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On Characterizations of W-Directional Curves of Null Curves in Minkowski 4-Space

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Abstract: In the present paper, we investigate the casual characterizations of W-directional curves of null curves in Minkowski 4-space. In section two, the basic concepts of curves with their Frenet equations in Minkowski 4-space are provided. In section three, the principal normal directional and donor curves of null curves in Minkowski 4-space are defined and their casual characterizations are also derived. In section four, we define the B_1 directional and donor curves of null curves and show their properties as well. In the last section, the B_2 directional and donor curves of null curves are also defined and their causal characterizations are provided.

Key words: Directional curve, Donor curve, Null curve, Minkowski space.

Minkowski 4-Uzayında Null Eğrilerin W-Yönlü Eğrilerinin Karakterizasyonları Üzerine

Özet: Bu makalede, Minkowski 4-uzayında null eğrilerin W-yönlü eğrilerinin basit karakterizasyonlarını inceliyoruz. İkinci bölümde, Minkowski 4-uzayında Frenet denklemleri ile eğrilerin temel kavramları verilmiştir. Üçüncü bölümde, Minkowski 4-uzayındaki null eğrilerinin temel normal yönü ve donör eğrileri tanımlanmış ve bunların basit karakterizasyonları da türetilmiştir. Dördüncü bölümde, null eğrilerin B_1 yönü ve donör eğrilerini tanımlıyoruz ve ayrıca temel özelliklerini gösteriyoruz. Son bölümde, null eğrilerin B_2 yönü ve donör eğrileri de tanımlanmış ve nedensel karakterizasyonları verilmiştir.

Anahtar kelimeler: Yön eğrisi, Donör eğrisi, Null eğrisi, Minkowski uzayı.

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1. Introduction

The theory of differential geometry in Minkowski space has been studied by numerous mathematicians and physicists since it has significant roles in the growth of modern physics, especially in the theory of gravitation and relativity. One of the fascinating topics in the theory of differential geometry is the differential geometry of curves in Minkowski space. Curves in Minkowski space can be spacelike, timelike, or null locally, depending on the causal properties of the tangent vector fields along the curve [8,9]. In general, the properties of classical differential geometry can be extended to those of properties of spacelike and timelike curves in Minkowski space. However, since the arc length parameter of null curves vanishes everywhere, we need a particular approach to find their properties.

The theory of null curves itself is very common in physics. Nersessian and Ramos demonstrated in 1998 that a geometric particle model associated with a null curve occurs in Minkowski space [10]. In addition, the classical relativistic string is a surface or world-sheet in Minkowski space that satisfies the Lorentzian analogue of the minimal surface equations. If string equations are reduced to the wave equation and a few extra basic equations, it turns out the string that is equal to pairs of null curves or a single null curve in the case of opening [11,12]. General studies of differential geometry of null curves in Minkowski space have been given by Duggal, Bejancu, and Jin in [5,13] and for additional sources, we refer to [1,2,3,4,12].

The notion of principal (binormal)-directional curve and princial-(binormal) donor curve of Frenet curves in E^3 was first introduced by Choi and Kim [14]. They present characterizations for the general and slant helices via their associated curves and provide the way to construct them from a planar curve. This theory was later extended to the directional curve and donor curve of Frenet curves in Minkowski 3-space such as those of a null curve in [16] an non null Frenet curves in [21,23]. Furthermore, associated curves of Frenet curves in three dimentional compact Lie group has been studied by Kiziltug and Önder and provided in [19]. Some other studies investigated the properties of directional curve and donor curve of Frenet curves can be seen in [15,17,18,20,22].

Motivated from the works above, this study aims to investigate the properties W-directional curves of null curves in Minkowski 4-space. We organized our paper as follow: In Section 2, we provide the basic theory of curves in Minkowski 4-space. In Section 3, we define the principal normal directional and donor curves of null curves in Minkowski 4 space and provide their causal characteristics. In the following two sections, the definitions and the casual characteristics of the B_1 and B_2 directional and donor curves of null curves in Minkowski 4-space are provided, respectively.

2. Preliminaries

Minkowski space $\mathbb{E}_1^4$ is the real vector space $\mathbb{R}^4$ equipped with the standard indefinite flat Lorenzian metric defined by

$$g(\,,) = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a coordinate system in $\mathbb{E}_1^4$.

Let $p = (p_1, p_2, p_3, p_4)$, $q = (q_1, q_2, q_3, q_4)$, and $r = (r_1, r_2, r_3, r_4)$ be vectors in $\mathbb{E}_1^4$. The vector product in Minkowski spacetime $\mathbb{E}_1^4$ is defined with the determinant

$$p \wedge q \wedge r = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ p_1 & p_2 & p_3 & p_4 \\ q_1 & q_2 & q_3 & q_4 \\ r_1 & r_2 & r_3 & r_4 \end{vmatrix}$$

where e_1, e_2, e_3 and e_4 are mutually orthonormal vectors satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4, \qquad e_2 \wedge e_3 \wedge e_4 = e_1, \qquad e_3 \wedge e_4 \wedge e_1 = e_2,$$
$$e_4 \wedge e_1 \wedge e_2 = -e_3.$$

A vector $v \in \mathbb{E}_1^4$ is timelike if $g(v,v) < 0$, spacelike if $g(v,v) > 0$ or $v = 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. The norm of $v \in \mathbb{E}_1^4$ is given by$\|v\| = \sqrt{|g(v,v)|}$. Locally, curves in Minkowski 4-space is timelike (resp. spacelike or null) if its tangent vector is timelike (resp. spacelike or null) along the curves.

Let $\alpha: I \to \mathbb{E}_1^4$ be curve in Minkowski 4-space. Suppose α be a spacelike curve with non null frame vectors parametrized by arc length s in spacetime$\mathbb{E}_1^4$. Then, we have

$$T(s) = \alpha'(s), \quad N = \frac{T'(s)}{\kappa(s)}, \qquad B_1 = \frac{N'(s) + \kappa(s)T(s)}{\tau(s)}, \qquad (1)$$
$$B_2(s) = \mu T(s) \wedge N(s) \wedge B_1(s)$$

and

$$\kappa(s) = \|T'(s)\| > 0, \qquad \tau(s) = \|N'(s) + \kappa(s)T(s)\| \qquad (2)$$

where $\mu = \pm 1$ which makes $\det\big(T(s), N(s), B_1(s), B_2(s)\big) = 1$.

