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Regularity of n-generalized Schützenberger product of monoids

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Abstract

In this work, regularity of n-generalized Schützenberger product of monoids from the point of Group Theory is studied. Here, it is determined necessary and sufficient conditions of the n-generalized Schützenberger product $A_1 \Diamond A_2 \Diamond \cdots \Diamond A_n$ to be regular while all $A_i\ (1 \le i \le n)$ are monoids. Also, by considering all $A_i\ (1 \le i \le n)$ to be groups, it is given another result for the regularity of this product.

Keywords: *Schützenberger product, monoid, regularity.*

Monoidlerin n-genelleştirilmiş Schützenberger çarpımının regülerliği

Öz

Bu çalışmada, monoidlerin n-genelleştirilmiş Schützenberger çarpımın regülerliği Grup Teori açısından incelenmiştir. Burada, bütün $A_i\ (1 \le i \le n)$'ler monoid iken $A_1 \Diamond A_2 \Diamond \cdots \Diamond A_n$ n-genelleştirilmiş Schützenberger çarpımın regüler olabilmesi için gerekli ve yeterli koşul elde edilmiştir. Ayrıca, bütün $A_i\ (1 \le i \le n)$'leri grup düşünerek bu çarpımın regülerliği için bir diğer sonuç verilmiştir.

Anahtar kelimeler: *Schützenberger çarpım, monoid, regülerlik.*

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1. Introduction and preliminaries

Recent developments in group and semigroup theory have raised the question of whether there exists a classification of some algebraic structures according to regularity, π-inverse, p-Cockcroft property. As an answer to the regularity of this algebraic structures, in [9], Skornjakov explained regularity of the wreath product of monoids. After that, in [7], it has been investigated regularity property of semidirect products of monoids. After these works, in [6], the authors determined necessary and sufficient conditions for Schützenberger product of monoids and the new version of the Schützenberger product of monoids to be regular and strongly π-inverse. Furthermore, in [3], the authors studied the regularity of a new monoid construction under crossed and Schützenberger product to any two monoids. In this study, as the main result of this paper, we give an answer by defining necessary and sufficient conditions of the n-generalized Schützenberger product $A_1 \Diamond A_2 \Diamond \cdots \Diamond A_n$ to be regular where all A_i $(1 \leq i \leq n)$ are any monoids.

We recall that a monoid M is called *regular* if, for every $a \in M$, there exists $b \in M$ such that $aba = a$ and $bab = b$ (or, equivalently, for the set of inverses of a in M, that is, $a^{-1} = \{b \in B : aba = a,\ bab = b\}$, M is regular if and only if, for all $a \in M$, the set a^{-1} is not equal to the emptyset.

The Schützenberger product is an operation on monoids that was originally introduced for solving questions in automata theory and to analyze the syntactic properties of the concatenation product in formal language theory. The Schützenberger product was originally defined by Schützenberger (1965) ([8]) for two monoids, and extended by Straubing (1981) ([10]) for any number of monoids. In [5], the authors obtained a presentation for Schützenberger product of two monoids and gave the normal form structure of the elements of this product.

Let M_1 and M_2 be monoids presented by $\langle X_1 \mid R_1 \rangle$ and $\langle X_2 \mid R_2 \rangle$, respectively. For $P \subseteq M_1 \times M_2, a \in M_1, b \in M_2$, we define

$$aP = \{(ac,d) \mid (c,d) \in P\}, Pb = \{(c,db) \mid (c,d) \in P\}.$$

The *Schützenberger product* of M_1 and M_2, denoted by $M_1 \Diamond M_2$, is the set

$$M_1 \times \mathrm{P}\,(M_1 \times M_2) \times M_2$$

with multiplication $(a_1, P_1, b_1)(a_2, P_2, b_2) = (a_1 a_2, P_1 b_2 \cup a_1 P_2, b_1 b_2)$, where $\mathrm{P}\,(M_1 \times M_2)$ is the power set obtained from the product sets M_1 and M_2. The Schützenberger product of M_1 and M_2 is presented by

$$\wp_{M_1 \Diamond M_2} = \Big\langle Z \mid R_1, R_2, z_{w_1,w_2}^2 = z_{w_1,w_2}, z_{w_1,w_2} z_{w_1',w_2'} = z_{w_1',w_2'} z_{w_1,w_2},$$

$$x_1 z_{w_1,w_2} = z_{x_1 w_1,w_2} x_1, z_{w_1,w_2} x_2 = x_2 z_{w_1,w_2 x_2}, x_1 x_2 = x_2 x_1 \Big\rangle,$$

where $x_i \in X_i$, $w_i, w_i' \in M_i$ $(i \in \{1,2\})$ and $Z = X_1 \cup X_2 \cup \{z_{w_1,w_2} \mid w_1 \in M_1, w_2 \in M_2\}$ (see [5]).

