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Research Article

Bimodules and matched pairs of noncommutative BiHom-(pre)-Poisson algebras

Ismail Laraiedh[1,2]

[1] *Departement of Mathematics, College of Sciences and Humanities Al Quwaiiyah, Shaqra University, 11971, Saudi Arabia*

[2] *Departement of Mathematics, Faculty of Sciences, Sfax University, BP 1171, 3000 Sfax, Tunisia*

Abstract

The purpose of this paper is to introduce the notion of noncommutative BiHom-pre-Poisson algebra. Also, we establish the bimodules and matched pairs of noncommutative BiHom-(pre)-Poisson algebras. Their related relevant properties are also given. Finally, we exploit the notion of $\mathcal{O}$-operator to illustrate the relations existing between noncommutative BiHom-Poisson and noncommutative BiHom pre-Poisson algebras.

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1. Introduction

Algebraic structure appeared in the Physics literature related to string theory, vertex models in conformal field theory, quantum mechanics and quantum field theory, such as the q-deformed Heisenberg algebras, q-deformed oscillator algebras, q-deformed Witt, q-deformed Virasoro algebras and related q-deformations of infinite-dimensional algebras [4, 14–20, 24, 25, 33–35].

Hom-type algebras satisfy a modified version of the Jacobi identity involving a homomorphism, and were called Hom-Lie algebras by Hartwig, Larsson and Silvestrov in [22], [29]. Afterwards, Hom-analogues of various classical algebraic structures have been introduced in the literature, such as Hom-associative algebras, Hom-dendriform algebras, Hom-pre-Lie algebras Hom-(pre)-Poisson algebras etc [1, 5–8, 10, 37–40, 43–48].

The notion of a noncommutative Poisson algebra was first given by Xu in [42]. A noncommutative Poisson algebra consists of an associative algebra together with a Lie algebra structure, satisfying the Leibniz identity. Noncommutative Poisson algebras are used in many fields in mathematics and physics. Aguiar introduced the notion of a pre-Poisson algebra in [3] and constructed many examples. A noncommutative pre-Poisson algebra contains a dendriform algebra and a pre-Lie algebra such that some compatibility conditions are satisfied. More applications of Poisson algebras, pre-Poisson algebra can be found in [11, 12, 26–28, 41].

Email address: ismail.laraiedh@gmail.com
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A generalization of this approach led the authors of [21] to introduce BiHom-associative and BiHom-Lie algebras. In these algebras, the associativity of the multiplication is twisted by two commuting homomorphisms. When these two homomorphisms are equal, one recovers Hom-associative and Hom-Lie algebras.

Noncommutative BiHom-Poisson algebras was first introduced in [36] and studied in [2, 9]. Let $(A, \cdot, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ be a noncommutative BiHom-Poisson algebra. A noncommutative BiHom-Poisson A-module V is simultaneously a BiHom-associative A-module $(\lambda, \beta_1, \beta_2, V)$ and a BiHom-Lie A-module $(\rho, \beta_1, \beta_2, V)$ satisfying the BiHom-Leibniz conditions:

$$\rho(\alpha_1\alpha_2(x), \lambda(y, v)) = \lambda(\{\alpha_2(x), y\}, \beta_2(v)) + \lambda(\alpha_2(y), \rho(\alpha_1(x), v)),$$
$$\rho(\mu(\alpha_2(x), y), \beta_2(v)) = \lambda(\alpha_1\alpha_2(x), \rho(y, v)) + \lambda(\alpha_2(y), \rho(\alpha_1(x), v)),$$

for $x, y \in A,\ v \in V$, (for more details see Definition 5.3 in[9]). A noncommutative BiHom-pre-Poisson algebra gives rise to a noncommutative BiHom-Poisson algebra naturally through the sub-adjacent BiHom-associative algebra of the BiHom-dendriform algebra [31] and the sub-adjacent BiHom-Lie algebra of the BiHom-pre-Lie algebra [32]. We also introduce the notion of $\mathcal{O}$-operators of noncommutative BiHom-Poisson algebra and we will prove that given a noncommutative BiHom-Poisson algebra and an $\mathcal{O}$-operator give rise to a noncommutative BiHom-pre-Poisson algebra. All that is illustrated by the following diagram

$$\begin{array}{ccc} \text{BiHom-dendrif. alg + BiHom-pre-Lie alg} & \longrightarrow & \text{noncomm BiHom-pre-Pois. alg} \\ \downarrow\uparrow & & \downarrow\uparrow \\ \text{BiHom-associative alg + BiHom-Lie alg} & \longrightarrow & \text{noncomm BiHom-Pois. alg} \end{array}$$

The paper is organized as follows. In section 2, we introduce the notions of representation and matched pair of noncommutative BiHom-Poisson algebra with a connection to a representations and matched pairs of BiHom-Lie algebra and BiHom-associative algebra. In section 3, we establish definition of noncommutative BiHom-pre-Poisson algebra and we give some key of constructions. Their bimodule and matched pair are defined and their related relevant properties are also given. In section 4, we study the notion of $\mathcal{O}$-operator and we illustrate the relations existing between noncommutative BiHom-Poisson and noncommutative BiHom pre-Poisson algebras.

Throughout this paper, all vector spaces are assumed to be over a field $\mathbb{K}$ of characteristic different from 2.

2. Representation and matched pair of noncommutative BiHom-Poisson algebras

In this section we recall the definition of noncommutative BiHom-Poisson algebra [36] and we study the representation and the matched pair of noncommutative BiHom-Poisson algebras with a connection to a representations and matched pairs of BiHom-associative algebra and BiHom-Lie algebra. Moreover we provide some key constructions.

Definition 2.1. A BiHom-module is a triple (V, α_V, β_V) consisting of a $\mathbb{K}$-vector space V and two linear maps $\alpha_V, \beta_V : V \longrightarrow V$ such that $\alpha_V\beta_V = \beta_V\alpha_V$. A morphism $f : (V, \alpha_V, \beta_V) \to (W, \alpha_W, \beta_W)$ of BiHom-modules is a linear map $f : V \longrightarrow W$ such that $f\alpha_V = \alpha_W f$ and $f\beta_V = \beta_W f$.

Definition 2.2. A BiHom-algebra is a quadruple $(A, \mu, \alpha_1, \alpha_2)$ in which (A, α_1, α_2) is a BiHom-module and $\mu : A^{\otimes 2} \to A$ is a bilinear map. The BiHom-algebra $(A, \mu, \alpha_1, \alpha_2)$ is said to be multiplicative if $\alpha_1 \circ \mu = \mu \circ \alpha_1^{\otimes 2}$ and $\alpha_2 \circ \mu = \mu \circ \alpha_2^{\otimes 2}$ (BiHom-multiplicativity).

Let us recall now the definition and the notion of bimodule of a BiHom-associative given in [21].

Definition 2.3. A BiHom-associative algebra is a quadruple $(A, \mu, \alpha_1, \alpha_2)$ consisting of a vector space A on which the operation $\mu : A\otimes A \to A$ and $\alpha_1, \alpha_2 : A \to A$ are linear maps satisfying

$$\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1, \tag{2.1}$$
$$\alpha_1 \circ \mu(x, y) = \mu(\alpha_1(x), \alpha_1(y)), \tag{2.2}$$
$$\alpha_2 \circ \mu(x, y) = \mu(\alpha_2(x), \alpha_2(y)), \tag{2.3}$$
$$\mu(\alpha_1(x), \mu(y, z)) = \mu(\mu(x, y), \alpha_2(z)), \tag{2.4}$$

for any $x, y, z \in A$.

Remark 2.4. Clearly, a Hom-associative algebra (A, μ, α) can be regarded as a BiHom-associative algebra (A, μ, α, α).

Definition 2.5 ([21])**.** Let $(A, \mu, \alpha_1, \alpha_2)$ be a BiHom-associative algebra. A left A-module is a triple (M, β_1, β_2), where M is a linear space, β_1 , $\beta_2 : M \to M$ are linear maps, with, in addition, another linear map: $A \otimes M \to M, a \otimes m \mapsto a \cdot m$, such that, for all $a, a' \in A, m \in M$:

$$\beta_1 \circ \beta_2 = \beta_2 \circ \beta_1,\ \beta_1(a \cdot m) = \alpha_1(a) \cdot \beta_1(m),$$
$$\beta_2(a \cdot m) = \alpha_2(a) \cdot \beta_2(m),\ \alpha_1(a) \cdot (a' \cdot m) = (aa') \cdot \beta_2(m).$$

Definition 2.6. Let $(A, \cdot, \alpha_1, \alpha_2)$ be a BiHom-associative algebra, and let (V, β_1, β_2) be a BiHom-module. Let $l, r : A \to gl(V)$, be two linear maps. The quintuple $(l, r, \beta_1, \beta_2, V)$ is called a bimodule of A if

$$l(x \cdot y)\beta_2(v) = l(\alpha_1(x))l(y)v, \tag{2.5}$$
$$r(x \cdot y)\beta_1(v) = r(\alpha_2(y))r(x)v, \tag{2.6}$$
$$l(\alpha_1(x))r(y)v = r(\alpha_2(y))l(x)v, \tag{2.7}$$
$$\beta_1(l(x)v) = l(\alpha_1(x))\beta_1(v), \tag{2.8}$$
$$\beta_1(r(x)v) = r(\alpha_1(x))\beta_1(v), \tag{2.9}$$
$$\beta_2(l(x)v) = l(\alpha_2(x))\beta_2(v), \tag{2.10}$$
$$\beta_2(r(x)v) = r(\alpha_2(x))\beta_2(v), \tag{2.11}$$

for all $x, y \in A, v \in V$.

Proposition 2.7. *Let $(l, r, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-associative algebra $(A, \cdot, \alpha_1, \alpha_2)$. Then, the direct sum $A \oplus V$ of vector spaces is turned into a BiHom-associative algebra by defining multiplication in $A \oplus V$ by*

$$(x_1 + v_1) \cdot' (x_2 + v_2) := x_1 \cdot x_2 + (l(x_1)v_2 + r(x_2)v_1),$$
$$(\alpha_1 \oplus \beta_1)(x_1 + v_1) := \alpha_1(x_1) + \beta_1(v_1),$$
$$(\alpha_2 \oplus \beta_2)(x_1 + v_1) := \alpha_2(x_1) + \beta_2(v_1),$$

for all $x_1, x_2 \in A, v_1, v_2 \in V$.

We denote such a BiHom-associative algebra by $A \times_{l,r,\alpha_1,\alpha_2,\beta_1,\beta_2} V$.

Theorem 2.8 ([23])**.** *Let $(A, \cdot_A, \alpha_1, \alpha_2)$ and $(B, \cdot_B, \beta_1, \beta_2)$ be two BiHom-associative algebras. Suppose that there are linear maps $l_A, r_A : A \to gl(B)$ and $l_B, r_B : B \to gl(A)$ such that $(l_A, r_A, \beta_1, \beta_2, B)$ is a bimodule of A and $(l_B, r_B, \alpha_1, \alpha_2, A)$ is a bimodule of B satisfy*

$$l_A(\alpha_1(x))(a \cdot_B b) = l_A(r_B(a)x)\beta_2(b) + (l_A(x)a) \cdot_B \beta_2(b) \tag{2.12}$$

$$r_A(\alpha_2(x))(a \cdot_B b) = r_A(l_B(b)x)\beta_1(a) + \beta_1(a) \cdot_B (r_A(x)b) \tag{2.13}$$

$$l_A(l_B(a)x)\beta_2(b)+(r_A(x)a)\cdot_B\beta_2(b)-r_A(r_B(b)x)\beta_1(a)-\beta_1(a)\cdot_B(l_A(x)b)=0 \quad (2.14)$$

$$l_B(\beta_1(a))(x\cdot_A y)=l_B(r_A(x)a)\alpha_2(y)+(l_B(a)x)\cdot_A\alpha_2(y) \quad (2.15)$$

$$r_B(\beta_2(a))(x\cdot_A y)=r_B(l_A(y)a)\alpha_1(x)+\alpha_1(x)\cdot_A(r_B(a)y) \quad (2.16)$$

$$l_B(l_A(x)a)\alpha_2(y)+(r_B(a)x)\cdot_A\alpha_2(y)-r_B(r_A(y)a)\alpha_1(x)-\alpha_1(x)\cdot_A(l_B(a)y)=0, \quad (2.17)$$

for any, $x,y\in A,\ a,b\in B$. Then $(A,B,l_A,r_A,\beta_1,\beta_2,l_B,r_B,\alpha_1,\alpha_2)$ is called a matched pair of BiHom-associative algebras. In this case, there is a BiHom-associative algebra structure on the direct sum $A\oplus B$ of the underlying vector spaces of A and B given by

$$\begin{array}{lcl}(x+a)\cdot(y+b) & := & x\cdot_A y+(l_A(x)b+r_A(y)a)+a\cdot_B b+(l_B(a)y+r_B(b)x),\\ (\alpha_1\oplus\beta_1)(x+a) & := & \alpha_1(x)+\beta_1(a),\\ (\alpha_2\oplus\beta_2)(x+a) & := & \alpha_2(x)+\beta_2(a).\end{array}$$

Proof. We prove the axiom (2.4) in $A\oplus B$ as the others relations are proved analogously. For any $x,y,z\in A$ and $a,b,c\in B$ we have

$$\begin{array}{cl} & (\alpha_1+\beta_1)(x+a)\cdot((y+b)\cdot(z+c))\\ = & (\alpha_1(x)+\beta_1(a))[y\cdot_A z+l_B(b)z+r_B(c)y+b\cdot c+l_A(y)c+r_A(z)b)\\ = & \alpha_1(x)\cdot_A(y\cdot_A z)+\alpha_1(x)\cdot_A l_B(b)z+\alpha_1(x)\cdot_A r_B(c)y+l_B(\beta_1(a))(y\cdot_A z)\\ + & l_B(\beta_1(a))l_B(b)z+l_B(\beta_1(a))r_B(c)y+r_B(b\cdot_B c)\alpha_1(x)+r_B(l_A(y)c)\alpha_1(x)\\ + & r_B(r_A(z)b)\alpha_1(x)+\beta_1(a)\cdot_B(b\cdot_B c)+\beta_1(a)\cdot_B l_A(y)c+\beta_A(\alpha_1(x))l_A(y)c\\ + & l_A(\alpha_1(x))r_A(z)b+r_A(y\cdot_A z)\beta_1(a)+r_A(l_A(b)z)\beta_1(a)+r_A(r_B(c)y)\beta_1(a).\end{array}$$

In the other hand, we have

$$\begin{array}{cl} & ((x+a)\cdot(y+b))\cdot(\alpha_2+\beta_2)(z+c)\\ = & (x\cdot_A y+l_B(a)y+r_B(b)x+a\cdot_B b+l_A(x)b+r_A(y)a)\cdot(\alpha_2(z)+\beta_2(c))\\ = & (x\cdot_A y)\cdot_A\alpha_2(z)+l_B(a)y\cdot_A\alpha_2(z)+r_B(b)x\cdot_A\alpha_2(z)+l_B(a\cdot_B b)\alpha_2(z)\\ + & l_B(l_A(x)b)\alpha_2(z)+l_B(r_A(y)a)\alpha_2(z)+r_B(\beta_2(c))(x\cdot_A y)+r_A(\beta_2(c))l_B(a)y\\ + & r_B(\beta_2(c))r_B(b)x+(a\cdot_B b)\cdot_B\beta_2(c)+(l_A(x)b)\cdot_B\beta_2(c)+(r_A(y)a)\cdot_B\beta_2(c)\\ + & r_A(\alpha_2(z))(a\cdot_B b)+r_A(\alpha_2(z))(l_A(x)b)+r_A(\alpha_2(z))(r_A(y)a)+l_A(x\cdot_A y)\beta_2(c)\\ + & l_A(l_B(a)y)\beta_2(c)+(r_B(b)x)\beta_2(c).\end{array}$$

Then by (2.4) and (2.12)-(2.17), we deduce that

$$(\alpha_1+\beta_1)(x+a)\cdot((y+b)\cdot(z+c))=((x+a)\cdot(y+b))\cdot(\alpha_2+\beta_2)(z+c).$$

This finishes the proof. □

We denote this BiHom-associative algebra by $A\bowtie^{l_A,r_A,\beta_1,\beta_2}_{l_B,r_B,\alpha_1,\alpha_2}B$.