Unit speed spacelike curve α with non null frame vectors satisfies the following equation [1]

$$\begin{bmatrix} T' \\ N' \\ B_1{}' \\ B_2{}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \mu_1\kappa & 0 & \mu_2\tau & 0 \\ 0 & \mu_3\tau & 0 & \mu_4\sigma \\ 0 & 0 & \mu_5\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}. \qquad (3)$$

Let $\beta: I \to \mathbb{E}_1^4$ be a unit speed pseudo null curve that is a unit speed spacelike curve where its principal normal and its second binormal vector fields are null vectors. The Frenet equation of $\beta(s)$ is given by

$$\begin{bmatrix} T' \\ N' \\ B_1{}' \\ B_2{}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ 0 & 0 & \tau & 0 \\ 0 & \sigma & 0 & -\tau \\ \kappa & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \qquad (4)$$

where $\{T, N, B_1, B_2\}$ is orthonormal basis vectors satisfyig

$$g(T,T) = g(B_1,B_1) = 1,\ g(N,N) = g(B_2,B_2) = 0,\ g(N,B_2) = 1,$$
$$g(T,N) = g(T,B_1) = g(T,B_2) = g(N,B_1) = g(B_1,B_2) = 0 .$$

Here, κ, τ and σ are curvature, torsion and bitorsion of curves, respectively. The value of curvature κ is 0 when curve is straight line and 1 in all other case for pseudo null curve [7].

Let $\gamma: I \to \mathbb{E}_1^4$ be an arbitrary null curve in Minkowski spacetime $\mathbb{E}_1^4$. Then, the Frenet frame of $\gamma(s)$ is given by

$$T(s) = \frac{\gamma'(s)}{\sqrt{g(\gamma''(s),\gamma''(s))}} = \frac{1}{\varphi(s)}\gamma'(s), \tag{5}$$

$$N(S) = T'(s) = \left(\frac{1}{\varphi(s)}\right)'\gamma'(s) + \left(\frac{1}{\varphi(s)}\right)\gamma''(s), \tag{6}$$

$$B_1(s) = \frac{1}{g(T,\gamma'''(s))}\left\{\gamma'''(s) - \frac{g(\gamma'''(s),\gamma'''(s))}{2g(T,\gamma'''(s))}T\right\}, \tag{7}$$

$$= -\frac{1}{\varphi(s)}\gamma'''(s) - \frac{g(\gamma'''(s),\gamma'''(s))}{2\varphi(s)^3}\gamma'(s),$$

$$B_2(s) = T(s) \wedge B_1(s) \wedge N(s) = \frac{1}{\varphi(s)^3}\left(\gamma'(s) \wedge \gamma''(s) \wedge \gamma'''(s)\right), \tag{8}$$

where $\varphi(s) = \sqrt{g(\gamma''(s),\gamma''(s))}$.

For arbitrary null curve $\gamma(s)$ such that $\varphi(s) = \sqrt{g(\gamma''(s),\gamma''(s))} \neq 0$, we have an orthonormal frame $\{T(s), N(s), B_1(s), B_2(s)\}$ of $\gamma(s)$ such that it satisfies

$$g(T,T) = g(B_1,B_1) = 0,\ g(N,N) = g(B_2,B_2) = g(T,B_1) = 1,$$
$$g(T,N) = g(T,B_2) = g(N,B_1) = g(N,B_2) = g(B_1,B_2) = 0.$$

Frenet equations associated with the Frenet frame $\{T(s), N(s), B_1(s), B_2(s)\}$ are given by

$$\begin{aligned} T'(s) &= N(s), \\ N'(s) &= -k_1(s)T(s) - B_1(s), \\ B_1'(s) &= k_1(s)N(s) + k_2(s)B_2(s), \\ B_2'(s) &= -k_2(s)T(s), \end{aligned} \tag{9}$$

where

$$k_1(s) = \frac{1}{2\varphi^2(s)}\left(g(\gamma''',\gamma''') + 2\varphi(s)\varphi''(s) - 4\left(\varphi'(s)\right)^2\right)$$
$$k_2(s) = -\frac{1}{\varphi^4(s)}\det\left(\gamma'(s),\gamma''(s),\gamma'''(s),\gamma^{(4)}(s)\right). \tag{10}$$

Curve $C = \gamma(I)$, which satisfies the assumption above, is called a Cartan curve with a Cartan frame $\{T(s), N(s), B_1(s), B_2(s)\}$ and Cartan curvatures$\{k_1(s), k_2(s)\}$. Furthermore, it is easy to see that

$$N \wedge T \wedge B_1 = B_2,\ N \wedge B_2 \wedge T = T,\ N \wedge B_1 \wedge B_2 = B_1,\ T \wedge B_2 \wedge B_1 = N. \tag{11}$$

The pseudo-arc length parameter for arbitrary null curve is defined by $u(s) = \int_{s_0}^{s} \sqrt[4]{g\left(\gamma''(t), \gamma''(t)\right)}\, dt$, so that we have $u(s) = 1$ [2].

With analogues to the definition of the W-directional and donor curves in [6] we have the following definition.

Definition 1. Let $\gamma: I \to \mathbb{E}_1^4$ be a curve in Minkowski 4-space parametrized by arc length or pseudo arc length with Frenet frame $\{T, N, B_1, B_2\}$ and W be a unit vector field along γ. The curve $\tilde{\gamma}: I \to \mathbb{E}_1^4$ is said to be W-directional curve of γ if the tangent vector field $\tilde{T}$ of the curve $\tilde{\gamma}$ equals to the vector W, i.e.,$\tilde{T} = W$. On the other hand, the curve γ is said to be W donor curve of $\tilde{T}$.

If $W = T$, then the tangent directional curve $\tilde{\gamma}$ of the null curve γ is trivially γ. By this notion, in the next part, we will not discuss further about the tangent directional-donor curves of null curves in Minkowski 4-space.

3. Principal Normal Directional Curves of Null Curves

In this section, we define the principal normal directional curves of null curves and investigate their casual characteristics.

Definition 2. *Let γ be a null curve in $\mathbb{E}_1^4$. The curve $\tilde{\gamma}$ is said to be the principal normal directional curve of γ if the tangent $\tilde{T}$ of $\tilde{\gamma}$ equals to the principal normal vector N of γ. Conversely, γ is said to be the principal normal donor curve of $\tilde{\gamma}$.*

By the definition 2, we have $\tilde{\gamma} = \int N(s)ds$. Note also that $\tilde{\gamma}$ is a spacelike curve since its tangent vector is spacelike along the curve.

