemma follows from the paper [2] and hence we omit it.

Lemma 2.2. *The curvature tensor* R *of a non - Sasakian* (k,μ) *- contact metric manifold with respect to the Riemannian connection* ∇ *is given by*

(2.6)
$$\begin{aligned} g(R(X,Y)Z,W) = {} & (1-\tfrac{\mu}{2})[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)]+g(Y,Z)g(hX,W) \\ & -g(X,Z)g(hY,W)+g(hY,Z)g(X,W)-g(hX,Z)g(Y,W) \\ & +\tfrac{1-\frac{\mu}{2}}{1-k}[g(hY,Z)g(hX,W)-g(hX,Z)g(hY,W)] \\ & -\tfrac{\mu}{2}[g(\phi Y,Z)g(\phi X,W)-g(\phi X,Z)g(\phi Y,W)] \\ & +\tfrac{k-\frac{\mu}{2}}{1-k}[g(\phi hY,Z)g(\phi hX,W)-g(\phi hX,Z)g(\phi hY,W)] \\ & +\mu g(\phi X,Y)g(\phi Z,W)+\eta(X)\eta(W)[(k-1+\tfrac{\mu}{2})g(Y,Z) \\ & +(\mu-1)g(hY,Z)]-\eta(X)\eta(Z)[(k-1+\tfrac{\mu}{2})g(Y,W) \\ & +(\mu-1)g(hY,W)]+\eta(Y)\eta(Z)[(k-1+\tfrac{\mu}{2})g(X,W) \end{aligned}$$

$$+(\mu-1)g(hX,W)]-\eta(Y)\eta(W)[(k-1+\tfrac{\mu}{2})g(X,Z) \\ +(\mu-1)g(hX,Z)]$$

for all vector fields X, Y, Z, W on M.

Proof. The proof of this Lemma is given in the paper of Boeckex [3] and hence we omit it.

Lemma 2.3. *A 3- dimensional complete non-Sasakian (k, μ) - contact metric manifold is locally isometric to one of the following Lie groups with a left invariant metric :*

$SU(2)$: the group of 2×2 unitary matrices of determinant 1,

$SO(3)$: the rotation group of 3 - space,

$SL(2, R)$: the group of 2×2 real matrices of determinant 1,

$E(2)$: the group of rigid motions of the Euclidean 2-space,

$E(1,1)$: the group of rigid motions of the Minkowski 2-space,

$O(1,2)$: the Lorentz group consisting of linear transformations preserving the quadratic form $t^2 - x^2 - y^2$.

Proof. Since the (k, μ)-contact metric manifold is non-Sasakian, we must have $k < 1$ and $h \neq 0$. Let X be a unit eigenvector of h orthogonal to ξ with corresponding eigenvalue $\lambda = \sqrt{1-k} > 0$. Then there exist three mutually orthonormal vector fields X, ϕX, ξ such that [2]

(2.7) $\quad [X, \phi X] = c_1\xi,\ [\phi X, \xi] = c_2 X,\ [\xi, X] = c_3 X$

where $c_1 = 2$, $c_2 = \frac{r}{2} + \frac{(\lambda-1)^2}{2} =$ constant, $c_3 = \frac{r}{2} + \frac{(\lambda+1)^2}{2}$ =constant.

The vector field ξ is defined globally on M^3. Going to the universal covering space $\tilde{M}^3$, if necessary, we have global vector fields on $\tilde{M}^3$ satisfying (2.7). By a well known result [10] we conclude that for each $P \in \tilde{M}^3$, $\tilde{M}^3$ has a unique Lie group structure such that P is the identity and the vector fields are left invariant. In [9] J. Milnor gave a complete classification of 3 - dimensional Lie groups, which admit the Lie algebra structure (2.7).

Therefore, the signs of c_2 and c_3 vary. Since the manifold under consideration is non - Sasakian ($k < 1$), we must have $c_2 \neq c_3$. Since $c_2 = 2 > 0$, the possible combinations of the signs of c_2 and c_3, determines the corresponding Lie groups. Hence M^3 is locally isometric to $SU(2)$ or $SO(3)$, when $c_2 > 0$ and $c_3 > 0$, to $SL(2, R)$ or $O(1,2)$, when $c_2 > 0$ and $c_3 < 0$, to $E(2)$ when $c_2 > 0$ and $c_3 = 0$ and to $E(1,1)$ when $c_2 < 0$ and $c_3 = 0$.