Let us recall now the definition and the notion of bimodule of a BiHom-Lie algebra given in [21].

Definition 2.9. A BiHom-Lie algebra is a quadruple $(A,[\cdot,\cdot],\alpha_1,\alpha_2)$ consisting of a linear space A, a bilinear map $[\cdot,\cdot]:\wedge^2A\to A$ and two linear maps $\alpha_1,\alpha_2:A\to A$ satisfying $\alpha_1\circ\alpha_2=\alpha_2\circ\alpha_1$ and the following conditions, $\forall x,y,z\in A$,

(1) $\alpha_1([x,y])=[\alpha_1(x),\alpha_1(y)]$ and $\alpha_2([x,y])=[\alpha_2(x),\alpha_2(y)]$,
(2) $[\alpha_2(x),\alpha_1(y)]=-[\alpha_2(y),\alpha_1(x)]$,
(3) $[\alpha_2^2(x),[\alpha_2(y),\alpha_1(z)]]+[\alpha_2^2(z),[\alpha_2(x),\alpha_1(y)]]+[\alpha_2^2(y),[\alpha_2(z),\alpha_1(x)]]=0.$

Definition 2.10. Let $(A,[\cdot,\cdot],\alpha_1,\alpha_2)$ be a BiHom-Lie algebra and (V,β_1,β_2) be a BiHom-module. Let $\rho:A\to gl(V)$ be a linear map. The quadruple (ρ,β_1,β_2,V) is called a representation of A if for all $x,y\in A,\ v\in V$, we have

$$\rho([\alpha_2(x),y])\beta_2(v) = \rho(\alpha_1\alpha_2(x))\circ\rho(y)v-\rho(\alpha_2(y))\circ\rho(\alpha_1(x))v, \quad (2.18)$$

$$\beta_1(\rho(x)v) = \rho(\alpha_1(x))\beta_1(v), \quad (2.19)$$

$$\beta_2(\rho(x)v) = \rho(\alpha_2(x))\beta_2(v). \tag{2.20}$$

Proposition 2.11. *Let (ρ,β_1,β_2,V) be a representation of a BiHom-Lie algebra $(A,[\cdot,\cdot],\alpha_1,\alpha_2)$ such that α_1, β_2 are bijectives. Then, the direct sum $A\oplus V$ of vector spaces is turned into a BiHom-Lie algebra by defining the multiplication in $A\oplus V$ by*

$$\begin{aligned}
[x_1+v_1,x_2+v_2]_\rho &:= [x_1,x_2]+\rho(x_1)v_2-\rho(\alpha_1^{-1}\alpha_2(x_2))\beta_1\beta_2^{-1}(v_1),\\
(\alpha_1\oplus\beta_1)(x_1+v_1) &:= \alpha_1(x_1)+\beta_1(v_1),\\
(\alpha_2\oplus\beta_2)(x_1+v_1) &:= \alpha_2(x_1)+\beta_2(v_1),
\end{aligned}$$

for all $x_1,x_2\in A, v_1,v_2\in V$.

We denote such a BiHom-Lie algebra by $(A\oplus V,[\cdot,\cdot]_\rho,\alpha_1+\beta_1,\alpha_2+\beta_2)$, or $A\times_{\rho,\alpha_1,\alpha_2,\beta_1,\beta_2} V$.

Now, we introduce the notion of matched pair of BiHom-Lie algebra.

Theorem 2.12. *Let $(A,\{\cdot,\cdot\}_A,\alpha_1,\alpha_2)$ and $(B,\{\cdot,\cdot\}_B,\beta_1,\beta_2)$ be two BiHom-Lie algebras. Suppose that there are linear maps $\rho_A: A\rightarrow gl(B)$ and $\rho_B: B\rightarrow gl(A)$ such that $(\rho_A,\beta_1,\beta_2,B)$ is a bimodule of A and $(\rho_B,\alpha_1,\alpha_2,A)$ is a bimodule of B satisfies*

$$\begin{aligned}
\rho_B(\beta_1\beta_2(a))\{\alpha_1(x),\alpha_1^2(y)\}_A &= \{\rho_B(\beta_2(a))\alpha_1(x),\alpha_1^2\alpha_2(y)\}_A+\{\alpha_1\alpha_2(x),\rho_B(\beta_1(a))\alpha_1^2(y)\}_A\\
&+ \rho_B(\rho_A(\alpha_2^2(y))\beta_1(a))\alpha_1^2(x)-\rho_B(\rho_A(\alpha_2(x))\beta_1(a))\alpha_2\alpha_1^2(y),
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\rho_A(\alpha_1\alpha_2(x))\{\beta_1(a),\beta_1^2(b)\}_B &= \{\rho_A(\alpha_2(x))\beta_1(a),\beta_1^2\beta_2(b)\}_B+\{\beta_1\beta_2(a),\rho_A(\alpha_1(x))\beta_1^2(b)\}_B\\
&+ \rho_A(\rho_B(\beta_2^2(b))\alpha_1(x))\beta_1^2(a)-\rho_A(\rho_B(\beta_2(a))\alpha_1(x))\beta_2\beta_1^2(b),
\end{aligned} \tag{2.22}$$

for any $x,y\in A,\ a,b\in B$. Then $(A,B,\rho_A,\beta_1,\beta_2,\rho_B,\alpha_1,\alpha_2)$ is called a matched pair of BiHom-Lie algebras. Moreover, assume that $(A,\{\cdot,\cdot\}_A,\alpha_1,\alpha_2)$ and $(B,\{\cdot,\cdot\}_B,\beta_1,\beta_2)$ be two regular BiHom-Lie algebras, then there exists a BiHom-Lie algebra structure on the vector space $A\oplus B$ of the underlying vector spaces of A and B given by

$$\begin{aligned}
[x+a,y+b] &:= \{x,y\}_A+\rho_A(x)b-\rho_A(\alpha_1^{-1}\alpha_2(y))\beta_1\beta_2^{-1}(a)\\
&+ \{a,b\}_B+\rho_B(a)y-\rho_B(\beta_1^{-1}\beta_2(b))\alpha_1\alpha_2^{-1}(x),\\
(\alpha_1\oplus\beta_1)(x+a) &:= \alpha_1(x)+\beta_1(a),\\
(\alpha_2\oplus\beta_2)(x+a) &:= \alpha_2(x)+\beta_2(a).
\end{aligned}$$

Proof. First, we prove the BiHom-multiplicativity in $A\oplus B$. For all $x,y\in A,\ a,b\in B$, we have

$$\begin{aligned}
&(\alpha_1+\beta_1)[x+a,y+b]\\
=&(\alpha_1+\beta_1)(\{x,y\}_A+\rho_A(x)b-\rho_A(\alpha_1^{-1}\alpha_2(y))\beta_1\beta_2^{-1}(a)\\
&+\{a,b\}_B+\rho_B(a)y-\rho_B(\beta_1^{-1}\beta_2(b))\alpha_1\alpha_2^{-1}(x))\\
=&\alpha_1(\{x,y\}_A)+\alpha_1\rho_B(a)y-\alpha_1\rho_B(\beta_1^{-1}\beta_2(b))\alpha_1\alpha_2^{-1}(x)\\
&+\beta_1\rho_A(x)b-\beta_1\rho_A(\alpha_1^{-1}\alpha_2(y))\beta_1\beta_2^{-1}(a)+\beta_1\{a,b\}_B\\
=&\{a_1(x),\alpha_1(y)\}_A+\rho_B(\beta_1(a))\alpha_1(y)-\rho_B(\beta_1^{-1}\beta_2(\beta_1(b))\alpha_1\alpha_2^{-1}(\alpha_1(x))\\
&+\rho_A(\alpha_1(x))\beta_1(b)-\rho_A(\alpha_1^{-1}\alpha_2(\alpha_1(y))\beta_1\beta_2^{-1}(\beta_1(a))+\{b_1(a),\beta_1(b)\}_B\\
=&[\alpha_1(x)+\beta_1(a),\alpha_1(y)+\beta_1(b)]\\
=&[(\alpha_1+\beta_1)(x+a),(\alpha_1+\beta_1)(y+b)].
\end{aligned}$$

In the same way, we have $(\alpha_2+\beta_2)[x+a,y+b]=[(\alpha_2+\beta_2)(x+a),(\alpha_2+\beta_2)(y+b)]$.
Now, we prove the Bihom-skewsymmetry in $A\oplus B$. For all $x,y\in A,\ a,b\in B$, we have

$$[(\alpha_2+\beta_2)(x+a),(\alpha_1+\beta_1)(y+b)]$$

$$
\begin{aligned}
&=[\alpha_2(x)+\beta_2(a),\alpha_1(y)+\beta_1(b)]\\
&=\{\alpha_2(x),\alpha_1(y)\}_A+\rho_A(\alpha_2(x))\beta_1(b)-\rho_A(\alpha_1^{-1}\alpha_2(\alpha_1(y))\beta_1\beta_2^{-1}(\beta_2(a))\\
&\quad+\{\beta_2(a),\beta_1(b)\}_B+\rho_B(\beta_2(a))\alpha_1(y)-\rho_B(\beta_1^{-1}\beta_2(\beta_1(b)))\alpha_1\alpha_2^{-1}(\alpha_2(x))\\
&=-\{a_2(y),\alpha_1(x)\}_A-\rho_A(\alpha_2(y))\beta_1(a)+\rho_A(\alpha_1^{-1}\alpha_2(\alpha_1(x))\beta_1\beta_2^{-1}(\beta_2(b))-\{\beta_2(b),\beta_1(a)\}_B\\
&\quad-\rho_B(\beta_2(b))\alpha_1(x)+\rho_B(\beta_1^{-1}\beta_2(\beta_1(a)))\alpha_1\alpha_2^{-1}(\alpha_2(y))\\
&=-[(\alpha_2+\beta_2)(y+b),(\alpha_1+\beta_1)(x+a)].
\end{aligned}
$$

Finally, we prove the Bihom-Jacobi identity in $A\oplus B$. For any $x,y,z\in A,\ a,b,c\in B$, we have

$$
\begin{aligned}
&[(\alpha_2+\beta_2)^2(x+a),[(\alpha_2+\beta_2)(y+b),(\alpha_1+\beta_1)(z+c)]]\\
&=[\alpha_2^2(x)+\beta_2^2(a),[\alpha_2(y)+\beta_2(b),\alpha_1(z)+\beta_1(c)]]\\
&=[\alpha_2^2(x)+\beta_2^2(a),\{\alpha_2(y),\alpha_1(z)\}_1+\rho_A(\alpha_2(y))\beta_1(c)-\rho_A(\alpha_1^{-1}\alpha_2(\alpha_1(z))))\beta_1\beta_2^{-1}(\beta_2(b))\\
&\quad+\{\beta_2(b),\beta_1(c)\}_B+\rho_B(\beta_2(b))\alpha_1(z)-\rho_B(\beta_1^{-1}\beta_2(\beta_1(c)))\alpha_1\alpha_2^{-1}(\alpha_2(y))]\\
&=[\alpha_2^2(x)+\beta_2^2(a),\{\alpha_2(y),\alpha_1(z)\}_1+\rho_A(\alpha_2(y))\beta_1(c)-\rho_A(\alpha_2(z))\beta_1(b)\\
&\quad+\{\beta_2(b),\beta_1(c)\}_B+\rho_B(\beta_2(b))\alpha_1(z)-\rho_B(\beta_2(c))\alpha_1(y)]\\
&=\{\alpha_2^2(x),\{\alpha_2(y),\alpha_1(z)\}_A\}_A+\{\alpha_2^2(x),\rho_B(\beta_2(b))\alpha_1(z)\}_A-\{\alpha_2^2(x),\rho_B(\beta_2(c))\alpha_1(y)\}_A\\
&\quad+\rho_A(\alpha_2^2(x))\rho_A(\alpha_2(y))\beta_1(c)-\rho_A(\alpha_2^2(x))\rho_A(\alpha_2(z))\beta_1(b)+\rho_A(\alpha_2^2(x))\{\beta_2(b),\beta_1(c)\}_B\\
&\quad-\rho_A(\{\alpha_1^{-1}\alpha_2^2(y),\alpha_2(z)\}_A)\beta_1\beta_2(a)-\rho_A(\rho_B(\beta_1^{-1}\beta_2^2(b))\alpha_2(z))\beta_1\beta_2(a)\\
&\quad+\rho_A(\rho_B(\beta_1^{-1}\beta_2^2(c))\alpha_2(y))\beta_1\beta_2(a)+\{\beta_2^2(a),\rho_A(\alpha_2(y))\beta_1(c)\}_B-\{\beta_2^2(a),\rho_A(\alpha_2(z))\beta_1(b)\}_B\\
&\quad+\{\beta_2^2(a),\{\beta_2(b),\beta_1(c)\}_B\}_B+\rho_B(\beta_2^2(a)\{\alpha_2(y),\alpha_1(z)\}_A+\rho_B(\beta_2^2(a))\rho_B(\beta_2(b))\alpha_1(z)\\
&\quad-\rho_B(\beta_2^2(a))\rho_B(\beta_2(c))\alpha_1(y)-\rho_B(\rho_A(\alpha_1^{-1}\alpha_2^2(y))\beta_2(c))\alpha_1\alpha_2(x)\\
&\quad+\rho_B(\rho_A(\alpha_1^{-1}\alpha_2^2(z))\beta_1(b))\alpha_1\alpha_2(x)-\rho_B(\{\beta_1^{-1}\beta_2^2(b),\beta_2(c)\}_B)\alpha_1\alpha_2(x),
\end{aligned}
$$

by a direct computation we verify that $\circlearrowleft_{x,y,z}[(\alpha_2+\beta_2)^2(x+a),[(\alpha_2+\beta_2)(y+b),(\alpha_1+\beta_1)(z+c)]]=0$. This ends the proof. □

We denote this BiHom-Lie algebra by $A\bowtie^{\rho_A,\beta_1,\beta_2}_{\rho_B,\alpha_1,\alpha_2}B$.

Definition 2.13. A noncommutative BiHom-Poisson algebra is a 5-tuple $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$, where $(A,\cdot,\alpha_1,\alpha_2)$ is a BiHom-associative algebra and $(A,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ is a BiHom-Lie algebra, such that the BiHom-Leibniz identity:

$$\{\alpha_1\alpha_2(x),y\cdot z\}=\{\alpha_2(x),y\}\cdot\alpha_2(z)+\alpha_2(y)\cdot\{\alpha_1(x),z\}. \tag{2.23}$$

Proposition 2.14 ([2])**.** *Let $(A,\cdot,\alpha_1,\alpha_2)$ be a regular BiHom-associative algebra. Then $A^-=(A,\{\cdot,\cdot\},\cdot,\alpha,\beta)$ is a regular noncommutative BiHom-Poisson algebra, where for all $x,y\in A$, $\{x,y\}=x\cdot y-\alpha_1^{-1}\alpha_2(y)\cdot\alpha_1\alpha_2^{-1}(x)$.*

In the following we introduce the notions of representation and matched pair of noncommutative BiHom-Poisson algebras.