In [4], the authors gave presentations of the Schützenberger product of n groups $G_1,\cdots,G_n$, given a monoid presentation $\langle X_i \mid R_i \rangle$ of each group G_i, by using matrix theory. In [2], the authors studied a new n-generalized monoid construction of Schützenberger product from view of Combinatorial Group Theory and found a presentation of this generalized product.

Let $A_1, A_2, \cdots, A_{n-1}$ and A_n be monoids. For $P_{i,i+1} \subseteq A_i \times A_{i+1} (1 \le i \le n-1)$, and $a_i \in A_i \ (1 \le i \le n)$, we define

$$a_i P_{i,i+1} = \{(a_i x_i, x_{i+1});\ \ (x_i, x_{i+1}) \in P_{i,i+1}\},$$
$$P_{i,i+1} a_{i+1} = \{(x_i, x_{i+1} a_{i+1});\ \ (x_i, x_{i+1}) \in P_{i,i+1}\}.$$

n-generalized Schützenberger product of monoids $A_1, A_2, \cdots A_{n-1}$ and A_n, denoted by $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$, is the set $A_1 \times \mathrm{P}(A_1 \times A_2) \times A_2 \times \mathrm{P}(A_2 \times A_3) \times A_3 \times \cdots \times \mathrm{P}(A_{n-1} \times A_n) \times A_n$ with the multiplication

$$\left(a_1, P_{1,2}, a_2, P_{2,3}, a_3, \cdots, P_{n-1,n}, a_n\right)\left(a_1^{'}, P_{1,2}^{'}, a_2^{'}, P_{2,3}^{'}, a_3^{'}, \cdots, P_{n-1,n}^{'}, a_n^{'}\right)$$
$$= \left(a_1 a_1^{'},\ a_1 P_{1,2}^{'} \cup P_{1,2} a_2^{'},\ a_2 a_2^{'},\ a_2 P_{2,3}^{'} \cup P_{2,3} a_3^{'}, \cdots, a_{n-1} a_{n-1}^{'},\ a_{n-1} P_{n-1,n}^{'} \cup P_{n-1,n} a_n^{'},\ a_n a_n^{'}\right).$$

For all $a_i \in A_i$ and $P_{i,i+1} \in \mathrm{P}\left(A_i \times A_{i+1}\right)$, n-generalized Schützenberger product $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ defines a monoid with the identity $\left(1_{A_1}, \varnothing, 1_{A_2}, \cdots, \varnothing, 1_{A_n}\right)$[2].

We finally note that the reader is referred to [1,2] and [4] for a detailed survey on n-generalized Schützenberger product.

2. Regularity of n-generalized Schützenberger product

In this section, we proved by two different proofs for the regularity of n-generalized Schützenberger product of monoids $A_1, A_2, \cdots, A_{n-1}$ and A_n. Firstly, we gave necessary and sufficient conditions for n-generalized Schützenberger product $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ to be regular while all $A_i \left(1 \le i \le n\right)$ are arbitrary monoids. Then, by considering all $A_i \left(1 \le i \le n\right)$ to be groups, we give another result for the regularity of this product.

Theorem 2.1 *Let $A_1, A_2, \cdots, A_{n-1}$ and A_n be any monoids. Then n-generalized Schützenberger product $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ is regular if and only if,*

(i) $\forall A_i \left(1 \le i \le n\right)$ are regular,

(ii) for $(a_1, P_{1,2}, a_2, P_{2,3}, a_3, \cdots, P_{n-1,n}, a_n) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$, either

$$P_{i,i+1} = a_i Q_{i,i+1} a_{i+1} = \bigcup_{(x_i, x_{i+1}) \in Q_{i,i+1}} \{(a_i x_i,\ x_{i+1} a_{i+1})\} (1 \le i \le n-1),$$

or

$$P_{i,i+1} = b_i a_i Q_{i,i+1} a_{i+1} b_{i+1} = \bigcup_{(x_i, x_{i+1}) \in Q_{i,i+1}} \{(b_i a_i x_i, x_{i+1} a_{i+1} b_{i+1})\} (1 \le i \le n-1)$$

where $P_{i,i+1} \subseteq A_i \times A_{i+1} (1 \le i \le n-1)$ and $b_i \in a_i^{-1} (1 \le i \le n)$.