Theorem 3. *Let $\tilde{\gamma}$ be the principal normal directional curve of null curve γ in $\mathbb{E}_1^4$. Suppose $\{T, N, B_1, B_2\}$ and $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2\}$ are the Frenet frames of γ and $\tilde{\gamma}$, respectively. Then,*

$$\tilde{T} = N,\ \tilde{N} = -\frac{k_1T + B_1}{\sqrt{2k_1}},\ \tilde{B}_1 = \frac{-k_1k_1'T + k_1'B_1 - 2k_1k_2B_2}{\sqrt{\left|-2k_1(k_1')^2 + 4k_1^2k_2^2\right|}},\ \tilde{B}_2 = \frac{k_1k_2T - k_2B_1 - k_1'B_2}{\sqrt{\left|-2k_1(k_1')^2 + 4k_1^2k_2^2\right|}}. \tag{12}$$

Proof. Let $\tilde{\gamma}$ be the principal normal directional curve of null curve γ, then $\tilde{T} = N$. Therefore,

$$\tilde{T}' = N' = -k_1 T - B_1. \tag{13}$$

Taking the norm of equation (12) yields

$$\kappa^2 = 2k_1 \Longrightarrow \left\|\tilde{T}'\right\| = \sqrt{2k_1}. \tag{14}$$

As a consequence,

$$\widetilde{N} = \frac{\tilde{T}'}{\|\tilde{T}'\|} = \frac{-k_1 T - B_1}{\sqrt{2k_1}}. \tag{15}$$

Differentiating $\widetilde{N}$ gives us

$$\begin{aligned} \widetilde{N}' &= \left(\frac{-k_1 T - B_1}{\sqrt{2k_1}}\right)' \\ &= \frac{\left(-k_1' T - k_1 N - k_1 N - k_2 B_2\right)\sqrt{2k_1}}{2k_1} - \frac{k_1'(-k_1 T - B_1)}{2k_1\sqrt{2k_1}} \\ &= \frac{-2k_1 k_1' T - 4k_1^2 N - 2k_1 k_2 B_2 + k_1 k_1' T + k_1' B_1}{2k_1\sqrt{2k_1}} \\ &= \frac{-k_1 k_1' T - 4k_1^2 N + k_1' B_1 - 2k_1 k_2 B_2}{2k_1\sqrt{2k_1}}. \end{aligned}$$

Furthermore,

$$\widetilde{N}' + \kappa\tilde{T} = \frac{-k_1 k_1' T - 4k_1^2 N + k_1' B_1 - 2k_1 k_2 B_2 + 4k_1^2 N}{2k_1\sqrt{2k_1}} = \frac{-k_1 k_1' T + k_1' B_1 - 2k_1 k_2 B_2}{2k_1\sqrt{2k_1}}, \tag{16}$$

and

$$\left\|\widetilde{N}' + \kappa\tilde{T}\right\| = \frac{-\left(k_1'\right)^2 + 2k_1 k_2^2}{4k_1^2}, \tag{17}$$

Therefore,

$$\tilde{B}_1 = \frac{\widetilde{N}' + \kappa\tilde{T}}{\|\widetilde{N}' + \kappa\tilde{T}\|} = \frac{-k_1 k_1' T + k_1' B_1 - 2k_1 k_2 B_2}{\sqrt{\left|-2k_1\left(k_1'\right)^2 + 4k_1^2 k_2^2\right|}}.$$

Taking $\mu = -1$ and equations (1) and (11), we have

$$\begin{aligned}
\tilde{B}_2 &= -\tilde{T} \times \tilde{N} \times \tilde{B}_1 \\
&= -N \times \left(\frac{-k_1 T - B_1}{\sqrt{2k_1}}\right) \times \left(\frac{-k_1 k_1' T + k_1' B_1 - 2k_1 k_2 B_2}{\sqrt{\left|-2k_1 (k_1')^2 + 4k_1^2 k_2^2\right|}}\right) \\
&= \frac{k_1 k_1' B_2 - 2k_1^2 k_2 (-T) - k_1 k_1' (-B_2) - 2k_1 k_2 B_1}{2k_1 \sqrt{\left|-(k_1')^2 + 2k_1 k_2^2\right|}} \\
&= \frac{2k_1^2 k_2 T - 2k_1 k_2 B_1 + 2k_1 k_1' B_2}{k_1 \sqrt{\left|-(k_1')^2 + 2k_1 k_2^2\right|}} .
\end{aligned}$$

Hence, the proof is completed.

Corollary 4. *Let $\tilde{\gamma}$ be the principal normal directional curve of null curve γ in $\mathbb{E}_1^4$. Then $\tilde{\gamma}$ is a spacelike curve with spacelike principal normal vector field.*

Proof. From equation (14), we have $\kappa^2 = 2k_1$ which implies $k_1 > 0$. Therefore, taking the norm of $\tilde{N}$ in (15) gives us $\|\tilde{N}\| = 1 > 0$. Hence, $\tilde{N}$ is spacelike and it completes the proof.

Theorem 5. *Let $\tilde{\gamma}$ be the principal normal directional curve of null curve γ in $\mathbb{E}_1^4$. The curvature κ, torsion τ and bitorsion σ of $\tilde{\gamma}$ can be written in terms of k_1 and k_2 which are the first null curvature and the second null curvature of γ, respectively by*

$$\kappa = \sqrt{2k_1}, \ , \tag{18}$$

$$\tau = \frac{1}{2k_1}\sqrt{|-(k_1')^2 + 2k_1 k_2^2|}, \tag{19}$$

$$\sigma = \frac{-2(k_1')^2 k_2 + 2k_1 k_1' k_2' + 2k_1 k_2^3 + 2k_1 k_1'' k_2}{\sqrt{\left|-(k_1')^2 + 2k_1 k_2^2\right|}} - \frac{2k_1 k_1' k_2 \left|-k_1'' + k_1' k_2^2 + 2k_1 k_2 k_2'\right|}{\sqrt{\left|-(k_1')^2 + 2k_1 k_2^2\right|^3}}. \tag{20}$$