This proves the Lemma.

3. EXISTENCE OF A QUARTER-SYMMETRIC METRIC CONNECTION

This section is devoted to the existence of the quarter - symmetric metric connection on a Riemannian manifold.

Proposition. *On a Riemannian manifold (M, g), for any skew - symmetric tensor field $T \in \chi^{1,2}(M)$ of bidegree (1,2) there exists a unique connection $\tilde{\nabla}$ with torsion tensor T and $\tilde{\nabla} g = 0$.*

Proof. If such a connection $\tilde{\nabla}$ exists, it must satisfy

$$Xg(Z,Y) = g(\tilde{\nabla}_X Z, Y) + g(Z, \tilde{\nabla}_X Y)$$
$$Yg(Z,X) = g(\tilde{\nabla}_Y Z, X) + g(Z, \tilde{\nabla}_Y X)$$
$$Zg(X,Y) = g(\tilde{\nabla}_Z X, Y) + g(X, \tilde{\nabla}_Z Y)$$

for any $X, Y, Z \in \chi(M)$. Hence

$$Xg(Z,Y) + Yg(Z,X) - Zg(X,Y) = g(Z, 2\tilde{\nabla}_X Y - [X,Y]_T) + g(Y, [X,Z]_T) + g(X, [Y,Z]_T)$$

in view of $\tilde{\nabla}_X Y - \tilde{\nabla}_Y X = [X,Y] + T(X,Y)$
where $[X,Y]_T = [X,Y] + T(X,Y)$ and consequently

(3.1) $$g(Z, \tilde{\nabla}_X Y) = \tfrac{1}{2}[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]_T) - g(Y,[X,Z]_T) + g(Z,[X,Y]_T)]$$

Then from (3.1) it can be easily seen that $\tilde{\nabla}$ is a linear connection such that

$$T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] \text{ and } \tilde{\nabla} g = 0.$$

The uniqueness of $\tilde{\nabla}$ can easily be shown from (3.1). This proves the proposition.

Proof of Theorem 1:

Let (M^m, g) be a Riemannian manifold and π be a 1-form on it.
Especially, if we take $T(X,Y) = \pi(Y)hX - \pi(X)hY$ for all vector fields $X, Y \in \chi(M)$, then the mapping $(X,Y) \to \tilde{\nabla}_X Y$ is defined by virtue of (3.1) that

(3.2) $$2g(\tilde{\nabla}_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y) + g(\pi(Y)hX - \pi(X)hY, Z) + g(\pi(Y)hZ - \pi(Z)hY, X) + g(\pi(X)hZ - \pi(Z)hX, Y)$$

for all vector field Z on M.
Then it can be easily seen that the mapping $(X,Y) \to \tilde{\nabla}_X Y$ satisfies the following relations :

(i) $\tilde{\nabla}_X(Y+Z) = \tilde{\nabla}_X Y + \tilde{\nabla}_X Z$,
(ii) $\tilde{\nabla}_{X+Y} Z = \tilde{\nabla}_X Z + \tilde{\nabla}_Y Z$,
(iii) $\tilde{\nabla}_{fX} Y = f\tilde{\nabla}_X Y$,
(iv) $\tilde{\nabla}_X fY = f\tilde{\nabla}_X Y + (Xf)Y$

for all $f \in C^\infty(M^m)$ and for all vector fields X, Y, Z on M where $C^\infty(M^m)$ denotes the set of all smooth functions over M^m.
Hence $\tilde{\nabla}$ determines a linear connection on (M^m, g).
Also from (3.2) it follows that

$$g(\tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_Y X, Z) = g([X,Y],Z) + g(\pi(Y)hX - \pi(X)hY, Z)$$

which yields

$$\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \pi(Y)hX - \pi(X)hY$$

and hence

(3.4) $$T(X,Y) = \pi(Y)hX - \pi(X)hY.$$

Also we have from (3.2) that

$$g(\tilde{\nabla}_X Y, Z) + g(\tilde{\nabla}_X Z, Y) = Xg(Y,Z)$$

which implies that $(\tilde{\nabla}_X g)(Y,Z) = 0$ i.e.,

(3.3) $$\tilde{\nabla} g = 0.$$

From (3.3) and (3.4), it follows that $\tilde{\nabla}$ determines a quarter-symmetric metric connection on (M^m, g).
The uniqueness of the quarter-symmetric metric connection $\tilde{\nabla}$ can easily be ensured by virtue of (3.2). This proves the Theorem 1.