Definition 2.15. Let $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ be a noncommutative BiHom-Poisson algebra. A representation of A is a 6-tuple $(l,r,\rho,\beta_1,\beta_2,V)$ such that (l,r,β_1,β_2,V) is a bimodule of the BiHom-associative algebra $(A,\cdot,\alpha_1,\alpha_2)$ and (ρ,β_1,β_2,V) is a representation of the BiHom-Lie algebra $(A,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ satisfying, for all $x,y\in A,\ v\in V$.

$$
\begin{aligned}
l(\{\alpha_2(x),y\})\beta_2(v)&=\rho(\alpha_1\alpha_2(x))l(y)v-l(\alpha_2(y))\rho(\alpha_1(x))v, &(2.24)\\
r(\{\alpha_1(x),y\})\beta_2(v)&=\rho(\alpha_1\alpha_2(x))r(y)v-r(\alpha_2(y))\rho(\alpha_2(x))v, &(2.25)\\
\rho(x\cdot y)\beta_1\beta_2(v)&=l(\alpha_1(x))\rho(y)\beta_1(v)+r(\alpha_1(y))\rho(x)\beta_2(v). &(2.26)
\end{aligned}
$$

Proposition 2.16. *Let $(l,r,\rho,\beta_1,\beta_2,V)$ be a representation of noncommutative BiHom-Poisson algebra $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ such that $\alpha_1,\ \beta_2$ are bijectives. Then $(A\oplus V,\cdot',\{\cdot,\cdot\}',\alpha_1+\beta_1,\alpha_2+\beta_2)$ is a noncommutative BiHom-Poisson algebra, where $(A\oplus V,\cdot',\alpha_1+\beta_1,\alpha_2+\beta_2)$ is* *the semi-direct product BiHom-associative algebra* $A\times_{l,r,\alpha_1,\alpha_2,\beta_1,\beta_2}V$ *and* $(A\oplus V,\{\cdot,\cdot\}',\alpha_1+\beta_1,\alpha_2+\beta_2)$ *is the semi-direct product BiHom-Lie algebra* $A\times_{\rho,\alpha_1,\alpha_2,\beta_1,\beta_2}V$

Proof. We prove only the BiHom-Leibniz identity in $A\oplus V$. For all $x_1,x_2,x_3\in A$, $v_1,v_2,v_3\in V$.

$$\begin{aligned}
&\{(\alpha_1\alpha_2+\beta_1\beta_2)(x_1+v_1),(x_2+v_2)\cdot'(x_3+v_3)\}'\\
&-\{(\alpha_2+\beta_2)(x_2+v_2)\cdot'\{(\alpha_1+\beta_1)(x_1+v_1),x_3+v_3\}'\\
&-(\alpha_2+\beta_2)(x_2+v_2)\cdot'\{(\alpha_1+\beta_1)(x_1+v_1),x_3+v_3\}'\\
=&\{\alpha_1\alpha_2(x_1)+\beta_1\beta_2(v_1),x_2\cdot x_3+l(x_2)v_3+r(x_3)v_2\}'\\
&-\Big(\{\alpha_2(x_1),x_2\}+\rho(\alpha_2(x_1))v_2-\rho(\alpha_1^{-1}\alpha_2(x_2))\beta_1(v_1)\Big)\cdot'(\alpha_2(x_1)+\beta_2(v_3))\\
&-(\alpha_2(x_2)+\beta_2(v_2))\cdot'\Big(\{\alpha_1(x_1),x_3\}+\rho(\alpha_1(x_1))v_3-\rho(\alpha_1^{-1}\alpha_2(x_3))\beta_1^2\beta_2^{-1}(v_1)\Big)\\
=&\{\alpha_1\alpha_2(x_1),x_2\cdot x_3\}+\rho(\alpha_1\alpha_2(x_2))l(x_2)v_3\\
&+\rho(\alpha_1\alpha_2(x_1))r(x_3)v_2-\rho(\alpha_1^{-1}\alpha_2(x_2(x_3))\beta_1^2(v_1)\\
&-\{\alpha_2(x_1),x_2\}\cdot\alpha_2(x_3)-l(\{a_2(x_1),x_2\})\beta_2(v_3))\\
&-r(\alpha_2(x_3))\rho(\alpha_2(x_1))v_2+r(\alpha_2(x_3))\rho(\alpha_1^{-1}\alpha_2(x_2))\beta_1(v_1)\\
&-\alpha_2(x_2)\cdot\{\alpha_1(x_1),x_3\}-l(\alpha_2(x_2))\rho(\alpha_1(x_1))v_3\\
&+l(\alpha_2(x_2))\rho(\alpha_1^{-1}\alpha_2(x_3))\beta_1^2\beta_2^1(v_1)-r(\{\alpha_1(x_1),x_3\})\beta_2\\
=&\Big(\{\alpha_1\alpha_2(x_1),x_2\cdot x_3\}-\{`\alpha_2(x_1),x_2\}\cdot\alpha_2(x_3)-\alpha_2(x_2)\{\alpha_1(x_1),x_3\}\Big)\\
&+\Big(\rho(\alpha_1\alpha_2(x_1))l(x_2)v_3-l(\{\alpha_2(x_1),x_2\})\beta_2(v_3)-l(\alpha_2(x_2))\rho(\alpha_1(x_1))v_3\Big)\\
&+\Big(\rho(\alpha_1\alpha_2(x_1))r(x_3)v_2-r(\alpha_2(x_3))\rho(\alpha_2(x_1))v_2-r(\{\alpha_1(x_1)),x_3\})\beta_2(v_2)\Big)\\
&-\Big(\rho(\alpha_1^{-1}\alpha_2(x_2\cdot x_3))\beta_1^2(v_1)+r(\alpha_2(x_3))\rho(\alpha_1^{-1}\alpha_2(x_2))\beta_1(v_1)\\
&+l(\alpha_2(x_2))\rho(\alpha_1^{-1}\alpha_2(x_3))\beta_1^2\beta_2^{-1}(v_1)\Big)=0+0+0+0=0.
\end{aligned}$$

Then $(A\oplus V,\cdot',\{\cdot,\cdot\}',\alpha_1+\beta_1,\alpha_2+\beta_2)$ is a noncommutative BiHom-Poisson algebra and we denote by $A\times_{l,r,\rho,\alpha_1,\alpha_2,\beta_1,\beta_2}V$. □

Example 2.17. Let $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ be a noncommutative BiHom-Poisson algebra. Then $(L.,R.,ad,\alpha_1,\alpha_2,A)$ is a regular representation of A, where $L.(x)y=x\cdot y$, $R.(x)y=y\cdot x$ and $ad(x)y=[x,y]$, for all $x,y\in A$.

Theorem 2.18. *Let* $(A,\cdot_A,\{\cdot,\cdot\}_A,\alpha_1,\alpha_2)$ *and* $(B,\cdot_B,\{\cdot,\cdot\}_B,\beta_1,\beta_2)$ *be two noncommutative BiHom-Poisson algebras. Suppose that there are linear maps* $l_A,r_A,\rho_A:A\to gl(B)$ *and* $l_B,r_B,\rho_B:B\to gl(A)$ *such that* $A\bowtie^{\rho_A,\beta_1,\beta_2}_{\rho_B,\alpha_1,\alpha_2}B$ *is a matched pair of BiHom-Lie algebras and* $A\bowtie^{l_A,r_A,\beta_1,\beta_2}_{l_B,r_B,\alpha_1,\alpha_2}B$ *is a matched pair of BiHom-associative algebras and for all* $x,y\in A$, $a,b\in B$, *the following equalities hold:*

$$\begin{aligned}\rho_A(\alpha_1\alpha_2^2(x))(\beta_1(a)\cdot_B\beta_1(b)) &= (\rho_A(\alpha_2^2(x))\beta_1(a))\cdot_B\beta_1\beta_2(b)+\beta_1\beta_2(a)\cdot_B\rho_A(\alpha_1\alpha_2(x))\beta_1(b)\\ &- l_A(\rho_B(\beta_2(a))\alpha_2(x))\beta_1\beta_2(b)-r_A(\rho_B(\beta_2(b))\alpha_1^2(x))\beta_1\beta_2(a),\end{aligned}\tag{2.27}$$

$$\begin{aligned}l_A(\alpha_1\alpha_2(x))\{\beta_1(a),\beta_1(b)\}_B &= \{\beta_1\beta_2(a),l_A(\alpha_1(x))\beta_1(b)\}_B-\rho_A(r_B(\beta_2(b))\alpha_2^2(x))\beta_1^2(a)\\ &- l_A(\rho_B(\beta_2(a))\alpha_1(x))\beta_1\beta_2(b)+(\rho_A(\alpha_2(x))\beta_1(a))\cdot_B\beta_1\beta_2(b),\end{aligned}\tag{2.28}$$

$$\rho_B(\beta_1\beta_2^2(a))(\alpha_1(x)\cdot_A\alpha_1(y)) = (\rho_B(\beta_2^2(a))\alpha_1(x))\cdot_A\alpha_1\alpha_2(y)+\alpha_1\alpha_2(x)\cdot_A\rho_B(\beta_1\beta_2(a))\alpha_1(y)$$

$$- \quad l_B(\rho_A(\alpha_2(x))\beta_2(a))\alpha_1\alpha_2(y) - r_B(\rho_A(\alpha_2(y))\beta_1^2(a))\alpha_1\alpha_2(x), \tag{2.29}$$

$$\begin{aligned} l_B(\beta_1\beta_2(a))\{\alpha_1(x),\alpha_1(y)\}_A &= \{\alpha_1\alpha_2(x), l_B(\beta_1(a))\alpha_1(y)\}_A - \rho_B(r_A(\alpha_2(y))\beta_2^2(a))\alpha_1^2(x) \\ &- l_B(\rho_A(\beta_2(x))\beta_1(a))\alpha_1\alpha_2(y) + (\rho_B(\beta_2(a))\alpha_1(x)) \cdot_A \alpha_1\alpha_2(y). \end{aligned} \tag{2.30}$$

Then $(A, B, l_A, r_A, \rho_A, \beta_1, \beta_2, l_B, r_B, \rho_B, \alpha_1, \alpha_2)$ is called a matched pair of noncommutative BiHom-Poisson algebras. Moreover, assume that $(A, \{\cdot,\cdot\}_A, \alpha_1, \alpha_2)$ and $(B, \{\cdot,\cdot\}_B, \beta_1, \beta_2)$ be two regular noncommutative BiHom-Poisson algebras, then, there exists a noncommutative BiHom-Poisson algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$\begin{aligned} (x+a)\cdot(y+b) &:= x \cdot_A y + (l_A(x)b + r_A(y)a) + a \cdot_B b + (l_B(a)y + r_B(b)x), \\ [x+a, y+b] &:= \{x,y\}_A + \rho_A(x)b - \rho_A(\alpha_1^{-1}\alpha_2(y))\beta_1\beta_2^{-1}(a) \\ &+ \{a,b\}_B + \rho_B(a)y - \rho_B(\beta_1^{-1}\beta_2(b))\alpha_1\alpha_2^{-1}(x), \\ (\alpha_1 \oplus \beta_1)(x+a) &:= \alpha_1(x) + \beta_1(a), \\ (\alpha_2 \oplus \beta_2)(x+a) &:= \alpha_2(x) + \beta_2(a), \end{aligned}$$

for any $x, y \in A,\ a, b \in B$.

Proof. By Theorem 2.8 and Theorem 2.12, we deduce that $(A \oplus B, \cdot, \alpha_1 + \beta_1, \alpha_2 + \beta_2)$ is a BiHom-associative algebra and $(A \oplus B, [\cdot,\cdot], \alpha_1 + \beta_1, \alpha_2 + \beta_2)$ is a BiHom-Lie algebra. Now, the rest, it is easy (in a similar way as for Proposition 2.12 and Proposition 2.8) to verify the BiHom-Leibniz identity satisfied. □

We denote this noncommutative BiHom-Poisson algebra by $A \bowtie^{l_A, r_A, \rho_A, \beta_1, \beta_2}_{l_B, r_B, \rho_B, \alpha_1, \alpha_2} B$.

3. Bimodule and matched pair of noncommutative BiHom-pre-Poisson algebras

In this section we introduce the definition and bimodule of a noncommutative BiHom-pre-Poisson algebra. We also establish the matched pair of noncommutative BiHom-pre-Poisson algebra and equivalently link them to a matched pair of their underlying noncommutative BiHom-Poisson algebras.

Definition 3.1 ([31])**.** A BiHom-pre-Lie algebra $(A, *, \alpha_1, \alpha_2)$ is a vector space A equipped with a bilinear product $* : A \otimes A \to A$, and two linear maps $\alpha_1, \alpha_2 \in End(A)$, such that for all $x, y, z \in A,\ \alpha_1(x * y) = \alpha_1(x) * \alpha_1(y),\ \alpha_2(x * y) = \alpha_2(x) * \alpha_2(y)$ and the following equality is satisfied:

$$(\alpha_2(x)*\alpha_1(y))*\alpha_2(z) - \alpha_1\alpha_2(x)*(\alpha_1(y)*z) = (\alpha_2(y)*\alpha_1(x))*\alpha_2(z) - \alpha_1\alpha_2(y)*(\alpha_1(x)*z). \tag{3.1}$$

The equation (3.1) is called BiHom-pre-Lie identity.

Lemma 3.2 ([31])**.** *Let $(A, *, \alpha_1, \alpha_2)$ be a regular BiHom-pre-Lie algebra. Then $(A, [\cdot,\cdot], \alpha_1, \alpha_2)$ is a BiHom-Lie algebra with*

$$[x,y] = x * y - \alpha_1^{-1}\alpha_2(y) * \alpha_1\alpha_2^{-1}(x),$$

for any $x, y \in A$. We say that $(A, [\cdot,\cdot], \alpha_1, \alpha_2)$ is the sub-adjacent BiHom-Lie algebra of $(A, *, \alpha_1, \alpha_2)$ and denoted by A^c.

Let us recall now the notion of bimodule of a BiHom-pre-Lie algebra given in [13].

Definition 3.3. Let $(A, *, \alpha_1, \alpha_2)$ be a BiHom-pre-Lie algebra, and let (V, β_1, β_2) be a BiHom-module. Let $l_*, r_* : A \to gl(V)$ be two linear maps. The quintuple $(l_*, r_*, \beta_1, \beta_2, V)$ is called a bimodule of A if for all $x, y \in A, v \in V$

$$\begin{aligned} l_*(\{\alpha_2(x), \alpha_1(y)\})\beta_2(v) &= l_*(\alpha_1\alpha_2(x))l_*(\alpha_1(y))v \\ &- l_*(\alpha_1\alpha_2(y))l_*(\alpha_1(x))v, \end{aligned} \tag{3.2}$$

$$\begin{aligned}
r_*(\alpha_2(y))\rho(\alpha_2(x))\beta_1(v) &= l_*(\alpha_1\alpha_2(x))r_*(y)\beta_1(v)\\
&- r_*(\alpha_1(x)*y)\beta_1\beta_2(v), \quad (3.3)\\
\beta_1(l_*(x)v) &= l_*(\alpha_1(x))\beta_1(v), \quad (3.4)\\
\beta_1(r_*(x)v) &= r_*(\alpha_1(x))\beta_1(v), \quad (3.5)\\
\beta_2(l_*(x)v) &= l_*(\alpha_2(x))\beta_2(v), \quad (3.6)\\
\beta_2(r_*(x)v) &= r_*(\alpha_2(x))\beta_2(v), \quad (3.7)
\end{aligned}$$

where $\{\alpha_2(x),\alpha_1(y)\}=\alpha_2(x)*\alpha_1(y)-\alpha_2(y)*\alpha_1(x)$ and $(\rho\circ\alpha_2)\beta_1=(l_*\circ\alpha_2)\beta_1-(r_*\circ\alpha_1)\beta_2$.