Proof. By equation (2), (14) and (17), we have

$$\kappa = \|\tilde{T}'\| = \sqrt{2k_1} \text{ and } \tau = \|\tilde{N}' + \kappa \tilde{T}\| = \frac{1}{2k_1}\sqrt{|-(k_1')^2 + 2k_1 k_2^2|}.$$

From equations (18) and (19), we have

$$\sqrt{|-(k_1')^2 + 2k_1 k_2^2|} = \kappa^2 \tau ,$$

so that $\tilde{B}_2$ becomes

$$\tilde{B}_2 = \frac{k_1 k_2 T - k_2 B - k_1'}{\kappa^2 \tau}. \tag{21}$$

Taking derivative of equation (21) gives us

$$\tilde{B}_2' = \frac{2k_1' k_2 \kappa^2 \tau + k_1 k_2' \kappa^2 \tau - k_1 k_2 \kappa^2 \tau' - 2k_1 k_2 \kappa \kappa' \tau}{\kappa^4 \tau^2} T + \frac{-k_2' \kappa^2 \tau + k_2 \kappa^2 \tau' + 2k_2 \kappa \kappa' \tau}{\kappa^4 \tau^2} B_1$$
$$+ \frac{-k_2^2 \kappa^2 \tau - k_1'' \kappa^2 \tau + k_1' \kappa^2 \tau' + 2k_1' \kappa \kappa' \tau}{\kappa^4 \tau^2} B_2. \tag{22}$$

Applying equation (3), we have

$$\sigma = \frac{2(k_1')^2 k_2 + 2k_1 k_1' k_2' + 2k_1 k_2^3 + 2k_1 k_1'' k_2}{\kappa^2 \tau} - \frac{2k_1 k_1' k_2 \kappa^2 \tau' + 8k_1 k_1' k_2 \kappa \kappa' \tau}{\kappa^4 \tau^2}. \tag{23}$$

Substituting equations (18) and (19) into equation (23), we get equation (20).

Theorem 6. *Let γ be a null curve which is a principal normal donor curve of spacelike curve $\tilde{\gamma}$. Suppose κ, τ and σ are the curvature, torsion and bitorsion of $\tilde{\gamma}$, respectively, then*

$$k_1 = \frac{\kappa^2}{2} \text{ and } k_2 = \pm\sqrt{\kappa^2 \tau^2 + (\kappa')^2}. \tag{24}$$

Proof. By solving equation (18), we find

$$\kappa = \sqrt{2k_1} \Longrightarrow \kappa^2 = 2k_1 \Longrightarrow k_1 = \frac{\kappa^2}{2}$$

and from equation (18), we get

$$2k_1 \tau = \sqrt{|-(k_1')^2 + 2k_1 k_2^2|} \Longrightarrow \kappa^2 \tau^2 = -(k_1')^2 + 2k_1 k_2^2 \Longrightarrow \kappa^4 \tau^2 + (\kappa\kappa')^2 = \kappa^2 k_2^2$$
$$\Longrightarrow k_2 = \pm\sqrt{\kappa^2 \tau^2 + (\kappa')^2}.$$

This completes the proof.

4. B_1 Directional Curves of Null Curves

In this section, the casual characterizations of the B_1 directional curves of null curves are investigated.

Definition 7. *The curve $\tilde{\gamma}(s)$ is called the B_1 directional curve of a null curve $\gamma(s)$ if the tangent $\tilde{T}(s)$ of $\tilde{\gamma}(s)$ equals to the first binormal vector $B_1(s)$ of null curve $\gamma(s)$. Conversely, $\gamma(s)$ is called the first binormal donor curve of $\tilde{\gamma}(s)$.*

By definition 7, we have $\tilde{\gamma}(s) = \int B_1(s)ds$. Note also that $\tilde{\gamma}(s)$ is a null curve since its tangent vectors are null along the curve.

Theorem 8. *Let $\gamma(s)$ be a null curve in $\mathbb{E}_1^4$. The B_1 directional curve $\tilde{\gamma}(s)$ of $\gamma(s)$ is said to be parametrized by pseudo arc length s if and only if*

$$k_1^2 + k_2^2 = 1 \tag{25}$$

where k_1 and k_2 are the first and the second null curvatures of $\gamma(s)$.

Proof. Let $\tilde{\gamma}$ be the B_1 directional curve of null curve γ. Then, for any null curve $\tilde{\gamma}$ we have $\tilde{\gamma}' = B_1$ and $\tilde{\gamma}'' = k_1N + k_2B_2$. Consequently, $\tilde{\gamma}$ is a unit speed curve if and only if $g(\tilde{\gamma}'', \tilde{\gamma}'') = k_1^2 + k_2^2 = 1$. Thus, this completes the proof.

Theorem 9. *Let $\tilde{\gamma}$ be the B_1 directional curve of null curve γ in $\mathbb{E}_1^4$ and $\tilde{\gamma}$be parametrized by pseudo arc length s. Suppose $\{T, N, B_1, B_2\}$ and $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2\}$ are the Frenet frames of γ and $\tilde{\gamma}$, respectively. Then*

$$\tilde{T} = B_1, \tag{26}$$

$$\tilde{N} = k_1N + k_2B_2, \tag{27}$$

$$\tilde{B}_1 = T - k_1'N - \frac{(k_1')^2 + (k_2')^2}{2}B_1 - k_2'B_2\,, \tag{28}$$

$$\tilde{B}_2 = k_2N + (k_1'k_2 - k_1k_2')B_1 - k_1B_2. \tag{29}$$

Proof. Since $\tilde{\gamma}$ is the B_1 directional curve of null curve γ, $\tilde{T} = B_1$. Therefore,

$$\tilde{\gamma}' = B_1, \tag{30}$$

$$\tilde{\gamma}'' = k_1N + k_2B_2, \tag{31}$$

$$\tilde{\gamma}''' = -(k_1^2 + k_2^2)T + k_1'N - k_1B_1 + k_2'B_2 = -T + k_1'N - k_1B_1 + k_2'B_2, \tag{32}$$

and

$$g(\tilde{\gamma}'', \tilde{\gamma}'') = k_1^2 + k_2^2 = 1 \;\; \Longrightarrow \;\; \varphi = 1, \tag{33}$$

$$g(\tilde{\gamma}''', \tilde{\gamma}''') = 2k_1 + (k_1')^2 + (k_2')^2. \tag{34}$$