4. PROOF OF THE RESULTS

Proof of Theorem 2:

We first suppose that in a non-Sasakian (k, μ) - contact metric manifold (M, g), the linear connection $\tilde{\nabla}$ is a quarter-symmetric metric connection. Then we write

(4.1) $\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y)$

where ∇ and $\tilde{\nabla}$ denotes respectively the Riemannian connection and the quarter-symmetric metric connection of (M^m, g). From (3.4) it follows that

$$Xg(Y, Z) - g(\tilde{\nabla}_X(Y), Z) - g(Y, \tilde{\nabla}_X(Z)) = 0$$

which yields by virtue of (4.1) that

$$(\nabla_X g)(Y, Z) - g(U(X, Y), Z) - g(Y, U(X, Z)) = 0.$$

Since ∇ is the Riemannian connection, $(\nabla_X g)(Y, Z) = 0$ and hence the above relation implies that

(4.2) $g(U(X, Y), Z) + g(Y, U(X, Z)) = 0.$

Again from (4.1) we have

$$U(X, Y) - U(Y, X) = T(X, Y).$$

Using (1.6)(a) in the above relation we get

(4.3) $U(X, Y) - U(Y, X) = \eta(Y)hX - \eta(X)hY$

Also from (4.3), it follows that

(4.4) $g(U(X, Y), Z) - g(U(Y, X), Z) = g(hX, Z)\eta(Y) - g(hY, Z)\eta(X),$

(4.5) $g(U(X, Z), Y) - g(U(Z, X), Y) = g(hX, Y)\eta(Z) - g(hZ, Y)\eta(X),$

(4.6) $g(U(Y, Z), X) - g(U(Z, Y), X) = g(hY, X)\eta(Z) - g(hZ, X)\eta(Y).$

Adding (4.4) and (4.5) and then subtracting (4.6) from the result we obtain by virtue of (4.2) and the symmetry of h that

(4.7) $U(Z, Y) = \eta(Y)hZ - g(hY, Z)\xi.$

In view of (4.7), (4.1) can be written as

(4.8) $\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)hX - g(hX, Y)\xi.$

Conversely, we define a linear connection $\tilde{\nabla}$ given by (4.8) in a non-Sasakian (k, μ) - contact metric manifold. Then

$$\begin{aligned} T(X, Y) &= \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] \\ &= \eta(Y)hX - \eta(X)hY. \end{aligned}$$

This implies that $\tilde{\nabla}$ is a quarter-symmetric metric connection.
Also we have

$$(\tilde{\nabla}_X g)(Y, Z) = Xg(Y, Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z)$$

which implies by virtue of (4.8) and the symmetry of h that

$$(\tilde{\nabla}_X g)(Y, Z) = 0$$

and hence $\tilde{\nabla}$ is a metric connection. This proves the Theorem.

Proof of Theorem 3:

If $\tilde{R}$ denotes the curvature tensor of $\tilde{\nabla}$, then

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X\tilde{\nabla}_Y Z - \tilde{\nabla}_Y\tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]}Z$$

which yields by virtue of (4.8) and (2.1)

$$\begin{aligned}(4.9)\qquad \tilde{R}(X,Y)Z = {} & R(X,Y)Z - g(Z,\phi X + \phi hX)hY + g(Z,\phi Y + \phi hY)hX \\ & + g(hY,Z)(\phi X + \phi hX - hX) - g(hX,Z)(\phi Y + \phi hY - hY) \\ & - [g((\nabla_X h)(Y),Z) - g((\nabla_Y h)(X),Z)]\xi \\ & + [(\nabla_X h)(Y) - (\nabla_Y h)(X)]\eta(Z).\end{aligned}$$