Proposition 3.4. *Let $(l_*,r_*,\beta_1,\beta_2,V)$ be a bimodule of a BiHom-pre-Lie algebra $(A,*,\alpha_1,\alpha_2)$. Then, the direct sum $A\oplus V$ of vector spaces is turned into a BiHom-pre-Lie algebra by defining multiplication in $A\oplus V$ by*

$$\begin{aligned}
(x_1+v_1)*'(x_2+v_2) &:= x_1*x_2+(l_*(x_1)v_2+r_*(x_2)v_1),\\
(\alpha_1\oplus\beta_1)(x_1+v_1) &:= \alpha_1(x_1)+\beta_1(v_1),\\
(\alpha_2\oplus\beta_2)(x_1+v_1) &:= \alpha_2(x_1)+\beta_2(v_1),
\end{aligned}$$

for all $x_1,x_2\in A, v_1,v_2\in V$.

We denote such a BiHom-pre-Lie algebra by $(A\oplus V,*',\alpha_1+\beta_1,\alpha_2+\beta_2)$, or $A\times_{l_*,r_*,\alpha_1,\alpha_2,\beta_1,\beta_2}V$.

Proposition 3.5. *Let $(l_*,r_*,\beta_1,\beta_2,V)$ be a bimodule of a regular BiHom-pre-Lie algebra $(A,*,\alpha_1,\alpha_2)$ such that β_1 is bijective. Let $(A,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ be the subadjacent BiHom-Lie algebra of $(A,*,\alpha_1,\alpha_2)$. Then $(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2,\beta_1,\beta_2,V)$ is a representation of BiHom-Lie algebra $(A,\{\cdot,\cdot\},\alpha_1,\alpha_2)$.*

Proof. For all $x\in A$, $v\in V$

$$\begin{aligned}
&\beta_1\circ(l_*(x)-r_*(\alpha_1\alpha_2^{-1}(x))\beta_1^{-1}\beta_2)(v)\\
=&\beta_1\circ l_*(x)v-\beta_1\circ(r_*(\alpha_1\alpha_2^{-1}(x))\beta_1^{-1}\beta_2(v)\\
=&l_*(\alpha_1(x))\beta_1(v)-(r_*\circ\alpha_1\alpha_2^{-1}(\alpha_1(x)))\beta_1^{-1}\beta_2(\beta_1(v))\\
=&(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2)(\alpha_1(x))\beta_1(v).
\end{aligned}$$

In the same way,

$$\beta_2\circ(l_*-(r_*\circ\alpha_1\alpha_2^{-1}))(x)\beta_1^{-1}\beta_2)(v)=(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2)(\alpha_2(x))\beta_2(v).$$

Finally, for all $x,y\in A$, $v\in V$ we have

$$\begin{aligned}
&(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2)(\alpha_1\alpha_2(x))\circ(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2)(y)v\\
&-(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2)(\alpha_2(y)\circ(l_*-(r_*\circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2)(\alpha_1(x))v\\
=&l_*(\alpha_1\alpha_2(x))\circ l_*(y)v-(r_*\circ\alpha_1^2(x))\beta_1^{-1}\beta_2\circ l_*(y)v\\
&-l_*(\alpha_1\alpha_2(x))\circ(r_*\circ\alpha_1\alpha_2^{-1}(y))\beta_1^{-1}\beta_2(v)+(r_*\circ\alpha_1^2(x))\beta_1^{-1}\beta_2\circ(r_*\circ\alpha_1\alpha_2^{-1}(y))\beta_1^{-1}\beta_2(v)\\
&-l_*(\alpha_2(y))\circ l_*(\alpha_1(x))v+l_*(\alpha_2(y))(r_*\circ\alpha_1^2\alpha_2^{-1}(x))\beta_1^{-1}\beta_2(v)\\
&+(r_*\circ\alpha_1(y))\beta_1^{-1}\beta_2\circ l_*(\alpha_1(x))v-(r_*\circ\alpha_1(y))\beta_1^{-1}\beta_2\circ(r_*\circ\alpha_1^2\alpha_2^{-1}(x))\beta_1^{-1}\beta_2(v)\\
=&l_*(\alpha_1\alpha_2(x))\circ l_*(y)v-(r_*\circ\alpha_1^2(x))\circ l_*(\alpha_1^{-1}\alpha_2(y))\beta_1^{-1}\beta_2(v)\\
&-l_*(\alpha_1\alpha_2(x))\circ(r_*\circ\alpha_1\alpha_2^{-1}(y))\beta_1^{-1}\beta_2(v)+(r_*\circ\alpha_1^2(x))\circ(r_*(y))\beta_1^{-2}\beta_2^2(v)\\
&-l_*(\alpha_2(y))\circ l_*(\alpha_1(x))v+l_*(\alpha_2(y))(r_*\circ\alpha_1^2\alpha_2^{-1}(x))\beta_1^{-1}\beta_2(v)\\
&+(r_*\circ\alpha_1(y))\circ l_*(\alpha_2(x))\beta_1^{-1}\beta_2(v)-(r_*\circ\alpha_1(y))\circ(r_*\circ\alpha_1(x))\beta_1^{-2}\beta_2^2(v)\\
=&\Big(l_*(\alpha_1\alpha_2(x))\circ l_*(y)v-l_*(\alpha_2(y))\circ l_*(\alpha_1(x))v\Big)\\
&+\Big(-l_*(\alpha_1\alpha_2(x))\circ(r_*\circ\alpha_1\alpha_2^{-1}(y))\beta_1^{-1}\beta_2(v)+(r_*\circ\alpha_1(y))\circ l_*(\alpha_2(x))\beta_1^{-1}\beta_2(v)
\end{aligned}$$

$$- (r_* \circ \alpha_1(y)) \circ (r_* \circ \alpha_1(x))\beta_1^{-2}\beta_2^2(v)\Big) - \Big((r_* \circ \alpha_1^2(x)) \circ l_*(\alpha_1^{-1}\alpha_2(y))\beta_1^{-1}\beta_2(v)$$
$$- (r_* \circ \alpha_1^2(x)) \circ (r_*(y))\beta_1^{-2}\beta_2^2(v) - l_*(\alpha_2(y))(r_* \circ \alpha_1^2\alpha_2^{-1}(x))\beta_1^{-1}\beta_2(v)\Big)$$
$$= l_*(\{\alpha_2(x), y\})\beta_2(v) - r_*(\alpha_1(x) * \alpha_1\alpha_2^{-1}(y))\beta_1^{-1}\beta_2^2(v) + r_*(y * \alpha_1^2\alpha_2^{-1}(x))\beta_1^{-1}\beta_2^2(v)$$
$$= l_*(\{\alpha_2(x), y\})\beta_2(v) - r_*(\alpha_1(x) * \alpha_1\alpha_2^{-1}(y) - y * \alpha_1^2\alpha_2^{-1}(x))\beta_1^{-1}\beta_2^2(v)$$
$$= l_*(\{\alpha_2(x), y\})\beta_2(v) - r_*(\{\alpha_1(x), \alpha_1\alpha_2^{-1}(y)\})\beta_1^{-1}\beta_2^2(v)$$
$$= \rho(\{\alpha_2(x), y\})\beta_2(v).$$

Therefore, the axioms (2.18)-(2.20) are satisfied. □

Now, we introduce the notion of matched pair of BiHom-pre-Lie algebra:

Theorem 3.6. *Let $(A, *_A, \alpha_1, \alpha_2)$ and $(B, *_B, \beta_1, \beta_2)$ be two BiHom-pre-Lie algebras. Suppose that there are linear maps $l_{*_A}, r_{*_A} : A \to gl(B)$ and $l_{*_B}, r_{*_B} : B \to gl(A)$ such that $(l_{*_A}, r_{*_A}, \beta_1, \beta_2, B)$ is a bimodule of A and $(l_{*_B}, r_{*_B}, \alpha_1, \alpha_2, A)$ is a bimodule of B satisfy for any, $x, y \in A,\ a, b \in B$:*

$$\begin{aligned}
&r_{*_A}(\alpha_2(x))\{\beta_2(a), \beta_1(b)\}_B = r_{*_A}(l_{*_B}(\beta_1(b))x)\beta_1\beta_2(a)\\
&-r_{*_A}(l_{*_B}(\beta_1(a)x)\beta_1\beta_2(b) + \beta_1\beta_2(a) *_B r_{*_A}(x)\beta_1(b)\\
&-\beta_1\beta_2(b) *_B r_{*_A}(x)\beta_1(a), \qquad (3.8)\\
&l_{*_A}(\alpha_1\alpha_2(x))(\beta_1(a) *_B b) = (\rho_A(\alpha_2(x))\beta_1(a)) *_B \beta_2(b)\\
&-l_{*_A}(\rho_B(\beta_2(a))\alpha_1(x))\beta_2(b) + \beta_1\beta_2(a) *_B (l_{*_A}(\alpha_1(x))b)\\
&+r_{*_A}(r_{*_B}(b)\alpha_1(x))\beta_1\beta_2(a), \qquad (3.9)\\
&r_{*_B}(\beta_2(a))\{\alpha_2(x), \alpha_1(y)\}_A = r_{*_B}(l_{*_A}(\alpha_1(y))a)\alpha_1\alpha_2(x)\\
&-r_{*_B}(l_{*_A}(\alpha_1(x)a)\alpha_1\alpha_2(y) + \alpha_1\alpha_2(x) *_A r_{*_B}(a)\alpha_1(y)\\
&-\alpha_1\alpha_2(y) *_A r_{*_B}(a)\alpha_1(x), \qquad (3.10)\\
&l_{*_B}(\beta_1\beta_2(a))(\alpha_1(x) *_A y) = (\rho_B(\beta_2(a))\alpha_1(x)) *_A \alpha_2(y)\\
&-l_{*_B}(\rho_A(\alpha_2(x))\beta_1(a))\alpha_2(y) + \alpha_1\alpha_2(x) *_A (l_{*_B}(\beta_1(a))y)\\
&+r_{*_B}(r_{*_A}(y)\beta_1(a))\alpha_1\alpha_2(x), \qquad (3.11)
\end{aligned}$$

where

$$\begin{aligned}
\{\alpha_2(x), \alpha_1(y)\}_A &= \alpha_2(x) *_A \alpha_1(y) - \alpha_2(y) *_A \alpha_1(x),\\
(\rho_A \circ \alpha_2)\beta_1 &= (l_{*_A} \circ \alpha_2)\beta_1 - (r_{*_A} \circ \alpha_1)\beta_2,\\
\{\beta_2(a), \beta_1(b)\}_B &= \beta_2(a) *_B \beta_1(b) - \beta_2(b) *_B \beta_1(a),\\
(\rho_B \circ \beta_2)\alpha_1 &= (l_{*_B} \circ \beta_2)\alpha_1 - (r_{*_B} \circ \beta_1)\alpha_2.
\end{aligned}$$

Then $(A, B, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$ is called a matched pair of BiHom-pre-Lie algebras. In this case, there exists a BiHom-pre-Lie algebra structure on the vector space $A \oplus B$ of the underlying vector spaces of A and B given by

$$\begin{aligned}
(x + a) * (y + b) &:= x *_A y + (l_{*_A}(x)b + r_{*_A}(y)a) + a *_B b + (l_{*_B}(a)y + r_{*_B}(b)x),\\
(\alpha_1 \oplus \beta_1)(x + a) &:= \alpha_1(x) + \beta_1(a),\\
(\alpha_2 \oplus \beta_2)(x + a) &:= \alpha_2(x) + \beta_2(a).
\end{aligned}$$

Proof. The proof is obtained in a similar way as for Theorem 2.8. □

We denote this BiHom-pre-Lie algebra by $A \bowtie^{l_{*_A}, r_{*_A}, \beta_1, \beta_2}_{l_{*_B}, r_{*_B}, \alpha_1, \alpha_2} B$.

Proposition 3.7. *Let $(A, B, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$ be a matched pair of regular BiHom-pre-Lie algebras $(A, *_A, \alpha_1, \alpha_2)$ and $(B, *_B, \beta_1, \beta_2)$. Then, $(A, B, l_{*_A} - (r_{*_A} \circ\alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \beta_1, \beta_2, l_{*_B} - (r_{*_B} \circ \beta_1\beta_2^{-1})\alpha_1^{-1}\alpha_2, \alpha_1, \alpha_2)$ is a matched pair of the associated BiHom-Lie algebras $(A, \{\cdot, \cdot\}_A, \alpha_1, \alpha_2)$ and $(B, \{\cdot, \cdot\}_B, \beta_1, \beta_2)$.*

Proof. Let $(A, B, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$ be a matched pair of regular BiHom-pre-Lie algebras $(A, *_A, \alpha_1, \alpha_2)$ and $(B, *_B, \beta_1, \beta_2)$. In view of Proposition 3.5, the linear maps $l_{*_A} - (r_{*_A} \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2 : A \to gl(B)$ and $l_{*_B} - (r_{*_B} \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2 : B \to gl(A)$ are representations of the underlying BiHom-Lie algebras $(A, \{\cdot,\cdot\}_A, \alpha_1, \alpha_2)$ and $(B, \{\cdot,\cdot\}_B, \beta_1, \beta_2)$, respectively. Therefore, (2.21) is equivalent to (3.8)-(3.9) and (2.22) is equivalent to (3.10)-(3.11). □

Now, we recall the definition of BiHom-dendriform algebra [32] and their notions of bimodule and matched pair given in [23].