Therefore, by using equations (5) to (8) and (30) to (34), we have

$$\tilde{T} = \frac{\tilde{\gamma}'}{\varphi} = B_1,$$

$$\tilde{N} = \tilde{T}' = k_1 N + k_2 B_2,$$

$$\tilde{B}_1 = -\frac{1}{\varphi}\tilde{\gamma}''' - \frac{g(\tilde{\gamma}''',\tilde{\gamma}''')}{2\varphi^3}\tilde{\gamma}'$$

$$= T - k_1' N + k_1 B_1 - k_2' B_2 - \frac{1}{2}(2k_1 + (k_1')^2 + (k_2')^2)B_1$$

$$= T - k_1' N - \frac{(k_1')^2 + (k_2')^2}{2} B_1 - k_2' B_2 ,$$

$$\tilde{B}_2 = \frac{1}{\varphi^3}(\tilde{\gamma}' \wedge \tilde{\gamma}'' \wedge \tilde{\gamma}''')$$

$$= B_1 \wedge (k_1 N + k_2 B_2) \wedge (-T + k_1' N - k_1 B_1 + k_2' B_2)$$

$$= -k_1 B_2 + k_1 k_2'(-B_1) - k_2(-N) + k_1' k_2 B_1$$

$$= k_2 N + (k_1' k_2 - k_1 k_2')B_1 - k_1 B_2.$$

This completes the proof.

Corollary 10. *Let $\tilde{\gamma}$ be the B_1 directional curve of null curve γ in $\mathbb{E}_1^4$ and $\tilde{\gamma}$ be parametrized by pseudo arc length s. Then the first null curvature $\tilde{k}_1$ and the second null curvature $\tilde{k}_2$ of $\tilde{\gamma}$ are given by*

$$\tilde{k}_1 = k_1 + \frac{(k_1')^2 + (k_2')^2}{2}, \tag{35}$$

$$\tilde{k}_2 = k_2 - (k_1' k_2 - k_1 k_2')' - (k_1 k_1' + k_2 k_2')(k_1' k_2 - k_1 k_2'). \tag{36}$$

Proof. From equations (10) and (34), we have

$$\tilde{k}_1 = \frac{1}{2\varphi^2}(g(\tilde{\gamma}''',\tilde{\gamma}''') - 2\varphi\varphi'' - 4(\varphi')^2)$$

$$= k_1 + \frac{(k_1')^2 + (k_2')^2}{2} .$$

Now, differentiating equation (29) yields

$$\tilde{B}_2 = k_2' N + k_2 N' + (k_1' k_2 - k_1 k_2')' B_1 + (k_1' k_2 - k_1 k_2') B_1' - k_1' B_2 + k_1 B_2'$$

$$= (k_2' + (k_1' k_2 - k_1 k_2')k_1)N + ((k_1' k_2 - k_1 k_2')' - k_2)B_1 + (k_2(k_1' k_2 - k_1 k_2') - k_1)B_2.$$

From equation (9), we have

$$\tilde{k}_2 = -g(\tilde{B}_2, \tilde{B}_1)$$

$= k_2 - (k_1' k_2 - k_1\, k_2')' - k_1'\, k_2' - k_1\, k_1'(k_1'\, k_2 - k_1\, k_2') + k_1 k_2' - k_2 k_2'(k_1' k_2 - k_1 k_2')$

$= k_2 - (k_1'\, k_2 - k_1 k_2')' - (k_1 k_1' + k_2 k_2')(k_1' k_2 - k_1 k_2')$.

5. B_2 Directional Curves of Null Curves

In this section, the casual characterizations of the B_2 directional curves of null curves are investigated.

Definition 11. *The curve $\tilde{\gamma}(s)$ is called the B_2 directional curve of a null curve $\gamma(s)$ if the tangent vector field $\tilde{T}(s)$ of $\tilde{\gamma}(s)$ equals to the second binormal vector $B_2(s)$ of $\gamma(s)$. Conversely, $\gamma(s)$ said to be the second binormal donor curve of $\tilde{\gamma}(s)$.*

By definition 11, we have $\tilde{\gamma}(s) = \int B_2(s)ds$. Note that $\tilde{\gamma}(s)$ is a spacelike curve since its tangent vector is spacelike along the curve.

Theorem 12. *Let $\tilde{\gamma}$ be the B_2 directional curve of null curve γ in $\mathbb{E}_1^4$ and $\tilde{\gamma}$ be parametrized by pseudo arc length s. Suppose $\{T, N, B_1, B_2\}$ and $\{\tilde{T}, \tilde{N}, \tilde{B}_1, \tilde{B}_2\}$ are the Frenet frame of γ and $\tilde{\gamma}$, respectively. Then*

$$\tilde{T} = B_2, \tag{37}$$

$$\tilde{N} = -k_2 T, \tag{38}$$

$$\tilde{B}_1 = -\frac{k_2'}{k_2} T - N, \tag{39}$$

$$\tilde{B}_2 = -\frac{1}{k_2} B_1. \tag{40}$$

Proof.

Let $\tilde{\gamma}$ be the B_2 directional curve of null curve γ, then $\tilde{T} = \tilde{\gamma}' = B_2$. Therefore,

$$\tilde{N} = \tilde{T}' = \tilde{\gamma}'' = -k_2 T.$$

Note that T is a null vector field. As a result, the principal normal vector field $\tilde{N}$ of $\tilde{\gamma}$ must be null. Therefore, $\tilde{\gamma}$ is a pseudo null curve. Differentiating $\tilde{\gamma}''$ again, we have

$$\tilde{\gamma}''' = -k_2' T - k_2 N \text{ and } \|\tilde{\gamma}'''\| = k_2$$

such that

$$\tilde{B}_1 = \frac{\tilde{\gamma}'''}{\|\tilde{\gamma}'''\|} = -\frac{k_2'}{k_2} T - N$$

is a unit spacelike vector. Since $\tilde{\gamma}$ is spacelike where null principal normal vector field $\widetilde{N}$ is null and its binormal vector field $\tilde{B}_1$ is spacelike then $\tilde{B}_2$ is a null vector field which is orthogonal to $\{\tilde{T}, \tilde{B}_1\}$ and $g\left(\widetilde{N}, \tilde{B}_2\right) = 1$. Therefore, we can take

$$\tilde{B}_2 = -\frac{1}{k_2} B_1.$$

This completes the proof.

