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NOTES ABOUT EXTREMAL PROBLEMS IN THE SPACE OF ENTIRE FUNCTIONS OF FINITE DEGREE

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ABSTRACT. In this study we considered extremal problems about linear operators defined on the classes of entire functions of finite degree. Our results are sufficiently general and strongly characterized. In many cases the solutions of unsolved partical problems are included in this study. This means that those solutions can be obtained as the special cases of our results.

1. INTRODUCTION

The functional of the type $L = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ is often investigated subject to economy, production, management etc. [1]. This type of functionals are as well as subject of theory of functions.

In this paper we give a very significant theorem, which has not been published before (see, Theorem 5).

In the theory of complex variable functions some functionals such as

$$L = f(z_0), \qquad L = f^{'}(z_0), \qquad L = \sum_{k=0}^{n} a_k\lambda_k,$$

where $a_k = \dfrac{1}{k!}f^{(k)}(z_0)$, λ_k-are given numbers (real or complex), etc. were investigated by numerous authors (see, [2 − 7]).

Problems about to find the norm of these functionals have been investigated in the some classes of analytic functions defined in the bounded, and one and many connected regions, and many important results established by S. Ya. Khavinson, A. Shapiro, W. W. Rogosinsky, G. Ts. Tumarkin etc. [3 − 5].

At first, in 1949 Prof Khavinson S. Ya. showed that these problems can be solved by using general theory of functional analysis [3].

Beginning from 1956, İ. İ. İbragimov investigated in some works in several classes of entire functions and obtained a series of important results.

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But, some problems are still unsolved in these classes of entire functions.

Therefore, we began to investigate in the various classes of the entire functions of finite degree. We could established some essential results for these problems [7 – 10].

In this study, we give some new theorems about these kind of problems, therewith in our theorems we cancel some conditions and we substitute some other conditions in the previous theorems. Besides of this we give some results, which has not been published before (see, Theorem 5).

2. Expression of the problems

2.1. We denote the set of entire functions of finite degree $\leq \sigma$ by T_σ, the space of the functions bounded on the real axis $R = (-\infty, \infty)$ by B_σ ($B_\sigma \subset T_\sigma$) ([6, 7, 10]) :

$$B_\sigma := \{f \in T_\sigma \ : \ \|f\|_C = \sup_{x \in R} |f(x)| \leq M(f) < +\infty\}.$$

Besides we define spaces $W_{\sigma,p}$ ($p \geq 1$) :

$$W_{\sigma,p} := \{f \in T_\sigma \ : \ \|f\|_p = \{\int_R |f(t)|^p \, dt\}^{1/p} \leq M_p(f) < +\infty\},$$

i. e. $W_{\sigma,p} := T_\sigma \cap L_p$ ($p \geq 1$).

We also denote the class of functions $K(z)$, which are analytic in exterior of the $\overline{\Gamma}_\lambda \equiv \{z \in \mathbb{C} : |z| \leq \lambda, \lambda > 0$ arbitrary constants number$\}$, but may having finite number of polar points in the interior of Γ_λ (obviously, Γ_λ can be replaced by arbitrary closed Jordan curve C) by M.

Now, let us consider the operator

$$u(f; z) := \frac{1}{2\pi i} \int_{\Gamma_\lambda} K(\xi) f(\xi + z) d\xi, \tag{2.1}$$

where $z = x + iy_0$ ($x \in R$-arbitrary, $y_0 \in R$-fixed numbers), $f \in W_{\sigma,p}, K(z) \in M$;

$$W_{\sigma,p}[1] := \{f \in W_{\sigma,p} \ : \ \|f\|_p \leq 1\}$$

is unit sphere in $W_{\sigma,p}$.

Finally, we can define our general problem in the following form:

Problem. Find the norm of $u(f; z)$:

$$\|u\|_p = \sup_{f \in W_{\sigma,p}[1]} \{\sup_{x \in R} |u(f; x + iy_0)|\} \tag{2.2}$$

for a given arbitrary y_0 ($-\infty < y_0 < \infty$) and a given function $K(z) \in M$.

The space $W_{\sigma,p}$ has the following properties :

1) If $f(z) \in W_{\sigma,p}$, then $\forall t : f(z + t) \in W_{\sigma,p}$;

2) $\|f(x + t)\|_p = \|f(x)\|_p = \|f\|_p$.

Now we will consider the functional

$$L(f) = u(f;0) = \frac{1}{2\pi i} \int\limits_{\Gamma_\lambda} K(\xi) f(\xi) d\xi \tag{2.3}$$

together (association) with operator $u(f;z)$.