Proof. Let us suppose that $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ is regular. Thus for $(a_1, \varnothing, a_2, \varnothing, a_3, \cdots, \varnothing, a_n) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$, there exists $(b_1, Q_{1,2}, b_2, Q_{2,3}, b_3, \cdots, Q_{n-1,n}, b_n)$ such that

$$\begin{aligned}(a_1, \varnothing, a_2, \cdots, \varnothing, a_n) &= (a_1, \varnothing, a_2, \cdots, \varnothing, a_n)(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n)(a_1, \varnothing, a_2, \cdots, \varnothing, a_n) \\ &= (a_1 b_1, a_1 Q_{1,2}, a_2 b_2, \cdots, a_{n-1} Q_{n-1,n}, a_n b_n)(a_1, \varnothing, a_2, \cdots, \varnothing, a_n) \\ &= (a_1 b_1 a_1, a_1 Q_{1,2} a_2, a_2 b_2 a_2, \cdots, a_{n-1} Q_{n-1,n} a_n, a_n b_n a_n)\end{aligned}$$

and

$$\begin{aligned}(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n) &= (b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n)(a_1, \varnothing, a_2, \cdots, \varnothing, a_n)(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n) \\ &= (b_1 a_1,\ Q_{1,2} a_2,\ b_2 a_2, \cdots, Q_{n-1,n} a_n,\ b_n a_n)(b_1,\ Q_{1,2},\ b_2, \cdots, Q_{n-1,n},\ b_n) \\ &= (b_1 a_1 b_1, b_1 a_1 Q_{1,2} \cup Q_{1,2} a_2 b_2, b_2 a_2 b_2, \cdots, b_{n-1} a_{n-1} Q_{n-1,n} \cup Q_{n-1,n} a_n b_n, b_n a_n b_n).\end{aligned}$$

Therefore, we obtain that $a_i = a_i b_i a_i \ (1 \le i \le n)$ and $b_i = b_i a_i b_i \ (1 \le i \le n)$. This implies that (i) must hold.

By the assumption on the regularity of $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ for $(a_1,\ P_{1,2},\ a_2, \cdots, P_{n-1,n},\ a_n) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$, we have $(b_1,\ Q_{1,2},\ b_2, \cdots, Q_{n-1,n},\ b_n)$ such that

$$\begin{aligned}&(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n) \\ &= (a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n)(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n)(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n) \\ &= (a_1 b_1, a_1 Q_{1,2} \cup P_{1,2} b_2, a_2 b_2, ..., a_{n-1} Q_{n-1,n} \cup P_{n-1,n} b_n, a_n b_n)(a_1, P_{1,2}, a_2, ..., P_{n-1,n}, a_n) \\ &= (a_1 b_1 a_1, a_1 b_1 P_{1,2} \cup a_1 Q_{1,2} a_2 \cup P_{1,2} b_2 a_2,\ a_2 b_2 a_2, ..., a_{n-1} b_{n-1} P_{n-1,n} \cup a_{n-1} Q_{n-1,n} a_n \cup P_{n-1,n} b_n a_n,\ a_n b_n a_n)\end{aligned}$$

and

$$\begin{aligned}&(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n) \\ &= (b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n)(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n)(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n) \\ &= (b_1 a_1, b_1 P_{1,2} \cup Q_{1,2} a_2, b_2 a_2, \cdots, b_{n-1} P_{n-1,n} \cup Q_{n-1,n} a_n, b_n a_n)(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n) \\ &= (b_1 a_1 b_1, b_1 a_1 Q_{1,2} \cup b_1 P_{1,2} b_2 \cup Q_{1,2} a_2 b_2, b_2 a_2 b_2, \cdots, b_{n-1} a_{n-1} Q_{n-1,n} \cup b_{n-1} P_{n-1,n} b_n \cup Q_{n-1,n} a_n b_n, b_n a_n b_n)\end{aligned}$$