Corollary 13. *Let $\tilde{\gamma}$ be the B_2 directional curve of null curve γ in $\mathbb{E}_1^4$ and $\tilde{\gamma}$ be parametrized by pseudo arc length s. Then the curvature κ, torsion τ and bitorsion σ of $\tilde{\gamma}$ are given by*

$$\kappa = 1,\ \tau = k_2,\ \sigma = \frac{(k_2')^2}{k_2^3} - \frac{k_1}{k_2}. \tag{41}$$

Proof. From equation (4), we have $\kappa = 1$ by assuming that $\tilde{\gamma}$ is not a straight line. We also have

$$\tau = \|\tilde{\gamma}'''\| = k_2.$$

Furthermore, we have

$$\begin{aligned}
\sigma &= -g\left(\tilde{B}_2', \tilde{B}_1\right) \\
&= -g\left(\left(-\frac{B_1}{k_2}\right)', \left(-\frac{k_2'}{k_2} T - N\right)\right) \\
&= g\left(\left(\frac{k_1 k_2 N - k_2' B_1 + k_2^2 B_2}{k_2^2}\right), \left(-\frac{k_2'}{k_2} T - N\right)\right) \\
&= \frac{(k_2')^2}{k_2^3} - \frac{k_1}{k_2}.
\end{aligned}$$

Theorem 14. *Let γ be a null curve and a B_2 donor curve of pseudo null curve $\tilde{\gamma}$. Suppose κ, τ and σ are the curvature, torsion and bitorsion of $\tilde{\gamma}$, respectively. Then,*

$$k_1 = \tau\sigma - \left(\frac{\tau'}{\tau}\right)^2 \text{ and } k_2 = \tau. \tag{42}$$

Proof. By solving the equation (41), we find $k_2 = \tau$ and

$$\frac{(\tau')^2}{\tau^3} - \frac{k_1}{\tau} = \sigma \implies \frac{k_1}{\tau} = \sigma - \frac{(\tau')^2}{\tau^3} \implies k_1 = \tau\sigma - \left(\frac{\tau'}{\tau}\right)^2.$$

This completes the proof.

6. Numerical Example

In this section, a numerical example as the application of the theorems given in the previous sections is provided. Consider the null curve

$$\gamma(s)=\left(\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}}+\frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{14}}{28}\left(\frac{s^{2+\frac{3\sqrt{6}}{2}}}{2+\frac{3\sqrt{6}}{2}}-\frac{s^{2-\frac{3\sqrt{6}}{2}}}{2-\frac{3\sqrt{6}}{2}}\right),\frac{s^2\sqrt{14}}{63}\left(2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)+\right.$$
$$\left.\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right),\frac{s^2\sqrt{14}}{63}\left(2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right)-\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right).$$

Differentiating $\gamma(s)$with respect to s and using equations (5) to (8), we obtain

$$T(s)=\left(\frac{\sqrt{14}}{28}\left(s^{1+\frac{3\sqrt{6}}{2}}+s^{1-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{14}}{28}\left(s^{1+\frac{3\sqrt{6}}{2}}-s^{1-\frac{3\sqrt{6}}{2}}\right),\frac{s\sqrt{14}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right),\right.$$
$$\left.\frac{s\sqrt{14}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$

$$N(s)=\left(\frac{\sqrt{7}}{28s^2}\left((3\sqrt{3}+\sqrt{2})s^{2+\frac{3\sqrt{6}}{2}}-(3\sqrt{3}-\sqrt{2})s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{7}}{28s^2}\left((3\sqrt{3}+\sqrt{2})s^{2+\frac{3\sqrt{6}}{2}}+\right.\right.$$
$$\left.(3\sqrt{3}-\sqrt{2})s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{7}}{14}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)-\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$
$$\left.\frac{\sqrt{7}}{14}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)+\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

$$B_1(s)=\left(-\frac{3\sqrt{14}}{56s^3}\left((5+\sqrt{2})s^{2+\frac{3\sqrt{6}}{2}}+(5-\sqrt{2})s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{3\sqrt{14}}{56s^3}\left((5+\sqrt{2})s^{2+\frac{3\sqrt{6}}{2}}-\right.\right.$$
$$\left.(5-\sqrt{2})s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{7}}{28s}\left(13\sqrt{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$
$$\left.\frac{\sqrt{7}}{28s}\left(13\sqrt{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

$$B_2(s)=\left(-\frac{\sqrt{7}}{28s^2}\left(s^{2+\frac{3\sqrt{6}}{2}}-s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{\sqrt{7}}{28s^2}\left(s^{2+\frac{3\sqrt{6}}{2}}+s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{3\sqrt{21}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right),\right.$$
$$\left.\frac{3\sqrt{21}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right).$$

In addition, by using equation (10), we have

$\varphi(s)=1$, $k_1(s)=-\frac{6}{s^2}$, and $k_2(s)=-\frac{3\sqrt{3}}{2s^2}$.

a. Principal normal directional curve of γ.
   By using equation (12), we have

$$\tilde{\gamma}(s)=\Bigg(\frac{\sqrt{14}}{28s}\left(s^{2+\frac{3\sqrt{6}}{2}}+s^{2-\frac{3\sqrt{6}}{2}}\right),\ \frac{\sqrt{14}}{28s}\left(s^{2+\frac{3\sqrt{6}}{2}}-s^{2-\frac{3\sqrt{6}}{2}}\right),$$
$$\frac{\sqrt{14}}{14}s\cos\left(\frac{\sqrt{2}}{2}\ln s\right),\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg),$$

$$\tilde{T}(s)=\Bigg(\frac{\sqrt{7}}{28s^2}\left(\left(3\sqrt{3}+\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}}-\left(3\sqrt{3}-\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right),\frac{\sqrt{7}}{28s^2}\Bigg(\left(3\sqrt{3}+\right.$$
$$\left.\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}}+\left(3\sqrt{3}-\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\Bigg),\frac{\sqrt{7}}{14}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)-\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$
$$\frac{\sqrt{7}}{14}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)+\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg)$$

$$\tilde{N}(s)=\Bigg(\frac{\sqrt{21}}{112s^2}\left(\left(2\sqrt{3}+9\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}}-\left(2\sqrt{3}-9\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right),\ \frac{\sqrt{21}}{112s^2}\Bigg(\left(2\sqrt{3}+\right.$$
$$\left.9\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}}+\left(2\sqrt{3}-9\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\Bigg),-\frac{\sqrt{21}}{168}\Bigg(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+$$
$$2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg),\ -\frac{\sqrt{21}}{168}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),$$