Using (2.6) in (4.9) we obtain

$$\begin{aligned}(4.10)\qquad \tilde{R}(X,Y)Z = {} & R(X,Y)Z - g(Z,\phi X)hY - g(Z,\phi hX)hY + g(Z,\phi Y)hX \\ & + g(Z,\phi hY)hX + g(hY,Z)\phi X + g(hY,Z)\phi hX \\ & - g(hY,Z)hX - g(hX,Z)\phi Y - g(hX,Z)\phi hY + g(hX,Z)hY \\ & + (1-k)[\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X + g(\phi X,Z)\eta(Y)\xi \\ & - g(\phi Y,Z)\eta(X)\xi] - g(h\phi X,Z)\eta(Y)\xi + g(h\phi Y,Z)\eta(X)\xi \\ & + \eta(Y)\eta(Z)h\phi X - \eta(X)\eta(Z)h\phi Y + \mu[g(\phi hY,Z)\eta(X)\xi \\ & - g(\phi hX,Z)\eta(Y)\xi - \eta(X)\eta(Z)\phi hY + \eta(Y)\eta(Z)\phi hX].\end{aligned}$$

Again, using (2.11) in (4.10) we get

$$\begin{aligned}(4.11)\ g(\tilde{R}(X,Y)Z,W) = {} & (1-\tfrac{\mu}{2})[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + g(Y,Z)g(hX,W) \\ & - g(X,Z)g(hY,W) - g(Y,W)g(hX,Z) + g(X,W)g(hY,Z) \\ & + \tfrac{k-\frac{\mu}{2}}{1-k}[g(hY,Z)g(hX,W) - g(hX,Z)g(hY,W) \\ & - g(\phi hY,W)g(\phi hX,Z) + g(\phi hX,W)g(\phi hY,Z)] \\ & - \tfrac{\mu}{2}[g(\phi Y,Z)g(\phi X,W) - g(\phi X,Z)g(\phi Y,W)] + \mu g(\phi X,Y)g(\phi Z,W) \\ & + \eta(X)\eta(W)[(k-1+\tfrac{\mu}{2})g(Y,Z) + (\mu-1)g(hY,Z)] \\ & - \eta(X)\eta(Z)[(k-1+\tfrac{\mu}{2})g(Y,W) + (\mu-1)g(hY,W)] \\ & + \eta(Y)\eta(Z)[(k-1+\tfrac{\mu}{2})g(X,W) + (\mu-1)g(hX,W)] \\ & - \eta(Y)\eta(W)[(k-1+\tfrac{\mu}{2})g(X,Z) + (\mu-1)g(hX,Z)] \\ & - g(Z,\phi X)g(hY,W) - g(Z,\phi hX)g(hY,W) + g(Z,\phi Y)g(hX,W) \\ & + g(Z,\phi hY)g(hX,W) + g(hY,Z)g(\phi X,W) + g(hY,Z)g(\phi hX,W) \\ & - g(hX,Z)g(\phi Y,W) - g(hX,Z)g(\phi hY,W) \\ & + (1-k)[g(\phi Y,W)\eta(X)\eta(Z) - g(\phi X,W)\eta(Y)\eta(Z) \\ & + g(\phi X,Z)\eta(Y)\eta(W) - g(\phi Y,Z)\eta(X)\eta(W)] \\ & - g(h\phi X,Z)\eta(Y)\eta(W) + g(h\phi Y,Z)\eta(X)\eta(W) \\ & + g(h\phi X,W)\eta(Y)\eta(Z) - g(h\phi Y,W)\eta(X)\eta(Z) \\ & + \mu[g(\phi hY,Z)\eta(X)\eta(W) - g(\phi hX,Z)\eta(Y)\eta(W) \\ & - g(\phi hY,W)\eta(X)\eta(Z) + g(\phi hX,W)\eta(Y)\eta(Z)].\end{aligned}$$

In view of (1.1) - (1.3), it can be easily seen from (4.10) that the curvature tensor of $\tilde{\nabla}$ satisfies the following :

$$(4.12)\qquad \tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z,$$

$$(4.13)\qquad g(\tilde{R}(X,Y)Z,W) = -g(\tilde{R}(X,Y)W,Z),$$

$$(4.14)\qquad g(\tilde{R}(X,Y)Z,W) = g(\tilde{R}(Z,W)X,Y),$$

(4.15) $$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 2[d\eta(X,Z)hY - d\eta(Y,Z)hX - d\eta(X,Y)hZ + (1-k)\{d\eta(Y,Z)\eta(X)\xi + d\eta(X,Y)\eta(Z)\xi + d\eta(X,Z)\eta(Y)\xi\}]$$

for all vector fields X, Y, Z, $W \in \chi(M)$.
Hence the Theorem is proved.