Obviously we have:

$$\|u\|_p = \sup_{f \in W_{\sigma,p}[1]} \|u(f;x)\|_C = \sup_{f \in W_{\sigma,p}[1]} |L(f)| = \|L\|_p . \tag{2.4}$$

2.2. Definition 1. The function $f_0 \in W_{\sigma,p}[1]$, satisfying the condition

$$\|u\|_p = \|L\|_p = |L(f_0)|$$

is called the extremal function for the problem (2.2), i. e. for the functional $L(f)$ (or operator $u(f;x)$).

Together with it is necessary to answer the following questions related to problem (2.2) :

1) Does any extremal function exists for $p \geq 1$ and $K(z) \in M$?
2) If the extremal functions exist, how many?
3) Is it possible to find the extremal function f_0?
4) Is there any criterion for the fact that f_0 is extremal?
5) Is it possible to find the norm $\|u\|_p$ exactly?
6) If $\|u\|_p$ could not be found exactly are there any estimates for it?
7) If there are estimates, are they optimal ?

We would like to emphasize that this problem of quality investigation is our principal aim.

In this work, we try to answer these questions. We have removed some conditions of our previous theorems.

3. Results and their proofs

At first, we the give following very important theorem of M.G. Krein's (see, [9,10]).

Theorem 1. (M. G.Krein) In order that a normed linear space E be a strictly normalized, a necessary and sufficient condition is that every linear bounded functional $L \in E^*$ attains its norm at most once on $S(E)$, where $S(E)$ denotes the unit sphere in E.

Theorem 2. For each $p \in (1, \infty)$ the extremal functions $f_0(z)$ exists for every linear bounded functional $L(f)$, which is defined on $W_{\sigma,p}$. For every $L(f)$ extremal function $f_0(z)$ is unique in the sense that, if $f_0(z)$ is extremal for $L(f)$ element, then $\forall \alpha \in [0, 2\pi)$ function $f_1(z) = e^{i\alpha} f_0(z)$ as well as extremal function for $L(f)$.

Proof. $W_{\sigma,p}[1]$ is a closed subspace of L_p. The space L_p $(1 < p < \infty)$ is reflexive : $(L_p^*)^* = L_q^* = L_p$. Since L_p is a complete space, then there is an extremal function

$f_0(z) \in W_{\sigma,p}[1]$. Besides, $L_p(1 < p < \infty)$ is strongly normed, as well. Because, we can take into account the Theorem 2, therefore, extremal function $f_0(z) \in W_{\sigma,p}[1]$ $(1 < p < \infty)$ is unique.

Thus our Theorem 2 is proved.

Theorem 3. For all $p \in [1,2]$, $L(f)$, defined by Equality (2.3) is a linear and bounded functional on the space $W_{\sigma,p}$.

Besides, the following are true:

1) For all $p \in [1,2]$ and every $f \in W_{\sigma,p}$ the following inequalities hold:

$$|L(f)| \leq \|u(f;\cdot)\|_C \leq \frac{N(p,q)}{\sqrt{2\pi}} \|\Phi\|_p^0 \|f\|_p, \tag{3.1}$$

where $p + q = pq$, $N(p,q) = (\frac{p}{2\pi})^{1/2p}(\frac{2\pi}{q})^{1/2q}$, $\|\Phi\|_p^0 = (\int\limits_{-\sigma}^{\sigma} |\Phi(t)|^p\, dt)^{1/p}$.

2)

$$\|L\|_p = \|u\|_p \leq \frac{N(p,q)}{\sqrt{2\pi}} \|\Phi\|_p^0. \tag{3.2}$$

3) In the case $1 \leq p < 2$ Inequality (3.2) has the sign "$<$", and in the case $p = 2$ equalities hold

$$\|L\|_2 = \|u\|_2 = \frac{1}{\sqrt{2\pi}} \|\Phi\|_2^0, \tag{3.3}$$

where

$$\Phi(t) = \frac{1}{2\pi i} \int\limits_{\Gamma_\lambda} e^{i\xi t} K(\xi) d\xi. \tag{3.4}$$

4) In order for the function $f_0(z) \in W_{\sigma,2}[1]$ to be an extremal for $L(f)$, it is sufficient and necessary condition is that function $f_0(z)$ admit a representation of the form :

$$f_0(z) = \frac{e^{i\gamma}}{\sqrt{2\pi}\,\|\Phi\|_2^0} \int\limits_{-\sigma}^{\sigma} e^{izt}\overline{\Phi}(t)dt, \tag{3.5}$$

where $0 \leq \gamma < 2\pi$, $\overline{\Phi}(t)$- is complex conjugated of $\Phi(t)$.