$$\widetilde{B_1}(s)=\Bigg(-\frac{\sqrt{21}}{140s^2}\left(\left(\sqrt{3}+\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}}-\left(\sqrt{3}-\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right),-\frac{\sqrt{21}}{140s^2}\Bigg(\left(\sqrt{3}+\right.$$
$$\left.\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}}+\left(\sqrt{3}-\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\Bigg),\frac{\sqrt{21}}{210}\Bigg(25\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+$$
$$29\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg),\frac{\sqrt{21}}{210}\left(25\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-29\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),$$

$$\widetilde{B_2}(s)=\Bigg(\frac{\sqrt{21}}{3360s}\left(\left(3\sqrt{6}+2\right)s^{2+\frac{3\sqrt{6}}{2}}+\left(3\sqrt{6}-2\right)s^{2-\frac{3\sqrt{6}}{2}}\right),\ \frac{\sqrt{21}}{3360s}\Bigg(\left(3\sqrt{6}+\right.$$
$$\left.2\right)s^{2+\frac{3\sqrt{6}}{2}}-\left(3\sqrt{6}-2\right)s^{2-\frac{3\sqrt{6}}{2}}\Bigg),\frac{5s\sqrt{7}}{112}\Bigg(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)+$$
$$2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg),\frac{5s\sqrt{7}}{112}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)-2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg).$$

By using equations (18) to (20), we have

$\kappa=\frac{2\sqrt{3}}{s}, \tau=-\frac{5}{4s}$, and $\sigma=-\frac{9\sqrt{3}}{250\,s^5}(-2143+64s^3)$.

b. B_1 directional curve of γ.

$$\tilde{\gamma}(s)=\Bigg(-\frac{\sqrt{7}}{84}\left(\left(5\sqrt{3}+3\sqrt{2}\right)s^{\frac{3\sqrt{6}}{2}}-\left(5\sqrt{3}-3\sqrt{2}\right)s^{-\frac{3\sqrt{6}}{2}}\right),-\frac{\sqrt{7}}{84}\Bigg(\left(5\sqrt{3}+\right.$$
$$\left.3\sqrt{2}\right)s^{\frac{3\sqrt{6}}{2}}+\left(5\sqrt{3}-3\sqrt{2}\right)s^{-\frac{3\sqrt{6}}{2}}\Bigg),-\frac{\sqrt{7}}{14}\Bigg(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)-$$
$$13\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\Bigg),-\frac{\sqrt{7}}{14}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)+13\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\Bigg),$$

$$\tilde{T}(s) = \left(-\frac{3\sqrt{7}}{56\,s^3}\left(\left(5\sqrt{2}+2\sqrt{3}\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(5\sqrt{2}-2\sqrt{3}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{7}}{56\,s^3}\left(\left(5\sqrt{2}+ 2\sqrt{3}\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(5\sqrt{2}-2\sqrt{3}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{\sqrt{7}}{28s}\left(13\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{7}}{28s}\left(13\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right)$$

$$\tilde{N}(s) = \left(-\frac{3\sqrt{7}}{56\,s^4}\left(\left(11\sqrt{3}+4\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(11\sqrt{3} - 4\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{7}}{56\,s^4}\left(\left(11\sqrt{3}+4\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(11\sqrt{3} - 4\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{7}}{28s^2}\left(4\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) - 13\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), -\frac{3\sqrt{7}}{28s^2}\left(4\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) + 13\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

$$\widetilde{B_1}(s) = \left(\frac{\sqrt{7}}{112s^9}\left(4\sqrt{2}\left(s^{10+\frac{3\sqrt{6}}{2}} + s^{10-\frac{3\sqrt{6}}{2}}\right) - \left(132\sqrt{3}+48\sqrt{2}\right)s^{6+\frac{3\sqrt{6}}{2}} + \left(132\sqrt{3}-48\sqrt{2}\right)s^{6-\frac{3\sqrt{6}}{2}} + \left(1026\sqrt{3}+2565\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(1026\sqrt{3} - 2565\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{\sqrt{7}}{112s^9}\left(4\sqrt{2}\left(s^{10+\frac{3\sqrt{6}}{2}} - s^{10-\frac{3\sqrt{6}}{2}}\right) - \left(132\sqrt{3} + 48\sqrt{2}\right)s^{6+\frac{3\sqrt{6}}{2}} - \left(132\sqrt{3}-48\sqrt{2}\right)s^{6-\frac{3\sqrt{6}}{2}} + \left(1026\sqrt{3}+2565\sqrt{2}\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(1026\sqrt{3}-2565\sqrt{2}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{\sqrt{7}}{56s^7}\left(4s^8\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) - 48s^4\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 156s^4\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2223\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) - 342\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{7}}{56s^7}\left(4s^8\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 48s^4\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 156s^4\cos\left(\frac{\sqrt{2}}{2}\ln s\right) - 2223\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) + 342\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right)$$

$$\widetilde{B_2}(s) = \left(-\frac{3\sqrt{7}}{56s^4}\left(\left(13+\sqrt{6}\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(13-\sqrt{6}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{7}}{56s^4}\left(\left(13+ \sqrt{6}\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(13-\sqrt{6}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{21}}{28\,s^2}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 11\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), -\frac{3\sqrt{21}}{28\,s^2}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 11\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right).$$

By using equations (35) and (36), we have

$\tilde{k}_1 = -\frac{12s^4-171}{2s^6}$ and $\tilde{k}_2 = -\frac{3\sqrt{3}}{2s^2}$.

c. B_2 directional curve of γ.