Hence this gives us $a_i = a_i b_i a_i$, $b_i = b_i a_i b_i (1 \le i \le n)$ and

$$P_{i,i+1} = a_i b_i P_{i,i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup P_{i,i+1} b_{i+1} a_{i+1} \left(1 \le i \le n-1\right),$$
$$Q_{i,i+1} = b_i a_i Q_{i,i+1} \cup b_i P_{i,i+1} b_{i+1} \cup Q_{i,i+1} a_{i+1} b_{i+1} \left(1 \le i \le n-1\right).$$

To show the second condition (ii) given in theorem, let us suppose that $P_{i,i+1} \ne a_i R_{i,i+1} a_{i+1}$ for some $R_{i,i+1} \subseteq A_i \times A_{i+1}$ $\left(1 \le i \le n-1\right)$. Then there exists $\left(x_i, x_{i+1}\right) \in P_{i,i+1}$ $\left(1 \le i \le n-1\right)$ such that $x_i \ne a_i x_i^{'}$ and $x_{i+1} \ne a_{i+1} x_{i+1}^{'}$ where $x_i^{'} \in A_i$ and $x_{i+1}^{'} \in A_{i+1}$. Thus $P_{i,i+1}$ cannot be equal to $a_i b_i P_{i,i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup P_{i,i+1} b_{i+1} a_{i+1}$ for all $Q_{i,i+1} \subseteq A_i \times A_{i+1}$ $\left(1 \le i \le n-1\right)$. This gives a contradiction with the regularity of $A_1 \Diamond A_2 \Diamond \cdots \Diamond A_n$.

In fact, if someone choose $P_{i,i+1} = a_i R_{i,i+1} a_{i+1}$, then we obtain that

$$\begin{aligned}
& a_i b_i P_{i,i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup P_{i,i+1} b_{i+1} a_{i+1} \\
&= \underline{a_i b_i a_i} R_{i,i+1} a_{i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup a_i R_{i,i+1} \underline{a_{i+1} b_{i+1} a_{i+1}} \\
&= a_i R_{i,i+1} a_{i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup a_i R_{i,i+1} a_{i+1} \quad (\textit{by choosing } Q_{i,i+1} = b_i a_i R_{i,i+1} a_{i+1} b_{i+1}) \\
&= a_i R_{i,i+1} a_{i+1} \cup \underline{a_i b_i a_i} R_{i,i+1} \underline{a_{i+1} b_{i+1} a_{i+1}} \cup a_i R_{i,i+1} a_{i+1} \\
&= a_i R_{i,i+1} a_{i+1} \\
&= P_{i,i+1}
\end{aligned}$$

and

$$\begin{aligned}
& b_i a_i Q_{i,i+1} \cup b_i P_{i,i+1} b_{i+1} \cup Q_{i,i+1} a_{i+1} b_{i+1} \\
&= b_i a_i Q_{i,i+1} \cup b_i a_i R_{i,i+1} a_{i+1} b_{i+1} \cup Q_{i,i+1} a_{i+1} b_{i+1} \qquad (\textit{by choosing } Q_{i,i+1} = b_i a_i R_{i,i+1} a_{i+1} b_{i+1}) \\
&= b_i \underline{a_i b_i a_i} R_{i,i+1} a_{i+1} b_{i+1} \cup b_i a_i R_{i,i+1} a_{i+1} b_{i+1} \cup b_i a_i R_{i,i+1} \underline{a_{i+1} b_{i+1} a_{i+1}} b_{i+1} \\
&= b_i a_i R_{i,i+1} a_{i+1} b_{i+1} \\
&= Q_{i,i+1}.
\end{aligned}$$

We say that, by making similar calculation as above for the case $P_{i,i+1} = b_i a_i R_{i,i+1} a_{i+1} b_{i+1}$ in this theorem, where $R_{i,i+1} \subseteq A_i \times A_{i+1}$ $\left(1 \le i \le n-1\right)$ and $a_i \in b_i^{-1}$ $\left(1 \le i \le n\right)$, it is seen that condition (ii) must hold.