Proof. İ. İ. İbragimov proved that, $\forall p \in [1,2)$ $W_{\sigma,p} \subset W_{\sigma,2}$. In the space $W_{\sigma,2}$ there exist Paley-N. Wiener's Theorem (see, [10]) :

The necessary and sufficient condition for $f \in W_{\sigma,2}$ is that f admit a representation of the form :

$$f(z) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\sigma}^{\sigma} e^{izt}\varphi(t)dt, \tag{3.6}$$

where $\varphi(t)$ is Fourier Transform of the $f(z)$, besides $\varphi(t) = 0$ for $|t| > \sigma$ and $\|f\|_2 = \|\varphi\|_2^0$.

That is why we have :

$$u(f;x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{ixt} \Phi(t) \varphi(t) dt. \tag{3.7}$$

Using, here, Titchmarsh-Babenko inequality [10] we have:

$$|L(f)| \le \sup_{x \in R} |u(f;x)| \le \frac{N(p,q)}{\sqrt{2\pi}} \|f\|_p \|\Phi\|_p^0, \tag{3.8}$$

where $N(p,q)$ is Titchmarsh-Babenko constant.

By investigating how the inequality (3.8) was obtained, we must complete the proof of our Theorem 3.

Remark 1. In those theorems by choosing $K(z) \in M$ all of the previous results obtained by other investigators can be obtained.

This is seen from the following cases clearly.

1) If $K(z) = \frac{1}{z} \in M$, then $u(f;x) = f(x)$ and $L(f) = f(0)$ (in particular $L(f) = f(x_0)$);

2) For $K(z) = \frac{n!}{z^{n+1}}$, we have $u(f;x) = f^{(n)}(x)$ and $L(f) = f^{(n)}(0)$;

3) If $K(z) = A\frac{m!}{z^{m+1}} + B\frac{n!}{z^{n+1}}$, then $u(f;x) = Af^{(m)}(x) + Bf^{(n)}(x)$ and in particular, $L(f) = Af^{(m)}(0) + Bf^{(n)}(0)$;

4) If $K(z) = \sum_{m=0}^{n} \frac{m!}{z^{m+1}} \lambda_m$, then $u(f;x) = \sum_{m=0}^{n} \lambda_m c_m$, where $c_m = f^{(m)}(x)$, and in particular, $L(f) = u(f;0) = \sum_{m=0}^{n} \lambda_m f^{(m)}(0)$, the numbers λ_m $(m = 0, 1, ..., n)$ given before.

Now, let us demonstrate these as an example in one of the special cases.

Corollary 1. For the functional $L(f) = f^{(n)}(0)$ we have :

$$\|L\|_2 = \sup_{f \in W_{\sigma,2}[1]} \left| f^{(n)}(0) \right| = \sigma^n \sqrt{\frac{\sigma}{\pi(2n+1)}}.$$

Extremal functions $f_0(z)$ is determined completely by the formula :

$$f_0(z) = \frac{1}{2} e^{i\gamma} \sigma^{-n} (\frac{2n+1}{\pi\sigma})^{1/2} \int_{-\sigma}^{\sigma} e^{izt} \frac{|it|^{2n}}{(it)^n} dt \ (0 \le \gamma < 2\pi).$$