Proof of Theorem 4:

The curvature tensor of the quarter-symmetric metric connection satisfies the properties (4.12) - (4.15).
Again, if η is closed, i.e., if $d\eta(X,Y) = 0$ for all X, Y then (4.15) implies that
(4.16) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0.$
Conversely, if (4.16) holds, then (4.15) yields by setting $Z = \xi$ and then using (1.2) and (1.3) that $d\eta(X,Y) = 0$ and hence η is closed. This proves the Theorem.

Proof of Theorem 5:

From (4.11) it follows that
(4.17) $\tilde{S}(Y,Z) = S(Y,Z)+(1-k)[g(Y,Z)-g(\phi Y,Z)-\eta(Y)\eta(Z)]+(\mu-1)g(\phi hY,Z).$
Now from (4.17), it follows that $\tilde{S}$ is not symmetric and we have
$$\tilde{S}(Y,Z) - \tilde{S}(Z,Y) = 2(1-k)d\eta(Y,Z),$$
where (2.3) has been used. Hence the Theorem follows.

Proof of Theorem 6.

The relation (4.17) yields by virtue of (2.4) that
(4.18) $\tilde{S}(Y,Z) = (2n-2-n\mu)g(Y,Z) + [2n-2+\mu]g(hY,Z) + [2-2n+2nk+n\mu]\eta(Y)\eta(Z)$
$+(1-k)[g(Y,Z) - g(\phi Y,Z) - \eta(Y)\eta(Z)] + (\mu-1)g(\phi hY,Z),$
where $\tilde{S}$ denotes the Ricci tensor of $\tilde{\nabla}$.
Also from (4.17) we obtain by applying (2.5) that
(4.19) $(a)\ \tilde{r} = 2n(2n-1-n\mu)$, $(b)\ \tilde{r} = r + 2n(1-k)$
where $\tilde{r}$ is the scalar curvature of the manifold with respect to $\tilde{\nabla}$.
We now suppose that in a non-Sasakian (k,μ) - contact metric manifold admitting a quarter-symmetric metric connection $\tilde{\nabla}$, the Ricci tensor $\tilde{S}$ of $\tilde{\nabla}$ vanishes. Then we have $\tilde{r} = 0$ and hence (4.18) implies that
(4.20) $\mu = \frac{2n-1}{n}.$
Again from (4.13), it follows that
(4.21) $[2n-2-n\mu]g(Y,Z) + [2n-2+\mu]g(hY,Z) + [2-2n+2nk+n\mu]\eta(Y)\eta(Z)$
$+(1-k)[g(Y,Z)-g(\phi Y,Z)-\eta(Y)\eta(Z)]+(\mu-1)g(\phi hY,Z) = 0.$
Setting $Z = \xi$ in (4.21) we obtain by virtue of (1.3) and (1.4) that $k = 0$. Hence for $k = 0$, (4.21) takes the form
(4.22) $[2n-2-n\mu+1]g(Y,Z) + [2n-2+\mu]g(hY,Z) + [2-2n+n\mu-1]\eta(Y)\eta(Z)$
$-g(\phi Y,Z) + (\mu-1)g(\phi hY,Z) = 0.$
Substituting Y by hY in (4.22) and then using (2.2) (for $k = 0$) we get
(4.23) $[2n-2-n\mu+1]g(hY,Z) + (2n-2+\mu)[g(Y,Z) - \eta(Y)\eta(Z)]$
$-g(\phi hY,Z) + (\mu-1)g(\phi Y,Z) = 0,$

which yields by contraction

(4.24) $\mu = -(2n-2)$.

Again, replacing Z by ϕZ in (4.23) we get by virtue of (1.1) and (1.2) that

(4.25) $$[2n-2-n\mu+1]g(hY,\phi Z)+(2n-2+\mu)g(Y,\phi Z)-g(hY,Z) +(\mu-1)[g(Y,Z)-\eta(Y)\eta(Z)]=0.$$

Let $\{e_i : i = 1, 2, ..., 2n+1\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (4.25) and taking summation over i, $1 \leq i \leq 2n+1$, we obtain by virtue of (1.1) and (1.2) that

(4.26) $\mu = 1$.

Thus we obtain the three relations, namely, (4.20), (4.24) and (4.26).