Definition 3.8. A BiHom-dendriform algebra is a quintuple $(A, \prec, \succ, \alpha_1, \alpha_2)$ consisting of a vector space A on which the operations $\prec, \succ : A \otimes A \to A$, and $\alpha_1, \alpha_2 : A \to A$ are linear maps such that $\alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$ and for all $x, y, z \in A$ the following equalities are satisfied:

$$\begin{aligned}
\alpha_1(x \prec y) &= \alpha_1(x) \prec \alpha_1(y)\\
\alpha_2(x \prec y) &= \alpha_2(x) \prec \alpha_2(y)\\
\alpha_1(x \succ y) &= \alpha_1(x) \succ \alpha_1(y)\\
\alpha_2(x \succ y) &= \alpha_2(x) \succ \alpha_2(y)\\
(x \prec y) \prec \alpha_2(z) &= \alpha_1(x) \prec (y \cdot z),\\
(x \succ y) \prec \alpha_2(z) &= \alpha_1(x) \succ (y \prec z),\\
\alpha_1(x) \succ (y \succ z) &= (x \cdot y) \succ \alpha_2(z),
\end{aligned}$$

where

$$x \cdot y = x \prec y + x \succ y. \tag{3.12}$$

Lemma 3.9 ([32]). *Let $(A, \prec, \succ, \alpha_1, \alpha_2)$ be a BiHom-dendriform algebra. Then, $(A, \cdot := \prec + \succ, \alpha_1, \alpha_2)$ is a BiHom-associative algebra.*

Definition 3.10. Let $(A, \prec, \succ, \alpha_1, \alpha_2)$ be a BiHom-dendriform algebra, and V be a vector space. Let $l_\prec, r_\prec, l_\succ, r_\succ : A \to gl(V)$, and $\beta_1, \beta_2 : V \to V$ be six linear maps. Then, $(l_\prec, r_\prec, l_\succ, r_\succ, \beta_1, \beta_2, V)$ is called a bimodule of A if the following equations hold for any $x, y \in A$ and $v \in V$:

$$\begin{aligned}
l_\prec(x \prec y)\beta_2(v) &= l_\prec(\alpha_1(x))l_\cdot(y)v, &(3.13)\\
r_\prec(\alpha_2(x))l_\prec(y)v &= l_\prec(\alpha_1(y))r_\cdot(x)v, &(3.14)\\
r_\prec(\alpha_2(y))r_\prec(y)v &= r_\prec(x \cdot y)\beta_1(v), &(3.15)\\
l_\prec(x \succ y)\beta_2(v) &= l_\succ(\alpha_1(x))l_\prec(y)v, &(3.16)\\
r_\prec(\alpha_2(x))l_\succ(y)v &= l_\succ(\alpha_1(y))r_\prec(x)v, &(3.17)\\
r_\prec(\alpha_2(x))r_\succ(y)v &= r_\succ(y \prec x)\beta_1(v), &(3.18)\\
l_\succ(x \cdot y)\beta_2(v) &= l_\succ(\alpha_1(x))l_\succ(y)v, &(3.19)\\
r_\succ(\alpha_2(x))l_\cdot(y)v &= l_\succ(\alpha_1(y))r_\succ(x)v, &(3.20)\\
r_\succ(\alpha_2(x))r_\cdot(y)v &= r_\succ(y \succ x)\beta_1(v), &(3.21)\\
\beta_1(l_\prec(x)v) &= l_\prec(\alpha_1(x))\beta_1(v), &(3.22)\\
\beta_1(r_\prec(x)v) &= r_\prec(\alpha_1(x))\beta_1(v), &(3.23)\\
\beta_2(l_\prec(x)v) &= l_\prec(\alpha_2(x))\beta_2(v), &(3.24)\\
\beta_2(r_\prec(x)v) &= r_\prec(\alpha_2(x))\beta_2(v) &(3.25)\\
\beta_1(l_\succ(x)v) &= l_\succ(\alpha_1(x))\beta_1(v), &(3.26)\\
\beta_1(r_\succ(x)v) &= r_\succ(\alpha_1(x))\beta_1(v), &(3.27)\\
\beta_2(l_\succ(x)v) &= l_\succ(\alpha_2(x))\beta_2(v), &(3.28)\\
\beta_2(r_\succ(x)v) &= r_\succ(\alpha_2(x))\beta_2(v), &(3.29)
\end{aligned}$$

where $x \cdot y = x \prec y + x \succ y,\ l_{\cdot} = l_{\prec} + l_{\succ}$ and $r_{\cdot} = r_{\prec} + r_{\succ}$.

Proposition 3.11. *Let $(l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ}, \beta_1, \beta_2, V)$ be a bimodule of a BiHom-dendriform algebra $(A, \prec, \succ, \alpha_1, \alpha_2)$. Then, there exists a BiHom-dendriform algebra structure on the direct sum $A \oplus V$ of the underlying vector spaces of A and V given by*

$$\begin{aligned}(x+u) \prec' (y+v) &:= x \prec y + l_{\prec}(x)v + r_{\prec}(y)u,\\ (x+u) \succ' (y+v) &:= x \succ y + l_{\succ}(x)v + r_{\succ}(y)u,\\ (\alpha_1 \oplus \beta_1)(x+a) &:= \alpha_1(x) + \beta_1(a),\\ (\alpha_2 \oplus \beta_2)(x+a) &:= \alpha_2(x) + \beta_2(a),\end{aligned}$$

for all $x, y \in A, u, v \in V$. We denote it by $A \times_{l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ}, \alpha_1, \alpha_2, \beta_1, \beta_2} V$.

Proposition 3.12 ([23])**.** *Let $(l_{\prec}, r_{\prec}, l_{\succ}, r_{\succ}, \beta_1, \beta_2, V)$ be a bimodule of BiHom-dendriform algebra $(A, \prec, \succ, \alpha_1, \alpha_2)$. Let $(A, \cdot = \prec + \succ, \alpha_1, \alpha_2)$ be the BiHom associative algebra. Then $(l_{\prec} + l_{\succ}, r_{\prec} + r_{\succ}, \beta_1, \beta_2, V)$ is a bimodule of $(A, \cdot, \alpha_1, \alpha_2)$.*

Proof. We prove only the axiom (2.5). The others being proved similarly. For any $x, y \in A$ and $v \in V$, we have

$$\begin{aligned}&(l_{\prec} + l_{\succ})(x \cdot y)\beta_2(v)\\ =&(l_{\prec} + l_{\succ})(x \prec y + x \succ y)\beta_2(v)\\ =&l_{\prec}(x \prec y) + l_{\prec}(x \succ y)\beta_2(v) + l_{\succ}(x \cdot y)\beta_2(v)\\ =&l_{\prec}(\alpha_1(x))(l_{\prec} + l_{\succ})(y)v + l_{\succ}(\alpha_1(x))l_{\prec}(y)v + l_{\succ}(\alpha_1(x))l_{\succ}(y)v\\ =&l_{\prec}(\alpha_1(x))(l_{\prec} + l_{\succ})(y)v + l_{\succ}(\alpha_1(x))(l_{\prec} + l_{\succ})(y)v\\ =&(l_{\prec} + l_{\succ})(\alpha_1(x))(l_{\prec} + l_{\succ})(y)v.\end{aligned}$$

This finishes the proof. □

Theorem 3.13. *Let $(A, \prec_A, \succ_A, \alpha_1, \alpha_2)$ and $(B, \prec_B, \succ_B, \beta_1, \beta_2)$ be two BiHom-dendriform algebras. Suppose that there are linear maps $l_{\prec_A}, r_{\prec_A}$, $l_{\succ_A}, r_{\succ_A} : A \to gl(B)$ and $l_{\prec_B}, r_{\prec_B}, l_{\succ_B}$, $r_{\succ_B} : B \to gl(A)$ such that for all $x, y \in A,\ a, b \in B$, the following equalities hold:*

$$r_{\prec_A}(\alpha_2(x))(a \prec_B b) = \beta_1(a) \prec_B (r_A(x)b) + r_{\prec_A}(l_B(x)\beta_1(a)), \tag{3.30}$$

$$\begin{aligned}l_{\prec_A}(l_{\prec_B}(x))\beta_2(b) \quad &+ (r_{\prec_A}(x)a) \prec_B \beta_2(b) =\\ &\beta_1(a) \prec_B (l_{\prec_A}(x)b) + r_{\prec_A}(r_{\prec_B}(b)x)\beta_1(a),\end{aligned} \tag{3.31}$$

$$l_{\prec_A}(\alpha_1(x))(a *_B b) = (l_{\prec_A}(x)a) *_B \beta_2(b) + l_{\prec_A}(r_{\prec_A}(a)x)\beta_2(b), \tag{3.32}$$

$$r_{\prec_A}(\alpha_2(x))(a \succ_B b) = r_{\succ_A}(l_{\prec_B}(b)x)\beta_1(a) + \beta_1(a) \succ_B (r_{\prec_A}(x)b), \tag{3.33}$$

$$\begin{aligned}l_{\prec_A}(l_{\succ_B}(a)x)\beta_2(b) \quad &+ (r_{\succ_A}(x)a) \prec_B \beta_2(b) =\\ &\beta_1(a) \succ_B (l_{\prec_A}(x)b) + r_{\succ_A}(r_{\prec_B}(b)x)\beta_1(a)\end{aligned} \tag{3.34}$$

$$l_{\succ_A}(\alpha_1(x))(a \prec_B b) = (l_{\succ_A}(x)a) \prec_B \beta_2(b) + l_{\prec_A}(r_{\succ_B}(a)x)\beta_2(b), \tag{3.35}$$

$$r_{\succ_A}(\alpha_2(x))(a *_B b) = \beta_1(a) \succ_B (r_{\succ_A}(x)b) + r_{\succ_A}(l_{\succ_B}(b)x)\beta_1(a), \tag{3.36}$$

$$\begin{aligned}\beta_1(a) \succ_B (l_{\succ_A}(x)b) \quad &+ r_{\succ_A}(r_{\succ_B}(b)x)\beta_1(a) =\\ &l_{\succ_A}(l_B(a)x)\beta_2(b) + (r_A(x)a) \succ_B \beta_2(b),\end{aligned} \tag{3.37}$$

$$l_{\succ_A}(\alpha_1(x))(a \succ_B b) = (l_A(x)a) \succ_B \beta_2(b) + l_{\succ_A}(r_B(a)x)\beta_2(b), \tag{3.38}$$

$$r_{\prec_B}(\beta_2(a))(x \prec_A y) = \alpha_1(x) \prec_A (r_B(a)y) + r_{\prec_B}(l_A(y)a)\alpha_1(x), \tag{3.39}$$

$$\begin{aligned}l_{\prec_B}(l_{\prec_A}(x)a)\alpha_2(y) \quad &+ (r_{\prec_B}(a)x) \prec_A \alpha_2(y) =\\ &\alpha_1(x) \prec_A (l_B(a)y) + r_{\prec_B}(r_A(y)a)\alpha_1(x),\end{aligned} \tag{3.40}$$

$$l_{\prec_B}(\beta_1(a))(x *_A y) = (l_{\prec_B}(a)x) \prec_A \alpha_2(y) + l_{\prec_B}(r_{\prec_A}(x)a)\alpha_2(y), \tag{3.41}$$

$$r_{\prec_B}(\beta_2(a))(x \succ_A y) = r_{\succ_B}(l_{\prec_B}(y)a)\alpha_1(x) + \alpha_1(x) \succ_A (r_{\prec_B}(a)y), \tag{3.42}$$

$$\begin{aligned}l_{\prec_B}(l_{\succ_A}(x)a)\alpha_2(y) \quad &+ (r_{\succ_B}(a)x) \prec_A \alpha_2(y) =\\ &\alpha_1(x) \succ_A (l_{\prec_B}(a)y) + r_{\succ_B}(r_{\prec_A}(y)a)\alpha_1(x),\end{aligned} \tag{3.43}$$

$$l_{\succ_B}(\beta_1(a))(x \prec_A y) = (l_{\succ_B}(a)x) \prec_A \alpha_2(y) + l_{\prec_B}(r_{\succ_A}(x)a)\alpha_2(y), \quad (3.44)$$
$$r_{\succ_B}(\beta_2(a))(x *_A y) = \alpha_1(x) \succ_A (r_{\succ_B}(a)y) + r_{\succ_B}(l_{\succ_A}(y)a)\alpha_1(x), \quad (3.45)$$
$$\alpha_1(x) \succ_A (l_{\succ_B}(a)y) + r_{\succ_B}(r_{\succ_A}(y)a)\alpha_1(x) = l_{\succ_B}(l_A(x)a)\alpha_2(y) + (r_B(a)x) \succ_A \alpha_2(y), \quad (3.46)$$
$$l_{\succ_B}(\beta_1(a))(x \succ_A y) = (l_B(a)x) \succ_A \alpha_2(y) + l_{\succ_B}(r_A(x)a)\alpha_2(y), \quad (3.47)$$

where

$$x *_A y = x \dashv_A y + x \vdash_A y,\ l_A = l_{\dashv_A} + l_{\vdash_A},\ r_A = r_{\dashv_A} + r_{\vdash_A},$$
$$a *_B b = a \dashv_B b + a \vdash_B b,\ l_B = l_{\dashv_B} + l_{\vdash_B},\ r_B = r_{\dashv_B} + r_{\vdash_B}.$$

Then $(A, B, l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, \beta_1, \beta_2, l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, \alpha_1, \alpha_2)$ is called a matched pair of BiHom-dendriform algebras. In this case, there exists a BiHom-dendriform algebra structure on the direct sum $A \oplus B$ of the underlying vector spaces of A and B given by

$$\begin{array}{rcl}(x+a) \prec (y+b) &:=& x \prec_A y + (l_{\prec_A}(x)b + r_{\prec_A}(y)a) + a \prec_B b + (l_{\prec_B}(a)y + r_{\prec_B}(b)x),\\ (x+a) \succ (y+b) &:=& x \succ_A y + (l_{\succ_A}(x)b + r_{\succ_A}(y)a) + a \succ_B b + (l_{\succ_B}(a)y + r_{\succ_B}(b)x),\\ (\alpha_1 \oplus \beta_1)(x+a) &:=& \alpha_1(x) + \beta_1(a),\\ (\alpha_2 \oplus \beta_2)(x+a) &:=& \alpha_2(x) + \beta_2(a).\end{array}$$

We denote this BiHom-dendriform algebra by $A \bowtie^{l_A, r_A, \beta_1, \beta_2}_{l_B, r_B, \alpha_1, \alpha_2} B$.

Proposition 3.14 ([23]). *Let $(A, B, l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, \beta_1, \beta_2, l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, \alpha_1, \alpha_2)$ be a matched pair of a BiHom-dendriform algebras $(A, \prec_A, \succ_A, \alpha_1, \alpha_2)$ and $(B, \prec_B, \succ_B, \beta_1, \beta_2)$. Then, $(A, B, l_{\prec_A} + l_{\succ_A}, r_{\prec_A} + r_{\succ_A}, \beta_1, \beta_2, l_{\prec_B} + l_{\succ_B}, r_{\prec_B} + r_{\succ_B}, \alpha_1, \alpha_2)$ is a matched pair of the associated BiHom-associative algebras $(A, \cdot_A = \prec_A + \succ_A, \alpha_1, \alpha_2)$ and $(B, \cdot_B = \prec_B + \succ_B, \beta_1, \beta_2)$.*

Proof. Let $(A, B, l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, \beta_1, \beta_2, l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, \alpha_1, \alpha_2)$ be a matched pair of a BiHom-dendriform algebras $(A, \prec_A, \succ_A, \alpha_1, \alpha_2)$ and $(B, \prec_B, \succ_B, \beta_1, \beta_2)$. In view of Proposition 3.12, the linear maps $l_{\prec_A} + l_{\succ_A}, r_{\prec_A} + r_{\succ_A} : A \to gl(B)$ and $l_{\prec_B} + l_{\succ_B}, r_{\prec_B} + r_{\succ_B} : B \to gl(A)$ are bimodules of the underlying BiHom-associative algebras $(A, \cdot_A, \alpha_1, \alpha_2)$ and $(B, \cdot_B, \beta_1, \beta_2)$, respectively. Therefore, (2.12)-(2.14) are equivalents to (3.30)-(3.38) and (2.15)-(2.17) are equivalents to (3.39)-(3.47). □

Now, we introduce the definition of noncommutative BiHom-pre-Poisson algebra and we give some results.

Definition 3.15. A noncommutative BiHom-pre-Poisson algebra is a 6-tuple $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ such that $(A, \prec, \succ, \alpha_1, \alpha_2)$ is a BiHom-dendriform algebra and $(A, *, \alpha_1, \alpha_2)$ is a BiHom-pre-Lie algebra satisfying the following compatibility conditions:

$$(\alpha_2(x) * \alpha_1(y) - \alpha_2(y) * \alpha_1(x)) \prec \alpha_2(z) = \alpha_1\alpha_2(x) * (\alpha_1(y) \prec z) - \alpha_1\alpha_2(y) \prec (\alpha_1(x) * z), \quad (3.48)$$
$$\alpha_2(x) \succ (\alpha_1\alpha_2(y) * \alpha_1(z) - \alpha_2(z) * \alpha_1^2(y)) = \alpha_1\alpha_2^2(y) * (x \succ \alpha_1(z)) - (\alpha_2^2(y) * x) \succ \alpha_1\alpha_2(z), \quad (3.49)$$
$$(\alpha_2(x) \prec \alpha_1(y) + \alpha_2(x) \succ \alpha_1(y)) * \alpha_2(z) = (\alpha_2(x) * \alpha_1(z)) \succ \alpha_2(y) + \alpha_1\alpha_2(x) \prec (\alpha_1(y) * z). \quad (3.50)$$

where $x, y, z \in A$.