Conversely, for the other part of the proof, we take $\left(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n\right) \in A_1 \Diamond A_2 \Diamond \cdots \Diamond A_n$. Thus we definitely have $a_i \in A_i$ $\left(1 \le i \le n\right)$ such that $a_i \in b_i^{-1}$. Now let us consider the union of sets
$a_i b_i P_{i,i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup P_{i,i+1} b_{i+1} a_{i+1}$ and $b_i a_i Q_{i,i+1} \cup b_i P_{i,i+1} b_{i+1} \cup Q_{i,i+1} a_{i+1} b_{i+1}$ $\left(1 \le i \le n-1\right)$.
Here, by taking $P_{i,i+1} = a_i R_{i,i+1} a_{i+1}$ and by choosing
$Q_{i,i+1} = b_i a_i R_{i,i+1} a_{i+1} b_{i+1} \subseteq A_i \times A_{i+1}$ $(1 \le i \le n-1)$,
then we get

$$a_i b_i P_{i,i+1} \cup a_i Q_{i,i+1} a_{i+1} \cup P_{i,i+1} b_{i+1} a_{i+1} = a_i R_{i,i+1} a_{i+1} = P_{i,i+1} \left(1 \leq i \leq n-1\right)$$

and

$$b_i a_i Q_{i,i+1} \cup b_i P_{i,i+1} b_{i+1} \cup Q_{i,i+1} a_{i+1} b_{i+1} = b_i a_i R_{i,i+1} a_{i+1} b_{i+1} = Q_{i,i+1} \ (1 \leq i \leq n-1).$$

As a result of this, for every $\left(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n\right) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$, there exists $\left(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n\right) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ such that

$$\begin{aligned}&\left(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n\right)\left(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n\right)\left(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n\right)\\ &= \left(a_1 b_1 a_1, a_1 b_1 P_{1,2} \cup a_1 Q_{1,2} a_2 \cup P_{1,2} b_2 a_2, \cdots, a_{n-1} b_{n-1} P_{n-1,n} \cup a_{n-1} Q_{n-1,n} a_n \cup P_{n-1,n} b_n a_n, a_n b_n a_n\right)\end{aligned}$$

and

$$\begin{aligned}&\left(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n\right)\left(a_1, P_{1,2}, a_2, \cdots, P_{n-1,n}, a_n\right)\left(b_1, Q_{1,2}, b_2, \cdots, Q_{n-1,n}, b_n\right)\\ &= \left(b_1 a_1 b_1,\ b_1 a_1 Q_{1,2} \cup b_1 P_{1,2} b_2 \cup Q_{1,2} a_2 b_2, \cdots, b_{n-1} a_{n-1} Q_{n-1,n} \cup b_{n-1} P_{n-1,n} b_n \cup Q_{n-1,n} a_n b_n,\ b_n a_n b_n\right).\end{aligned}$$

Additionally, by applying the above similar arguments for the case $P_{i,i+1} = b_i a_i R_{i,i+1} a_{i+1} b_{i+1}$, where $R_{i,i+1} \subseteq A_i \times A_{i+1}$ $\left(1 \leq i \leq n-1\right)$ and $b_i \in a_i^{-1}$ $\left(1 \leq i \leq n\right)$, the proof of the regularity of $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ is completed.

Now, we give another result for the regularity of n-generalized Schützenberger product $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ while all A_i $(1 \leq i \leq n)$ are groups.

Theorem 2.2 The n-generalized Schützenberger product $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ of n monoids A_i $\left(1 \leq i \leq n\right)$ is regular if and only if every A_i $\left(1 \leq i \leq n\right)$ are groups.

Proof. Let A_i $\left(1 \leq i \leq n\right)$ be monoids and suppose that $A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ is regular. Then for each $a_i \in A_i$, the element $\left(a_1, \left\{\left(1_{A_1}, 1_{A_2}\right)\right\}, a_2, \left\{\left(1_{A_2}, 1_{A_3}\right)\right\}, a_3, ..., \left\{\left(1_{A_{n-1}}, 1_{A_n}\right)\right\}, a_n\right) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ is regular, that is, there exists $\left(a_1^{'}, P_{1,2}, a_2^{'}, P_{2,3}, \cdots, P_{n-1,n}, a_n^{'}\right) \in A_1 \lozenge A_2 \lozenge \cdots \lozenge A_n$ such that