In special, case $n = 0$ we have : $L(f) = f(0)$ and

$$f_0(z) = \frac{e^{i\gamma}}{\sqrt{\pi\sigma}} \frac{\sin \sigma z}{z},$$

$$\|L\|_2 = \sup_{f\in W_{\sigma,2}[1]} |f(0)| = \sqrt{\frac{\sigma}{\pi}}.$$

As well, in the case $n = 1$, $L(f) = f'(0)$ and

$$\begin{aligned}
f_0(z) &= \frac{e^{i\gamma}}{2\sigma}\left(\frac{3}{\pi\sigma}\right)^{1/2}\int_{-\sigma}^{\sigma} e^{izt}\frac{t^2}{it}dt \\
&= \frac{e^{i\gamma}}{2\sigma}\left(\frac{3}{\pi\sigma}\right)^{1/2}\int_{-\sigma}^{\sigma} e^{izt}(-it)dt \\
&= -\frac{e^{i\gamma}}{\sigma}\left(\frac{3}{2\sigma}\right)^{1/2}\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma} e^{izt}(it)dt \\
&= -\frac{e^{i\gamma}}{\sigma}\left(\frac{3}{2\sigma}\right)^{1/2}\sqrt{\frac{2}{\pi}}\left(\frac{\sin\sigma z}{z}\right)'; \\
f_0(z) &= -\frac{e^{i\gamma}}{\sigma}\sqrt{\frac{3}{\sigma\pi}}\left(\frac{\sin\sigma z}{z}\right)', \\
\|L\|_2 &= \sup_{f\in W_{\sigma,2}[1]}\left|f'(0)\right| = \sigma\sqrt{\frac{\sigma}{3\pi}} = \frac{\sigma^{3/2}}{\sqrt{3\pi}}.
\end{aligned}$$

Corollary 2. Let $a^2 + b^2 > 0$ $(a, b \in R)$, $\alpha, \beta \in N$. Then for

$$\|L\|_2 = \sup_{f\in W_{\sigma,2}[1]}\left|af^{(\alpha)}(0) + bf^{(\beta)}(0)\right|$$

extremal function may be given by the formula

$$f_0(z) = \frac{e^{i\gamma}}{\sqrt{2\pi}\,\|\Phi\|_2^0}\int_{-\sigma}^{\sigma} e^{izt}\frac{\left|a(it)^\alpha + b(it)^\beta\right|^2}{a(it)^\alpha + b(it)^\beta}dt,$$

and

$$\|L\|_2 = \frac{1}{\sqrt{2\pi}}\|\Phi\|_2^0 = \frac{1}{\sqrt{2\pi}}\left\{\int_{-\sigma}^{\sigma}\left|a(it)^\alpha + b(it)^\beta\right|^2 dt\right\}^{1/2}.$$

And, in the special case $\alpha = 0, \beta = 1$, $L(f) = af(0) + b\sigma f'(0)$, $\Phi(t) = a + b(it)\sigma$,

$$\|\Phi\|_2^0 = \sqrt{\frac{2\sigma}{3}(3a^2 + b^2\sigma^2)},$$

$$\begin{aligned} f_0(z) &= \frac{e^{i\gamma}}{2}\left[\frac{3}{\pi\sigma(3a^2+b^2\sigma^2)}\right]^{1/2}\int_{-\sigma}^{\sigma} e^{izt}(a - ib\sigma t)dt \\ &= e^{i\gamma}\sqrt{\frac{3}{\pi\sigma(3a^2+b^2\sigma^2)}}\{a\frac{\sin\sigma z}{z} - b\sigma(\frac{\sin\sigma z}{z})'\}. \end{aligned}$$

To obtain these results, it suffices to take

$$K(z) = a\frac{\alpha!}{z^{\alpha+1}} + b\frac{\beta!}{z^{\beta+1}} \in M.$$

Remark 2. In this results we can think the order of derivatives n, α, β as well as arbitrary nonnegative real numbers.

Theorem 4. If function $f(z)$ is changing in $W_{\sigma,2}$ [1], all the values of the functional $L(f)$ completes the circle with radius $R = \frac{1}{\sqrt{2\pi}} \|\Phi\|_2^0$; Furthermore, to the boundary points of this circle corresponds only to the function $f_0(z)$ defined by the formula (3.5).

Indeed, if $f_0(z) \in W_{\sigma,2}$ [1] is an extremal function for $L(f)$, then $\forall\gamma \in [0, 2\pi)$ and A, $|A| \le R \le 1$ the function $f_1(z) = e^{i\gamma}Af_0(z)$ is also extremal for $L(f)$, obviously.

Theorem 5. In the space $W_{\sigma,2}$ for all (2.2) type extremal problems an extremal function depends on the given operator is a function : $f_0(z) = A\frac{\sin\sigma z}{z}$ (where A-arbitrary constant numbers, for normalized) or is the different derivatives of this function.

To show this we must investigate the function $\Phi(t)$ and the function $f_0(z)$ and the expression which depends on those functions.

ÖZET Çalışmada sonlu dereceli tam fonksiyonların bir sınıfında tanımlı lineer operatörler için ekstremal problem araştırılmaktadır. Sonuçlarımız son derece genel ve kesin karakterdedir. Bir çok hallerde önceden çözümü bulunmamış olan problemlerin çözümü bu neticelerimizin kapsamında olup, özel durumlar olarak çıkarılabilirler. Çalışmamızda Fonksiyonel Analiz ilkeleri sık sık kullanılmaktadır.

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