Theorem 3.16. *Let $(A, \prec, \succ, *)$ be a noncommutative pre-Poisson algebra [30] and $\alpha_1, \alpha_2 : A \to A$ be two morphisms of A such that $\alpha_1\alpha_2 = \alpha_2\alpha_1$. Then $A_{\alpha_1,\alpha_2} := (A, \prec_{\alpha_1,\alpha_2} = \prec \circ(\alpha_1 \otimes \alpha_2), \succ_{\alpha_1,\alpha_2} = \succ \circ(\alpha_1 \otimes \alpha_2), *_{\alpha_1,\alpha_2} = * \circ(\alpha_1 \otimes \alpha_2), \alpha_1, \alpha_2)$ is a noncommutative BiHom-pre-Poisson algebra, called the Yau twist of A. Moreover, assume that $(A', \prec', \succ', *')$ is another noncommutative pre-Poisson algebra and $\alpha'_1, \alpha'_2 : A' \to A'$ be a two commuting* *noncommutative pre-Poisson algebra morphisms. Let $f : A \to A'$ be a pre-Poisson algebra morphism satisfying $f \circ \alpha_1 = \alpha'_1 \circ f$ and $f \circ \alpha_2 = \alpha'_2 \circ f$. Then $f : A_{\alpha_1,\alpha_2} \to A'_{\alpha'_1,\alpha'_2}$ is a noncommutative BiHom-pre-Poisson algebra morphism.*

Proof. We shall only prove axiom (3.48) the others being proved analogously. Then, for any $x, y, z \in A$,

$$\begin{aligned}
&(\alpha_2(x) *_{\alpha_1,\alpha_2} \alpha_1(y) - \alpha_2(y) *_{\alpha_1,\alpha_2} \alpha_1(x)) \prec_{\alpha_1,\alpha_2} \alpha_2(z)\\
=&(\alpha_1\alpha_2(x) * \alpha_1\alpha_2(y) - \alpha_1\alpha_2(y) * \alpha_1\alpha_2(x)) \prec_{\alpha_1,\alpha_2} \alpha_2(z)\\
=&(\alpha_1^2\alpha_2(x) * \alpha_1^2\alpha_2(y) - \alpha_1^2\alpha_2(y) * \alpha_1^2\alpha_2(x)) \prec \alpha_2^2(z)\\
=&\alpha_1^2\alpha_2(x) * (\alpha_1^2\alpha_2(y) \prec \alpha_2^2(z)) - \alpha_1^2\alpha_2(y) \prec (\alpha_1^2\alpha_2(x) * \alpha_2^2(z))\\
=&\alpha_1\alpha_2(x) *_{\alpha_1,\alpha_2} (\alpha_1(y) \prec_{\alpha_1,\alpha_2} z) - \alpha_1\alpha_2(y) \prec_{\alpha_1,\alpha_2} (\alpha_1(x) *_{\alpha_1,\alpha_2} z).
\end{aligned}$$

For the second assertion, we have

$$\begin{aligned}
f(x \prec_{\alpha_1,\alpha_2} y) =&f(\alpha_1(x) \prec \alpha_2(y))\\
=&f(\alpha_1(x)) \prec' f(\alpha_2(y))\\
=&\alpha'_1 f(x) \prec' \alpha'_2 f(y)\\
=&f(x) \prec'_{\alpha_1,\alpha_2} f(y).
\end{aligned}$$

Similarly, we have $f(x \succ_{\alpha_1,\alpha_2} y) = f(x) \succ'_{\alpha'_1,\alpha'_2} f(y)$ and $f(x *_{\alpha_1,\alpha_2} y) = f(x) *'_{\alpha'_1,\alpha'_2} f(y)$. This completes the proof. □

Proposition 3.17. *More generally, let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a noncommutative BiHom-pre-Poisson algebra and $\alpha'_1, \alpha'_2 : A \to A$ be a two noncommutative BiHom-pre-Poisson algebra morphisms such that any two of the maps $\alpha_1, \alpha_2, \alpha'_1, \alpha'_2$ commute. Then $(A, \prec_{\alpha'_1,\alpha'_2}, \succ_{\alpha'_1,\alpha'_2}, *_{\alpha'_1,\alpha'_2}, \alpha_1 \circ \alpha'_1, \alpha_2 \circ \alpha'_2)$ is a noncommutative BiHom-pre-Poisson algebra.*

Corollary 3.18. *Let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a noncommutative BiHom-pre-Poisson algebra and $n \in \mathbb{N}^*$. Then*

(1) *The nth derived noncommutative BiHom-pre-Poisson algebra of type 1 of A is defined by*

$$A_1^n = (A, \prec^{(n)}=\prec \circ(\alpha_1^n \otimes \alpha_2^n), \succ^{(n)}=\succ \circ(\alpha_1^n \otimes \alpha_2^n), *^{(n)} = * \circ (\alpha_1^n \otimes \alpha_2^n), \alpha_1^{n+1}, \alpha_2^{n+1}).$$

(2) *The nth derived noncommutative BiHom-pre-Poisson algebra of type 2 of A is defined by*

$$\begin{aligned}
&A_2^n = (A, \prec^{(2^n-1)}=\prec \circ(\alpha_1^{2^n-1} \otimes \alpha_2^{2^n-1}), \succ^{(2^n-1)}=\succ \circ(\alpha_1^{2^n-1} \otimes \alpha_2^{2^n-1}),\\
&*^{(2^n-1)} = * \circ (\alpha_1^{2^n-1} \otimes \alpha_2^{2^n-1}), \alpha_1^{2^n}, \alpha_2^{2^n}).
\end{aligned}$$

Proof. Apply Theorem 3.17 with $\alpha'_1 = \alpha_1^n$, $\alpha'_2 = \alpha_2^n$ and $\alpha'_1 = \alpha_1^{2^n-1}$, $\alpha'_2 = \alpha_2^{2^n-1}$ respectively. □

Theorem 3.19. *Let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a regular noncommutative BiHom-pre-Poisson algebra. Then $(A, \cdot, \{\cdot, \cdot\}, \alpha_1, \alpha_2)$ is a noncommutative BiHom-Poisson algebra with*

$$x \cdot y = x \prec y + x \succ y, \text{ and } \{x, y\} = x * y - \alpha_1^{-1}\alpha_2(y) * \alpha_1\alpha_2^{-1}(x),$$

for any $x, y \in A$. We say that $(A, \cdot, \{\cdot, \cdot\}, \alpha_1, \alpha_2)$ is the sub-adjacent noncommutative BiHom-Poisson algebra of $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ and denoted by A^c.

Proof. By Lemma 3.2 and Lemma 3.9, we deduce that $(A, \cdot, \alpha_1, \alpha_2)$ is a BiHom-associative algebra and $(A, \{\cdot, \cdot\}, \alpha_1, \alpha_2)$ is a BiHom-Lie algebra. Now, we show the BiHom-Leibniz identity

$$\begin{aligned}
&\{\alpha_1\alpha_2(x), y \cdot z\} - \{\alpha_2(x), y\} \cdot \alpha_2(z) - \alpha_2(y) \cdot \{\alpha_1(x), z\}\\
=&\{\alpha_1\alpha_2(x), y \prec z + y \succ z\} - \{\alpha_2(x), y\} \prec \alpha_2(z) - \{\alpha_2(x), y\} \succ \alpha_2(z)
\end{aligned}$$

$$
\begin{aligned}
&- \alpha_2(y) \prec \{\alpha_1(x), z\} - \alpha_2(y) \succ \{\alpha_1(x), z\}\\
=&\alpha_1\alpha_2(x) * (y \prec z) + \alpha_1\alpha_2(x) * (y \succ z) - \alpha_1^{-1}\alpha_2(y \prec z) * \alpha_1^2(x)\\
&- \alpha_1^{-1}\alpha_2(y \succ z) * \alpha_1^2(x) - (\alpha_2(x) * y) \prec \alpha_2(z) + (\alpha_1^{-1}\alpha_2(y) * \alpha_1(x))) \prec \alpha_2(z)\\
&- (\alpha_2(x) * y) \succ \alpha_2(z) - (\alpha_1^{-1}\alpha_2(y) * \alpha_1(x)) \succ \alpha_2(z) - \alpha_2(y) \prec (\alpha_1(x) * z)\\
&+ \alpha_2(y) \prec (\alpha_1^{-1}\alpha_2(z) * \alpha_1^2\alpha_2^{-1}(x)) - \alpha_2(y) \succ (\alpha_1(x) * z) + \alpha_2(y) \succ (\alpha_1^{-1}\alpha_2(z) * \alpha_1^2\alpha_2^{-1}(x))\\
=&\Big(\alpha_1\alpha_2(x) * (y \prec z) - \alpha_2(y) \prec (\alpha_1(x) * z) - (\alpha_2(x) * y - \alpha_1^{-1}\alpha_2(y) * \alpha_1(x)) \prec (z)\Big)\\
&+ \Big(\alpha_1\alpha_2(x) * (y \succ z) - (\alpha_2(x) * y) \succ \alpha_2(z) - \alpha_2(y) \succ (\alpha_1(x) * z - \alpha_1^{-1}\alpha_2(z) * \alpha_1^2\alpha_2^{-1}(x)\Big)\\
&+ \Big((\alpha_1^{-1}\alpha_2(y) * \alpha_1(x)) \succ \alpha_2(z) + \alpha_2(y) \prec (\alpha_1^{-1}\alpha_2(z) * \alpha_1^2\alpha_2^{-1}(x))\\
&- (\alpha_1^{-1}\alpha_2(y) \succ \alpha_1^{-1}\alpha_2(z)\alpha_1^{-1}\alpha_2(y) \succ \alpha_1^{-1}\alpha_2(z)) * \alpha_1^2(x)\Big)\\
=&0 + 0 + 0 = 0 \ (by\ (3.48) - (3.50))
\end{aligned}
$$

which implies that $(A, \cdot, \{\cdot, \cdot\}, \alpha_1, \alpha_2)$ is a noncommutative BiHom-Poisson algebra. □

The relation existing between a noncommutative BiHom-Poisson algebra and noncommutative BiHom-pre-Poisson algebra, as illustrated by the following diagram:

$$
\begin{array}{ccc}
\text{BiHom-dendriform alg+ BiHom-pre-Lie alg} & \longrightarrow & \text{BiHom-pre-Pois. alg}\\
{\scriptstyle x\prec y+x\succ y}\Big\downarrow{\scriptstyle x*y-\alpha_1^{-1}\alpha_2(y)\alpha_1\alpha_2^{-1}(x)} & & {\scriptstyle x\prec y+x\succ y}\Big\downarrow{\scriptstyle x*y-\alpha_1^{-1}\alpha_2(y)\alpha_1\alpha_2^{-1}(x)}\\
\text{BiHom-associative alg+BiHom-Lie alg} & \longrightarrow & \text{BiHom-Pois. alg.}
\end{array}
$$

In the following we introduce the notions of bimodule and matched pair of noncommutative BiHom-pre-Poisson algebras and related relevant properties are also given.

Definition 3.20. Let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a noncommutative BiHom-pre-Poisson algebra. A bimodule of A is a 9-tuple $(l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ such that $(l_*, r_*, \beta_1, \beta_2, V)$ is a bimodule of the BiHom-pre-Lie algebra $(A, *, \alpha_1, \alpha_2)$ and $(l_\prec, r_\prec, l_\succ, r_\succ, \beta_1, \beta_2, V)$ is a bimodule of the BiHom-dendriform algebra $(A, \prec, \succ, \alpha_1, \alpha_2)$ satisfying for all $x, y \in A$ and $v \in V$:

$$l_\prec(\{\alpha_2(x), \alpha_1(y)\})\beta_2(v) = l_*(\alpha_1\alpha_2(x))l_\prec(\alpha_1(y))v - l_\prec(\alpha_1\alpha_2(y))l_*(\alpha_1(x))v, \quad (3.51)$$
$$r_\prec(\alpha_2(x))\rho(\alpha_2(y))\beta_1(v) = l_*(\alpha_1\alpha_2(y))r_\prec(x)\beta_1(v) - r_\prec(\alpha_1(y) * x)\beta_1\beta_2(v), \quad (3.52)$$
$$-r_\prec(\alpha_2(x))\rho(\alpha_2(y))\beta_1(v) = r_*(\alpha_1(y) \prec x)\beta_1\beta_2(v) - l_\prec(\alpha_1\alpha_2(y))r_*(x)\beta_1(v), \quad (3.53)$$
$$l_\succ(\alpha_2(x))\rho(\alpha_1\alpha_2(y))\beta_1(v) = l_*(\alpha_1\alpha_2^2(y))l_\succ(x)\beta_1(v) - l_\succ(\alpha_2^2(y) * z)\beta_1\beta_2(v), \quad (3.54)$$
$$r_\succ(\{\alpha_1\alpha_2(x), \alpha_1(y)\})\beta_2(v) = l_*(\alpha_1\alpha_2^2(x))r_\succ(\alpha_1(y))v - r_\succ(\alpha_1\alpha_2(y))l_*(\alpha_2^2(y))v, \quad (3.55)$$
$$-l_\succ(\alpha_2(x))\rho(\alpha_2(y))\beta_1^2(v) = r_*(x \succ \alpha_1(y))\beta_1\beta_2^2(v) - r_\succ(\alpha_1\alpha_2(y))r_*(x)\beta_2^2(v), \quad (3.56)$$
$$l_*(\alpha_2(x) \cdot \alpha_1(y))\beta_2(v) = r_\succ(a_2(y))l_*(\alpha_2(x))\beta_1(v) + l_\prec(\alpha_1\alpha_2(x))l_*(\alpha_1(y))v, \quad (3.57)$$
$$r_*(\alpha_2(x))l_.(\alpha_2(y))\beta_1(v) = l_\succ(\alpha_2(y) * \alpha_1(x))\beta_2(v) + l_\prec(\alpha_1\alpha_2(y))r_*(x)\beta_1(v), \quad (3.58)$$
$$r_*(\alpha_2(x))r_.(\alpha_1(y))\beta_2(v) = r_\succ(\alpha_2(y))r_*(\alpha_1(x))\beta_2(v) + r_\prec(\alpha_1(y) * x)\beta_1\beta_2(v), \quad (3.59)$$

where

$$
\begin{aligned}
&x \cdot y = x \prec y + x \succ y,\ l_. = l_\prec + l_\succ,\ r_. = r_\prec + r_\succ,\\
&\{\alpha_2(x), \alpha_1(y)\} = \alpha_2(x) * \alpha_1(y) - \alpha_2(y) * \alpha_1(x),\\
&(\rho \circ \alpha_2)\beta_1 = (l_* \circ \alpha_2)\beta_1 - (r_* \circ \alpha_1)\beta_2.
\end{aligned}
$$

Proposition 3.21. *Let $(l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ be a bimodule of a noncommutative BiHom-pre-Poisson algebra $(A, \prec, \succ, *, \alpha_1, \alpha_2)$. Then, there exists a noncommutative*

BiHom-pre-Poisson algebra structure on the direct sum $A \oplus V$ of the underlying vector spaces of A and V given by

$$\begin{array}{rcl}(x+u)\prec'(y+v) & := & x\prec y+l_{\prec}(x)v+r_{\prec}(y)u,\\ (x+u)\succ'(y+v) & := & x\succ y+l_{\succ}(x)v+r_{\succ}(y)u,\\ (x+u)*'(y+v) & := & x*y+l_{*}(x)v+r_{*}(y)u,\\ (\alpha_1\oplus\beta_1)(x+u) & := & \alpha_1(x)+\beta_1(u),\\ (\alpha_2\oplus\beta_2)(x+u) & := & \alpha_2(x)+\beta_2(u),\end{array}$$

for all $x, y \in A, u, v \in V$. We denote it by $A\times_{l_{\prec},r_{\prec},l_{\succ},r_{\succ},l_*,r_*,\alpha_1,\alpha_2,\beta_1,\beta_2} V$.