$$\tilde{\gamma}(s) = \left(-\frac{\sqrt{7}}{700s}\left(\left(3\sqrt{6}-2\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(3\sqrt{6}+2\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{\sqrt{7}}{700s}\left(\left(3\sqrt{6}-2\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(3\sqrt{6}+2\right)s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{s\sqrt{21}}{14}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{s\sqrt{21}}{14}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right)$$

$$\tilde{T}(s) = \left(-\frac{\sqrt{7}}{28s^2}\left(s^{2+\frac{3\sqrt{6}}{2}} - s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{\sqrt{7}}{28s^2}\left(s^{2+\frac{3\sqrt{6}}{2}} + s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{3\sqrt{21}}{14}\sin\left(\frac{\sqrt{2}}{2}\ln s\right), \frac{3\sqrt{21}}{14}\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$

$$\tilde{N}(s) = \left(\frac{3\sqrt{21}}{56s^3}\left(s^{2+\frac{3\sqrt{6}}{2}} + s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{3\sqrt{21}}{56s^3}\left(s^{2+\frac{3\sqrt{6}}{2}} - s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{3\sqrt{21}}{28s}\cos\left(\frac{\sqrt{2}}{2}\ln s\right), \frac{3\sqrt{21}}{28s}\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right),$$

$$\tilde{B}_1(s) = \left(-\frac{\sqrt{14}}{56s^2}\left(\left(3\sqrt{6}-2\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(3\sqrt{6}+2\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{\sqrt{14}}{56s^2}\left(\left(3\sqrt{6}-2\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(3\sqrt{6}+2\right)s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{\sqrt{7}}{14}\left(\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + \sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{\sqrt{7}}{14}\left(\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - \cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right),$$

$$\tilde{B}_2(s) = \left(-\frac{\sqrt{21}}{84s}\left(\left(5\sqrt{2}+2\sqrt{3}\right)s^{2+\frac{3\sqrt{6}}{2}} + \left(5\sqrt{2}-2\sqrt{3}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), -\frac{\sqrt{21}}{84s}\left(\left(5\sqrt{2}+2\sqrt{3}\right)s^{2+\frac{3\sqrt{6}}{2}} - \left(5\sqrt{2}-2\sqrt{3}\right)s^{2-\frac{3\sqrt{6}}{2}}\right), \frac{s\sqrt{21}}{126}\left(13\sqrt{2}\cos\left(\frac{\sqrt{2}}{2}\ln s\right) + 2\sin\left(\frac{\sqrt{2}}{2}\ln s\right)\right), \frac{s\sqrt{21}}{126}\left(13\sqrt{2}\sin\left(\frac{\sqrt{2}}{2}\ln s\right) - 2\cos\left(\frac{\sqrt{2}}{2}\ln s\right)\right)\right).$$

By using equation (41), we have
$\kappa = 1$, $\tau = -\frac{3\sqrt{3}}{2s^2}$, and $\sigma = -\frac{20\sqrt{3}}{9}$.

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Conflict of Interest

The author states that there is no conflict of interest.

Ethics Committee Approval and Informed Consent

As the authors of this study, we declare that we do not have any ethics committee approval and/or informed consent statement.

References

[1] S. Yılmaz and M. Turgut, "On the differential geometry of the curves in Minkowski space-time I," *Int. J. Contemp. Math. Sci.*, 3(27), 1343–1349, 2008.

[2] X. Liu and Z. Wang, "On lightlike hypersurfaces and lightlike focal sets of null Cartan curves in Lorentz-Minkowski spacetime," *J. Nonlinear Sci. Appl.*, 8, 628–639, 2015.

[3] M. Sasaki, "Notes on null curves in Minkowski space," *Turk. J. Math.*, 34, 417–424, 2010.

[4] A. C. Çoken and U. Ciftci, "On the Cartan curvatures of a null curve in Minkowski spacetime," *Geom. Dedicata*, 114, 71–78, 2005.

[5] K. L. Duggal and A. Bejancu. *Lightlike Submanifold of Semi-Riemannian Manifolds and Applications*, The Netherland: Dordecht, 1996.

[6] J. H. Choi, Y. H. Kim and A. T. Ali, "Some associated curves of Frenet non-lightlike curves in $\mathbb{E}_1^3$," *J. Math. Anal. Appl.*, 394, 712–723, 2012.

[7] S. Yılmaz and M. Turgut, "Determination of Frenet apparatus of partially null and pseudo null curves in Minkowski space-time," *Int. J. Contemp. Math. Sci.*, 3(27), 1337–1341, 2008.

[8] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, New York: Adacemic Press, 1983.

[9] R. Lopez, "Differential geometry of curves and surfaces in Lorentz–Minkowski space," *Int. Electron. J. Geom.*, 7, 44–107, 2014.

[10] A. Nersessian and E. Ramos, "Massive spinning particles and geometry of null curves," *Phys. Lett. B.*, 445, 123–128, 1998.

[11] L. P. Hughston and W. T. Shaw, "Real classical string," *Proc. Roy. Soc. London Ser. A*, 414, 415–422, 1987.

[12] A. Ferrandez, A. Gimenez and P. Lucas, "Characterization of null curves in Lorentz–Minkowski spaces," *Publ. de la RSME*, 3, 221–226, 2001.

[13] K. L. Duggal and D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, Singapore: World Scientific, 2007.

[14] J. H. Choi and Y. H. Kim, "Associated curves of a Frenet curve and their applications," *Appl. Math. Comput.*, 218(18), 9116-9124, 2012.

[15] N. Macit and M. Düldül, "Some new associated curves of a Frenet curve in E^3 and E^4," *Turk J. Math.*, 38(6), 1023-1037, 2014.

[16] J. Qian and Y. H. Kim, "Directional associated curves of a null curve in Minkowski 3- space," *Bull. Korean Math. Soc.*, 52(1), 183-200, 2015.

[17] B. Şahiner, "Direction curves of tangent indicatrix of a curve," *Appl. Math. Comput.*, 343, 273-284, 2019.

[18] B. Sahiner, B. "Quaternionic direction curves," *Kyungpook Math. J.*, 58(2), 377-388, 2018.

[19] S. Kiziltug and M. Önder, "Associated curves of Frenet curves in three-dimensional compact Lie group," *Miskolc Math. Notes*, 16(2), 953-964, 2015.

[20] M. A. K. Mahmut and H. Altınbaş, "Some special associated curves of nondegenerate curve in anti de sitter 3-space," Math. Sci. Appl. E-Notes, 5(2), 89-97, 2017.

[21] S. Yurttançıkmaz, S. Kızıltuğ and A. Çakmak, "The directional curves of spacelike and timelike Frenet curves in E_1^3," *Journal of Advanced Mathematics and Mathematics Education*, 2(3), 1-12.

[22] B. Şahiner, "Some Special Dual Direction Curves," *Erzincan University Journal of Science and Technology*, 11(3), 509-517, 2019.

[23] J. H. Choi, Y.H. Kim, A.T. Ali, "Some associated curves of Frenet non-lightlike curves in E_1^3," *J. Math. Anal. Appl.*, 394, 712–723, 2012.