Now in view of (4.24), it follows from (2.4) that the manifold is η - Einstein. Also from (4.20) and (4.26) we get $n = 1$ and hence (for $k = 0$) the manifold reduces to a 3-dimensional non-Sasakian (k, μ) - contact metric manifold. Since the relations (4.24), (4.26) taken together and (4.24), (4.20) taken together gives us inadmissible value of n, we omit these cases. Hence the Theorem is proved.

Proof of Theorem 7:

We now suppose that a non-Sasakian (k, μ) - contact metric manifold is complete and is not η - Einstein. Then by Lemma 2.1 and Theorem 6, the Theorem is proved.

Proof of Theorem 8:

First we determine the Weyl conformal curvature tensor $\tilde{C}(X,Y)Z$ of the quarter-symmetric metric connection. We have

(4.27) $$\tilde{C}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{2n-1}[g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y + \tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] + \frac{\tilde{r}}{2n(2n-1)}[g(Y,Z)X - g(X,Z)Y]$$

where $\tilde{Q}$ is the Ricci operator with respect to $\tilde{\nabla}$ i.e., $g(\tilde{Q}X,Y) = \tilde{S}(X,Y)$.

Using (4.10), (4.17) and (4.19) (b) in (4.27) we get

(4.28) $$\begin{aligned} g(\tilde{C}(X,Y)Z,W) = {} & g(C(X,Y)Z,W)-g(Z,\phi X)g(hY,W)-g(Z,\phi hX)g(hY,W)\\ &+g(Z,\phi Y)g(hX,W)+g(Z,\phi hY)g(hX,W)+g(hY,Z)g(\phi X,W)\\ &+g(hY,Z)g(\phi hX,W)-g(hY,Z)g(hX,W)-g(hX,Z)g(\phi Y,W)\\ &-g(hX,Z)g(\phi hY,W)+g(hX,Z)g(hY,W)\\ &+(1-k)[g(\phi Y,W)\eta(X)\eta(Z)-g(\phi X,W)\eta(Y)\eta(Z)\\ &+g(\phi X,Z)\eta(Y)\eta(W)-g(\phi Y,Z)\eta(X)\eta(W)]\\ &-g(h\phi X,Z)\eta(Y)\eta(W)+g(h\phi Y,Z)\eta(X)\eta(W)\\ &+g(h\phi X,W)\eta(Y)\eta(Z)-g(h\phi Y,W)\eta(X)\eta(Z)\\ &+\mu[g(\phi hY,Z)\eta(X)\eta(W)-g(\phi hX,Z)\eta(Y)\eta(W)\\ &-g(\phi hY,W)\eta(X)\eta(Z)+g(\phi hX,W)\eta(Y)\eta(Z)]\\ &+\tfrac{1-k}{2n-1}[g(Y,W)g(X,Z)-g(Y,Z)g(X,W)+g(\phi X,W)g(Y,Z)\\ &-g(\phi Y,W)g(X,Z)+g(\phi Y,Z)g(X,W)-g(\phi X,Z)g(Y,W)\\ &+g(Y,Z)\eta(X)\eta(W)-g(X,Z)\eta(Y)\eta(W)+g(X,W)\eta(Y)\eta(Z)\\ &-g(Y,W)\eta(X)\eta(Z)]+\tfrac{\mu-1}{2n-1}[g(\phi hY,W)g(X,Z)\\ &-g(\phi hX,W)g(Y,Z)-g(\phi hY,Z)g(X,W)+g(\phi hX,Z)g(Y,W)] \end{aligned}$$

where $C(X,Y)Z$ is the conformal curvature tensor of the Riemannian connection ∇.

We suppose that $\tilde{C}(X,Y)Z = C(X,Y)Z$, i.e., $g(\tilde{C}(X,Y)Z,W) = g(C(X,Y)Z,W)$. Then (4.28) yields for $Z = \xi$

(4.29) $(1-k)[g(\phi Y,W)\eta(X)-g(\phi X,W)\eta(Y)]+g(h\phi X,W)\eta(Y)-g(h\phi Y,W)\eta(X) +\mu[g(\phi hX,W)\eta(Y) - g(\phi hY,W)\eta(X)] + \frac{1-k}{2n-1}[g(\phi X,W)\eta(Y) -g(\phi Y,W)\eta(X)]+\frac{\mu-1}{2n-1}[g(\phi hY,W)\eta(X)-g(\phi hX,W)\eta(Y)] = 0.$