Proof. We prove only the axiom (3.48) in $A\oplus V$. The axioms (3.49), (3.50) being proved similarly. For any $x_1, x_2, x_3 \in A$ and $v_1, v_2, v_3 \in V$, we have

$$\begin{aligned}&((\alpha_2+\beta_2)(x_1+v_1)*'(\alpha_1+\beta_1)(x_2+v_2))\prec'(\alpha_2+\beta_2)(x_3+v_3)\\ &-((\alpha_2+\beta_2)(x_2+v_2)*'(\alpha_1+\beta_1)(x_1+v_1))\prec'(\alpha_2+\beta_2)(x_3+v_3)\\ =&((\alpha_2(x_1)*\alpha_1(x_2))+l_*(\alpha_2(x_1))\beta_1(v_2)+r_*(\alpha_1(x_2))\beta_2(v_1))\prec'(\alpha_2+\beta_2)(x_3+v_3)\\ &-(\alpha_2(x_2)*\alpha_1(x_1)+l_*(\alpha_2(x_2)\beta_1(v_1)+r_*(\alpha_1(x_1))\beta_2(v_2))\prec'(\alpha_2+\beta_2)(x_3+v_3)\\ =&(\alpha_2(x_1)*\alpha_1(x_2))\prec\alpha_2(x_3)+l_{\prec}(\alpha_2(x_1)*\alpha_1(x_2))\beta_2(v_3)+r_{\prec}(\alpha_2(x_3))l_*(\alpha_2(x_2))\beta_1(v_1)\\ &-r_{\prec}(\alpha_2(x_3))r_*(\alpha_1(x_1))\beta_2(v_2)-(\alpha_2(x_2)*\alpha_1(x_1))\prec\alpha_2(x_3)-l_{\prec}(\alpha_2(x_2)*\alpha_1(x_1))\beta_2(v_3)\\ &+r_{\prec}(\alpha_2(x_3))l_*(\alpha_2(x_2))\beta_1(v_1)-r_{\prec}(\alpha_2(x_3))r_*(\alpha_1(x_1))\beta_2(v_2)\\ =&\{\alpha_2(x_1),\alpha_1(x_2)\}+l_{\prec}(\{\alpha_2(x_1),\alpha_1(x_2)\})\beta_2(v_3)\\ &+r_{\prec}(\alpha_2(x_3))\rho(\alpha_2(x_2))\beta_1(v_1)+r_{\prec}(\alpha_2(x_3))\rho(\alpha_2(x_2))\beta_1(v_1).\end{aligned}$$

On the other hand ,

$$\begin{aligned}&(\alpha_1+\beta_1)(\alpha_2+\beta_2)(x_1+v_1)*'(\alpha_1(x_2+v_2)\prec(x_3+v_3))\\ &-(\alpha_1+\beta_1)(\alpha_2+\beta_2)(x_2+v_2)\prec'((\alpha_1+\beta_1)(x_1+v_1)*'(x_3+v_3))\\ =&(\alpha_1\alpha_2(x_1)+\beta_1\beta_2(v_1))*'(\alpha_1(x_2)\prec x_3+l_{\prec}(\alpha_1(x_2))v_3+r_{\prec}(x_3)\beta_1(v_1)\\ &-(\alpha_1\alpha_2(x_2)+\beta_1\beta_2(v_2))\prec'(\alpha_1(x_1)*x_3+l_*(\alpha_1(x_1))v_3+r_*(x_3)\alpha_1(v_1))\\ =&\alpha_1\alpha_2(x_1)*\alpha_1(x_2)\prec x_3)+l_*(\alpha_1\alpha_2(x_1))l_{\prec}(\alpha_1(x_2))v_3+l_*(\alpha_1\alpha_2(x_1))r_{\prec}(x_3)\beta_1(v_1)\\ &+r_*(\alpha_1(x_2)\prec x_3)\beta_1\beta_2(v_1)-\alpha_1\alpha_2(x_2)\prec(\alpha_1(x_1)*x_3)-l_{\prec}(\alpha_1\alpha_2(x_2))l_*(\alpha_1(x_1))v_3\\ &-l_{\prec}(\alpha_1\alpha_2(x_2))r_*(x_3)\alpha_1(v_1)-r_{\prec}(\alpha_1(x_1)*x_3)\beta_1\beta_2(v_2).\end{aligned}$$

By equations (3.48), (3.51)-(3.53) we deduce that

$$\begin{aligned}&((\alpha_2+\beta_2)(x_1+v_1)*'(\alpha_1+\beta_1)(x_2+v_2))\prec'(\alpha_2+\beta_2)(x_3+v_3)\\ &-((\alpha_2+\beta_2)(x_2+v_2)*'(\alpha_1+\beta_1)(x_1+v_1))\prec'(\alpha_2+\beta_2)(x_3+v_3),\\ =&(\alpha_1+\beta_1)(\alpha_2+\beta_2)(x_1+v_1)*'(\alpha_1(x_2+v_2)\prec(x_3+v_3))\\ &-(\alpha_1+\beta_1)(\alpha_2+\beta_2)(x_2+v_2)\prec'((\alpha_1+\beta_1)(x_1+v_1)*(x_3+v_3)).\end{aligned}$$

□

There is an example of bimodule of noncommutative BiHom-pre-Poisson algebra

Example 3.22. Let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a noncommutative BiHom-pre-Poisson algebra. Then $(L_{\prec}, R_{\prec}, L_{\succ}, R_{\succ}, L_*, R_*, \alpha_1, \alpha_2, A)$ is called a regular bimodule of A, where $L_{\prec}(x)y = x \prec y,\ R_{\prec}(x)y = y \prec x,\ L_{\succ}(x)y = x \succ y,\ R_{\succ}(x)y = y \succ x$ and $L_*(x)y = x*y,\ R_*(x)y = y*x$, for all $x, y \in A$.

Proposition 3.23. *If $f : (A, \prec_1, \succ_1, *_1, \alpha_1, \alpha_2) \longrightarrow (A', \prec_2, \succ_2, *_2, \beta_1, \beta_2)$ is a morphism of noncommutative BiHom-pre-Poisson algebra, then $(l_{\prec_1}, r_{\prec_1}, l_{\succ_1}, r_{\succ_1}, l_{*_1}, r_{*_1}, \beta_1, \beta_2, A')$ becomes a bimodule of A via f, i.e, $l_{\prec_1}(x)y = f(x) \prec_2 y,\ r_{\prec_1}(x)y = y \prec_2 f(x),\ l_{\succ_1}(x)y = f(x) \succ_2 y,\ r_{\succ_1}(x)y = y \succ_2 f(x)$ and $l_{*_1}(x)y = f(x) *_2 y,\ r_{*_1}(x)y = y *_2 f(x)$ for all $(x, y) \in A \times A'$.*

Proof. We prove only the axiom (3.57). The others being proved similarly. For any $x, y \in A$ and $z \in A'$, we have

$$\begin{aligned}
& l_{*_1}(\alpha_2(x) \cdot_1 \alpha_1(y))\beta_2(z) \\
=& f(\alpha_2(x) \cdot_1 \alpha_1(y)) *_2 \beta_2(z)) \\
=& (\beta_2 f(x) \cdot_2 \beta_1 f(y)) *_2 \beta_2(z) \\
=& (\beta_2 f(x) *_2 \beta_1(z)) \succ_2 \beta_2 f(y) + \beta_1\beta_2 f(x) \prec_2 (\beta_1 f(y) *_2 z) \text{ (by (3.51))} \\
=& (f(\alpha_2(x)) *_2 \beta_1(z)) \succ_2 f(\alpha_2(y)) + f(\alpha_1\alpha_2(x)) \prec_2 (f(\alpha_1(y)) *_2 z) \\
=& r_{\succ_1}(\alpha_2(y))(f(\alpha_2(x)) *_2 \beta_1(z)) + l_{\prec_1}(\alpha_1\alpha_2(x))(f(\alpha_1(y)) *_2 z) \\
=& r_{\succ_1}(\alpha_2(y)) l_{*_1}(\alpha_2(x))\beta_1(z) + l_{\prec_1}(\alpha_1\alpha_2(x)) l_{*_1}(\alpha_1(y))z.
\end{aligned}$$

This finishes the proof. □

Corollary 3.24. *Let $(l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ be a bimodule of a regular noncommutative BiHom-pre-Poisson algebra $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ such that β_1 is bijective. Let $(A, \cdot, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ be the subadjacent of $(A, \prec, \succ, *, \alpha_1, \alpha_2)$. Then $(l_\prec + l_\succ, r_\prec + r_\succ, l_* - (r_* \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \beta_1, \beta_2, V)$ is a representation of $(A, \cdot, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$.*

Proof. It follows from the relation between the noncommutative BiHom-pre-Poisson algebra and the associated noncommutative BiHom-Poisson algebra. More precisely by Poposition 3.5 and Poposition 3.12, we deduce that $(l_* - (r_* \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \beta_1, \beta_2, V)$ is a representation of $(A, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ and $(l_\prec + l_\succ, r_\prec + r_\succ, \beta_1, \beta_2, V)$ is a bimodule of $(A, \cdot, \alpha_1, \alpha_2)$. Now, the rest, it is easy (in a similar way as for Poposition 3.5 and Poposition 3.12) to verify the axioms (2.24)-(2.26). □

Corollary 3.25. *Let $(l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ be a bimodule of a regular noncommutative BiHom-pre-Poisson algebra $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ such that β_1 is bijective. Let $(A, \cdot, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ be the subadjacent of $(A, \prec, \succ, *, \alpha_1, \alpha_2)$. Then*

1. *$(l_\prec, r_\succ, l_* - (r_* \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \beta_1, \beta_2, V)$ is bimodule of $(A, \cdot, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$;*
2. *$(l_\prec + l_\succ, 0, 0, r_\prec + r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ and $(l_\prec, 0, 0, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ are bimodules of $(A, \prec, \succ, *, \alpha_1, \alpha_2)$;*
3. *the noncommutative BiHom-pre-Poisson algebras*

   $A \times_{l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \alpha_1, \alpha_2, \beta_1, \beta_2} V$ *and* $A \times_{l_\prec + l_\succ, 0, 0, r_\prec + r_\succ, l_*, r_*, \alpha_1, \alpha_2, \beta_1, \beta_2} V$

   have the same associated noncommutative BiHom-Poisson algebra

$$A \times_{l_\prec + l_\succ, r_\prec + r_\succ, l_* - (r_* \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \alpha_1, \alpha_2, \beta_1, \beta_2} V.$$

Proof. It results from a direct computation. □

The following result gives a construction of a bimodule of a BiHom-pre-Poisson algebra by means of the Yau twist procedure

Theorem 3.26. *Let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a noncommutative BiHom-pre-Poisson algebra, $(l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ be a bimodule of A. Let α'_1, α'_2 be two endomorphisms of A such that any two of the maps $\alpha_1, \alpha'_1, \alpha_2, \alpha'_2$ commute and β'_1, β'_2 be linear maps of V such that any two of the maps $\beta_1, \beta'_1, \beta_2, \beta'_2$ commute. Suppose furthermore that*

$$\begin{cases}
\beta'_1 \circ l_\prec = (l_\prec \circ \alpha'_1)\beta'_1, \ \beta'_2 \circ l_\prec = (l_\prec \circ \alpha'_2)\beta'_2, \\
\beta'_1 \circ l_\succ = (l_\succ \circ \alpha'_1)\beta'_1, \ \beta'_2 \circ l_\succ = (l_\succ \circ \alpha'_2)\beta'_2, \\
\beta'_1 \circ l_* = (l_* \circ \alpha'_1)\beta'_1, \ \beta'_2 \circ l_* = (l_* \circ \alpha'_2)\beta'_2,
\end{cases}$$

and

$$\begin{cases}
\beta'_1 \circ r_\prec = (r_\prec \circ \alpha'_1)\beta'_1, \ \beta'_2 \circ r_\prec = (r_\prec \circ \alpha'_2)\beta'_2, \\
\beta'_1 \circ r_\succ = (r_\succ \circ \alpha'_1)\beta'_1, \ \beta'_2 \circ r_\succ = (r_\succ \circ \alpha'_2)\beta'_2, \\
\beta'_1 \circ r_* = (r_* \circ \alpha'_1)\beta'_1, \ \beta'_2 \circ r_* = (r_* \circ \alpha'_2)\beta'_2,
\end{cases}$$

and write $A_{\alpha'_1,\alpha'_2}$ for the noncommutative BiHom-pre-Poisson algebra $(A, \prec_{\alpha'_1,\alpha'_2}, \succ_{\alpha'_1,\alpha'_2}, *_{\alpha'_1,\alpha'_2}, \alpha_1\alpha'_1, \alpha_2\alpha'_2)$ *and* $V_{\beta'_1,\beta'_2} = (\widetilde{l}_\prec, \widetilde{r}_\prec, \widetilde{l}_\succ, \widetilde{r}_\succ, \widetilde{l}_*, \widetilde{r}_*, \beta_1\beta'_1, \beta_2\beta'_2, V)$, *where*

$$\widetilde{l}_\prec = (l_\prec \circ \alpha'_1)\beta'_2,\ \widetilde{r}_\prec = (r_\prec \circ \alpha'_2)\beta'_1,\ \widetilde{l}_\succ = (l_\succ \circ \alpha'_1)\beta'_2,$$
$$\widetilde{r}_\succ = (r_\succ \circ \alpha'_2)\beta'_1,\ \widetilde{l}_* = (l_* \circ \alpha'_1)\beta'_2,\ \widetilde{r}_* = (r_* \circ \alpha'_2)\beta'_1.$$

Then $V_{\beta'_1,\beta'_2}$ is a bimodule of $A_{\alpha'_1,\alpha'_2}$.