$$\begin{aligned}&\left(a_1, \left\{(1_{A_1}, 1_{A_2})\right\}, a_2, \left\{(1_{A_2}, 1_{A_3})\right\}, a_3, ..., \left\{(1_{A_{n-1}}, 1_{A_n})\right\}, a_n\right)\\ &= \left(a_1, \left\{\left(1_{A_1}, 1_{A_2}\right)\right\}, a_2, \left\{\left(1_{A_2}, 1_{A_3}\right)\right\}, \cdots, \left\{\left(1_{A_{n-1}}, 1_{A_n}\right)\right\}, a_n\right)\left(a_1^{'}, P_{1,2}, a_2^{'}, P_{2,3}, \cdots, P_{n-1,n}, a_n^{'}\right)\\ &\qquad \left(a_1, \left\{\left(1_{A_1}, 1_{A_2}\right)\right\}, a_2, \left\{\left(1_{A_2}, 1_{A_3}\right)\right\}, \cdots, \left\{\left(1_{A_{n-1}}, 1_{A_n}\right)\right\}, a_n\right)\\ &= \left(a_1 a_1^{'}, a_1 P_{1,2} \cup \left\{\left(1_{A_1}, 1_{A_2}\right)\right\} a_2^{'}, a_2 a_2^{'}, a_2 P_{2,3} \cup \left\{\left(1_{A_2}, 1_{A_3}\right)\right\} a_3^{'}, \cdots, a_{n-1} P_{n-1,n} \cup \left\{\left(1_{A_{n-1}}, 1_{A_n}\right)\right\} a_n^{'}, a_n a_n^{'}\right)\\ &\qquad \left(a_1, \left\{\left(1_{A_1}, 1_{A_2}\right)\right\}, a_2, \left\{\left(1_{A_2}, 1_{A_3}\right)\right\}, \cdots, \left\{\left(1_{A_{n-1}}, 1_{A_n}\right)\right\}, a_n\right)\end{aligned}$$

$$=\Big(a_1a_1^{'}a_1,\ a_1a_1^{'}\{(1_{A_1},1_{A_2})\}\cup a_1P_{1,2}a_2\cup\{(1_{A_1},1_{A_2})\}a_2^{'}a_2,\ a_2a_2^{'}a_2,\ a_2a_2^{'}\{(1_{A_2},1_{A_3})\}\cup a_2P_{2,3}a_3$$
$$\cup\{(1_{A_2},1_{A_3})\}a_3^{'}a_3,\cdots,a_{n-1}a_{n-1}^{'}\{(1_{A_{n-1}},1_{A_n})\}\cup a_{n-1}P_{n-1,n}a_n\cup\{(1_{A_{n-1}},1_{A_n})\}a_n^{'}a_n,a_na_n^{'}a_n\Big).$$

So,

$$\{(1_{A_1},1_{A_2})\}=a_1a_1^{'}\{(1_{A_1},1_{A_2})\}\cup a_1P_{1,2}a_2\cup\{(1_{A_1},1_{A_2})\}a_2^{'}a_2,$$
$$\{(1_{A_2},1_{A_3})\}=a_2a_2^{'}\{(1_{A_2},1_{A_3})\}\cup a_2P_{2,3}a_3\cup\{(1_{A_2},1_{A_3})\}a_3^{'}a_3,$$
$$\cdots$$
$$\{(1_{A_{n-1}},1_{A_n})\}=a_{n-1}a_{n-1}^{'}\{(1_{A_{n-1}},1_{A_n})\}\cup a_{n-1}P_{n-1,n}a_n\cup\{(1_{A_{n-1}},1_{A_n})\}a_n^{'}a_n.$$

Hence, we obtain

$$\left(a_{i-1}a_{i-1}^{'},1_{A_i}\right)=\{(1_{A_{i-1}},1_{A_i})\}\ (2\le i\le n)\qquad\text{and}\qquad\left(1_{A_{i-1}},a_i^{'}a_i\right)=\{(1_{A_{i-1}},1_{A_i})\}(2\le i\le n).$$

Altogether, the monoids A_i $(1\le i\le n)$ satisfy

$$\forall a_i\quad\text{and}\quad\exists a_i^{'}:\qquad a_ia_i^{'}=1_{A_i}\qquad\text{and}\qquad a_i^{'}a_i=1_{A_i},$$

all of which statements imply that the monoids A_i $(1\le i\le n)$ in assumptions are groups.

Conversely, let us suppose that each A_i $(1\le i\le n)$ be groups. Then, the regularity of n-generalized Schützenberger product of n groups is obvious.

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