Again replacing Y by ξ in (4.29) we obtain

(4.30) $(n-1)[(k-1)d\eta(X,W) + (\mu-1)d\eta(hX,W)] = 0,$

which yields either $n = 1$ or

(4.31) $(k-1)d\eta(X,W) + (\mu-1)d\eta(hX,W) = 0.$

Replacing X by hX in (4.31) and then using (2.2) we get

(4.32) $d\eta(hX,W) = (\mu-1)d\eta(X,W).$

By virtue of (4.32), (4.31) yields $d\eta(X,W) = 0$ for $k + \mu^2 - 2\mu \neq 0$.

Hence the Theorem is proved.

Proof of Theorem 9:

From Theorem 8 and Lemma 2.1 the Theorem 9 immediately follows.

Proof of Theorem 10:

If $d\eta(X,Y) = 0$ for all X, Y then (4.28) implies that

(4.33) $\tilde{C}(X,Y)\xi = C(X,Y)\xi$ for all X, Y.

Hence the Theorem is proved.

Proof of Theorem 11:

The generalized projective curvature tensor [6] with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ is defined by

(4.34) $\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{2n+2}[\tilde{S}(X,Y)Z - \tilde{S}(Y,X)Z] + \frac{1}{(2n+1)^2-1}[\{(2n+1)\tilde{S}(X,Z) +\tilde{S}(Z,X)\}Y - \{(2n+1)\tilde{S}(Y,Z) + \tilde{S}(Z,Y)\}X].$

Using (4.17) in (4.34) we obtain

(4.35) $\tilde{P}(X,Y)Z = P(X,Y)Z + \frac{\mu-1}{2n(2n+2)}[g(\phi hX,Z)Y - g(\phi hY,Z)X] +\frac{1-k}{2n(2n+2)}[g(X,Z)Y - g(Y,Z)X - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X] -\frac{1-k}{2n+2}[g(\phi X,Z)Y + g(\phi Y,Z)X] + \frac{1-k}{n+1}g(X,\phi Y)Z -g(Z,\phi X)hY - g(Z,\phi hX)hY + g(Z,\phi Y)hX + g(Z,\phi hY)hX + g(hY,Z)\phi X +g(hY,Z)\phi hX - g(hY,Z)hX - g(hX,Z)\phi Y - g(hX,Z)\phi hY + g(hX,Z)hY +(1-k)[\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X + g(\phi X,Z)\eta(Y)\xi - g(\phi Y,Z)\eta(X)\xi] -g(h\phi X,Z)\eta(Y)\xi + g(h\phi Y,Z)\eta(X)\xi - \eta(X)\eta(Z)h\phi Y + \eta(Y)\eta(Z)h\phi X +\mu[g(\phi hY,Z)\eta(X)\xi - g(\phi hX,Z)\eta(Y)\xi - \eta(X)\eta(Z)\phi hY + \eta(Y)\eta(Z)\phi hX],$

where $P(X,Y)Z$ is the projective curvature tensor of the manifold.

We now suppose that $\tilde{P}(X,Y)Z = P(X,Y)Z$. Then putting $Z = \xi$ in (4.35) we get

(4.36) $\frac{1-k}{n+1}g(X,\phi Y)\xi + (1-k)[\eta(X)\phi Y - \eta(Y)\phi X] = 0.$

Taking the inner product on both sides of (4.36) by ξ we obtain

$d\eta(X,Y) = 0$ as $k \neq 1$.

Conversely, if $d\eta(X,Y) = 0$ for all X, Y then (4.35) implies that $\tilde{P}(X,Y)Z = P(X,Y)Z$. Thus the Theorem follows.

Proof of Theorem 12:

If $\tilde{P}(X,Y)Z = P(X,Y)Z$ then we have $d\eta = 0$ i.e., η is closed. In a contact metric manifold, (2.1) implies that $\delta\eta = 0$ i.e., η is co-closed. This proves the Theorem.

ÖZET: Bu çalışmanın amacı, bir Riemann manifoldu üzerinde quarter-simetrik metrik konneksiyonunun varlığını ispat etmek ve Sasakian-olmayan bir (k,μ)-kontakt metrik manifold üzerinde böyle bir konneksiyonun bazı özeliklerini incelemektir.

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