Proof. We prove only one axiom. The others being proved similarly. For any $x, y \in A$ and $v \in V$, we have

$$\begin{aligned}
&\widetilde{l_\prec}(\{\alpha_2\alpha'_2(x), \alpha_1\alpha'_1(y)\}_{\alpha'_1,\alpha'_2}\beta_2\beta'_2(v)\\
=&\widetilde{l_\prec}(\{\alpha_2\alpha'_2\alpha'_1(x), \alpha_1\alpha'_1\alpha'_2(y)\}\beta_2\beta_2^{'2}(v)\\
=&l_\prec(\{\alpha_2\alpha'_2\alpha_1^{'2}(x), \alpha_1\alpha_1^{'2}\alpha'_2(y)\}\beta_2^{'2}(v)\\
=&l_*(\alpha_1\alpha_2\alpha_1^{'2}\alpha'_2(x))l_\prec(\alpha_1\alpha_1^{'2}\alpha'_2(y))\beta_2^{'2}(v) - l_\prec(\alpha_1\alpha_2\alpha_1^{'2}\alpha'_2(y))l_*(\alpha_1\alpha_1^{'2}\alpha'_2(x))\beta_2^{'2}(v)\ (by\ (3.51))\\
=&\widetilde{l}_*(\alpha_1\alpha'_1\alpha_2\alpha'_2(x))\widetilde{l_\prec}(\alpha_1\alpha'_1(y))v - \widetilde{l_\prec}(\alpha_1\alpha'_1\alpha_2\alpha'_2(y))l_*(\alpha_1\alpha'_1(x))v.
\end{aligned}$$

□

Taking $\alpha'_1 = \alpha_1^{p_1},\ \alpha'_2 = \alpha_2^{p_2}$ and $\beta'_1 = \beta_1^{q_1},\ \beta'_2 = \beta_2^{q_2}$ leads to the following statement:

Corollary 3.27. *Let $(A, \prec, \succ, *, \alpha_1, \alpha_2)$ be a noncommutative BiHom-pre-Poisson algebra and $(l_\prec, r_\prec, l_\succ, r_\succ, l_*, r_*, \beta_1, \beta_2, V)$ a bimodule of A. Then $V_{\beta_1^{q_1},\beta_2^{q_2}}$ is a bimodule of $A_{\alpha_1^{p_1},\alpha_1^{p_2}}$ for any nonnegative integers $p_1,\ p_2,\ q_1$ and q_2.*

Let $(l_\prec, r_\prec, l_\succ, r_\succ, l_\diamond, r_\diamond, \beta_1, \beta_2, V)$ be a bimodule of a noncommutative BiHom-pre-Poisson algebra $(A, \prec, \succ, \diamond, \alpha_1, \alpha_2)$ and let $l^*_\prec, r^*_\prec, l^*_\succ, r^*_\succ, l^*_\diamond, r^*_\diamond : A \to gl(V^*)$, furthermore $\alpha_1^*, \alpha_2^* : A^* \to A^*,\ \beta_1^*, \beta_2^* : V^* \to V^*$ be the dual maps of respectively $\alpha_1, \alpha_2, \beta_1$ and β_2 such that

$$\begin{array}{ll}
\langle l^*_\prec(x)u^*, v\rangle = \langle u^*, l_\prec(x)v\rangle, & \langle r^*_\prec(x)u^*, v\rangle = \langle u^*, r_\prec(x)v\rangle\\
\langle l^*_\succ(x)u^*, v\rangle = \langle u^*, l_\succ(x)v\rangle, & \langle r^*_\succ(x)u^*, v\rangle = \langle u^*, r_\succ(x)v\rangle\\
\langle l^*_\diamond(x)u^*, v\rangle = \langle u^*, l_\diamond(x)v\rangle, & \langle r^*_\diamond(x)u^*, v\rangle = \langle u^*, r_\diamond(x)v\rangle\\
\alpha_1^*(x^*(y)) = x^*(\alpha_1(y)), & \alpha_2^*(x^*(y)) = x^*(\alpha_2(y))\\
\beta_1^*(u^*(v)) = u^*(\beta_1(v)), & \beta_2^*(u^*(v)) = u^*(\beta_2(v))
\end{array}$$

Proposition 3.28. *Let $(l_\prec, r_\prec, l_\succ, r_\succ, l_\diamond, r_\diamond, \beta_1, \beta_2, V)$ be a bimodule of a noncommutative BiHom-pre-Poisson algebra $(A, \prec, \succ, \diamond, \alpha_1, \alpha_2)$. Then $(l^*_\prec, r^*_\prec, l^*_\succ, r^*_\succ, l^*_\diamond, r^*_\diamond, \beta_1^*, \beta_2^*, V^*)$ is a bimodule of $(A, \prec, \succ, \diamond, \alpha_1, \alpha_2)$ provided that*

$$\beta_2(l_\diamond(\alpha_2(x)\cdot\alpha_1(y))u = \beta_1 l_\diamond(\alpha_2(x))r_\succ(a_2(y))u + l_\diamond(\alpha_1(y))l_\prec(\alpha_1\alpha_2(x))u, \quad (3.60)$$
$$\beta_1 l.(\alpha_2(y))r_\diamond(\alpha_2(x))u = \beta_2(l_\succ(\alpha_2(y)\diamond\alpha_1(x)))u + \beta_1 r_\diamond(x)l_\prec(\alpha_1\alpha_2(y))u, \quad (3.61)$$
$$\beta_2 r.(\alpha_1(y))r_\diamond(\alpha_2(x))u = \beta_2 r_\diamond(\alpha_1(x))r_\succ(\alpha_2(y))u + \beta_1\beta_2(r_\prec(\alpha_1(y)\diamond x))u, \quad (3.62)$$
$$\beta_1\rho(\alpha_1\alpha_2(y))l_\succ(\alpha_2(x))u = \beta_1 l_\succ(x)l_\diamond(\alpha_1\alpha_2^2(y))u - \beta_1\beta_2(l_\succ(\alpha_2^2(y)\diamond z))u, \quad (3.63)$$
$$\beta_2(r_\succ(\{\alpha_1\alpha_2(x), \alpha_1(y)\}))u = r_\succ(\alpha_1(y))l_\diamond(\alpha_1\alpha_2^2(x))u - l_\diamond(\alpha_2^2(y))r_\succ(\alpha_1\alpha_2(y))u, \quad (3.64)$$
$$-\beta_1^2\rho(\alpha_2(y))l_\succ(\alpha_2(x))u = \beta_1\beta_2^2(r_\diamond(x\succ\alpha_1(y)))u - \beta_2^2 r_\diamond(x)r_\succ(\alpha_1\alpha_2(y))u, \quad (3.65)$$
$$\beta_2(l_\prec(\{\alpha_2(x), \alpha_1(y)\})u = l_\prec(\alpha_1(y))l_\diamond(\alpha_1\alpha_2(x))u - l_\diamond(\alpha_1(x))l_\prec(\alpha_1\alpha_2(y))u, \quad (3.66)$$
$$\beta_1\rho(\alpha_2(y))r_\prec(\alpha_2(x))u = \beta_1 r_\prec(x)l_\diamond(\alpha_1\alpha_2(y))u - \beta_1\beta_2(r_\prec(\alpha_1(y)\diamond x))u, \quad (3.67)$$
$$-\beta_1\rho(\alpha_2(y))r_\prec(\alpha_2(x))u = \beta_1\beta_2(r_\diamond(\alpha_1(y)\prec x))u - \beta_1 r_\diamond(x)l_\prec(\alpha_1\alpha_2(y))(u), \quad (3.68)$$

for all $x, y \in A$ and $u \in V$.

Proof. Straightforward. □

Theorem 3.29. *Let* $(A, \prec_A, \succ_A, *_A, \alpha_1, \alpha_2)$ *and* $(B, \prec_B, \succ_B, *_B, \beta_1, \beta_2)$ *be two noncommutative BiHom-pre-Poisson algebra. Suppose that there are linear maps* $l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, l_{*_A}, r_{*_A} : A \to gl(B)$, *and* $l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, l_{*_B}, r_{*_B} : B \to gl(A)$ *such that* $A \bowtie^{l_{*_A}, r_{*_A}, \beta_1, \beta_2}_{l_{*_B}, r_{*_B}, \alpha_1, \alpha_2} B$ *is a matched pair of BiHom-pre-Lie algebras and* $A \bowtie^{l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, \beta_1, \beta_2}_{l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, \alpha_1, \alpha_2} B$ *is a matched pair of BiHom-dendriform algebra and for all* $x, y \in A,\ a, b \in B$, *the following equalities hold:*

$$\begin{aligned}
& -l_{\prec_A}(\rho_B(\beta_2(a))\alpha_1(x))\beta_2(b) + \rho_A(\alpha_2(x))\beta_1(a) \prec_B \beta_2(b) \\
= \ & l_{*_A}(\alpha_1\alpha_2(x))(\beta_1(a) \prec_B b) - \beta_1\beta_2(a) \prec_B (l_{*_A}(\alpha_1(x))b) \\
& -r_{\prec_A}(r_{*_B}(b)\alpha_1(x))\beta_1\beta_2(a), \qquad (3.69) \\
& l_{\prec_A}(\rho_B(\beta_2(a))\alpha_1(x))\beta_2(b) - (\rho_A(\alpha_2(x))\beta_1(a)) \prec_B \beta_2(b) \\
= \ & \beta_1\beta_2(a) *_B \rho(\alpha_1(x))b + r_{*_A}(r_{\prec_B}(b)\alpha_1(x))\beta_1\beta_2(a) \\
& -l_{\prec_A}(\alpha_1\alpha_2(x))(\beta_1(a) *_B b), \qquad (3.70) \\
& r_{\prec_A}(\alpha_2(x))(\{\beta_2(a), \beta_1(b)\}_B) = \beta_1\beta_2(a) *_B (r_{\prec_A}(x)\beta_1(b)) \\
& +r_{*_A}(l_{\prec_B}(\beta_1(b))x)\beta_1\beta_2(a) - l_{\prec_A}(\alpha_1\alpha_2(x))(\beta_1(a) *_B b), \qquad (3.71) \\
& l_{\succ_A}(\alpha_2(x))\{\beta_1\beta_2(a), \beta_1(b)\} = \beta_1\beta_2^2(a) *_B (l_{\prec_A}(x)\beta_1(b)) \\
& +r_{*_A}(r_{\prec_B}(\beta_1(b))x)\beta_1\beta_2^2(a) - (r_{*_A}(x)\beta_2^2(a)) \prec_B \beta_1\beta_2(b) \\
& -l_{\prec_A}(l_{*_A}(\beta_2^2(a))x)\beta_1\beta_2(b), \qquad (3.72) \\
& \beta_2(a) \prec_B (\rho_A(\alpha_1\alpha_2(x)))\beta_1(b) - r_{\prec_A}(\rho_B(\beta_2(b))\alpha_1^2(x))\beta_2(a) \\
= \ & l_{*_A}(\alpha_1\alpha_2^2(x))(a \succ_B \beta_1(b)) - (l_{*_A}(\alpha_2^2(x))a) \prec_A \beta_1\beta_2(b) \\
& -l_{\succ_A}(r_{*_B}(a)\alpha_2^2(x))\beta_1\beta_2(b), \qquad (3.73) \\
& -\beta_2(a) \succ_B (\rho(\alpha_2(x))\alpha_1^2(b)) + r_{\succ_A}(\rho(\beta_1\beta_2(b))\alpha_1(x))\beta_2(a) \\
= \ & \beta_1\beta_2^2(b) *_B (r_{\succ_A}(\alpha_1(x))b) + r_{*_A}(l_{\succ_B}(a)\alpha_1(x)) \\
& -r_{\succ_A}(\alpha_1\alpha_2(x))(\beta_2^2(b) *_B a), \qquad (3.74) \\
& (l_{\cdot_B}(\alpha_2(x))\beta_1(a)) *_B \beta_2(b) + l_{*_A}(r_{\cdot_B}(\beta_1(a))\alpha_2(x))\beta_2(b) \\
= \ & (l_{*_A}(\alpha_2(x))\beta_1(b)) \succ_B \beta_2(a) + l_{\succ_A}(r_{*_B}(\beta_1(b))\alpha_2(x))\beta_2(a) \\
& +l_{\prec_A}(\alpha_1\alpha_2(x))(\beta_1(a) *_B b), \qquad (3.75) \\
& l_{*_A}(r_{\cdot_A}(\alpha_1(x))\beta_2(a))\beta_2(b) + (r_{\cdot_A}(\alpha_1(x))\beta_2(a)) *_B \beta_2(b) \\
= \ & r_{\succ_A}(\alpha_2(x))(\beta_2(a) *_B \beta_1(b)) + \beta_1\beta_2(a) \prec_B (l_{*_A}(\alpha_1(x))b) \\
& +r_{\prec_A}(r_{*_B}(b)\alpha_1(x))\beta_1\beta_2(a), \qquad (3.76) \\
& r_{*_A}(\alpha_2(x))(\beta_2(a) \cdot_B \beta_1(b)) = (r_{*_A}(\alpha_1(x))\beta_2(a)) \succ_B \beta_2(b) \\
& +l_{\succ_A}(l_{*_B}(\beta_2(a))\alpha_1(x))\beta_2(b) + \beta_1\beta_2(a) \prec_B (r_{*_A}(x)\beta_1(b)) \\
& +r_{\prec_A}(l_{*_B}(\beta_2(a))\alpha_1(x))\beta_2(b) + \beta_1\beta_2(a) \prec_B (r_{*_A}(x)\beta_1(b)) \\
& +r_{\prec_A}(l_{*_B}(\beta_1(b))x)\beta_1\beta_2(x), \qquad (3.77) \\
& -l_{\prec_B}(\rho_A(\alpha_2(x))\beta_1(a))\alpha_2(y) + \rho_B(\beta_2(a))\alpha_1(x) \prec_A \alpha_2(y) \\
= \ & l_{*_B}(\beta_1\beta_2(a))(\alpha_1(x) \prec_A y) - \alpha_1\alpha_2(x) \prec_A (l_{*_B}(\beta_1(a))y) \\
& -r_{\prec_B}(r_{*_A}(y)\beta_1(a))\alpha_1\alpha_2(x), \qquad (3.78) \\
& l_{\prec_B}(\rho_A(\alpha_2(x))\beta_1(a))\alpha_2(y) - (\rho_B(\beta_2(a))\alpha_1(x)) \prec_A \alpha_2(y) \\
= \ & \alpha_1\alpha_2(x) *_A \rho(\beta_1(a))y + r_{*_B}(r_{\prec_A}(y)\beta_1(a))\alpha_1\alpha_2(x) \\
& -l_{\prec_B}(\beta_1\beta_2(a))(\alpha_1(x) *_A y), \qquad (3.79) \\
& r_{\prec_B}(\beta_2(a))(\{\alpha_2(x), \alpha_1(y)\}_A) = \alpha_1\alpha_2(x) *_A (r_{\prec_B}(a)\alpha_1(y)) \\
& +r_{*_B}(l_{\prec_A}(\alpha_1(y))a)\alpha_1\alpha_2(x) - l_{\prec_B}(\beta_1\beta_2(a))(\alpha_1(x) *_A y), \qquad (3.80)
\end{aligned}$$

$$\begin{aligned}
& l_{\succ_B}(\beta_2(a))\{\alpha_1\alpha_2(x),\alpha_1(y)\} = \alpha_1\alpha_2^2(x) *_A (l_{\prec_B}(x)\alpha_1(y)) \\
& +r_{*_B}(r_{\prec_A}(\alpha_1(y))a)\alpha_1\alpha_2^2(x) - (r_{*_B}(a)\alpha_2^2(x)) \prec_A \alpha_1\alpha_2(y) \\
& -l_{\prec_B}(l_{*_B}(\alpha_2^2(x))a)\alpha_1\alpha_2(y), \qquad (3.81) \\
& \alpha_2(x) \prec_A (\rho_B(\beta_1\beta_2(a)))\alpha_1(y) - r_{\prec_B}(\rho_A(\alpha_2(y))\beta_1^2(a))\alpha_2(x) \\
= \ & l_{*_B}(\beta_1\beta_2^2(x))(a \succ_B \alpha_1(b)) - (l_{*_A}(\beta_2^2(a))x) \prec_B \alpha_1\alpha_2(y) \\
& -l_{\succ_B}(r_{*_A}(x)\beta_2^2(a))\alpha_1\alpha_2(y), \qquad (3.82) \\
& -\alpha_2(x) \succ_A (\rho(\beta_2(a))\beta_1^2(y)) + r_{\succ_B}(\rho(\alpha_1\alpha_2(y))\beta_1(a))\alpha_2(x) \\
= \ & \alpha_1\alpha_2^2(y) *_A (r_{\succ_B}(\beta_1(a))y) + r_{*_B}(l_{\succ_A}(x)\beta_1(a)) \\
& -r_{\succ_B}(\beta_1\beta_2(a))(\alpha_2^2(y) *_A x), \qquad (3.83) \\
& (l_{\cdot_A}(\beta_2(a))\alpha_1(x)) *_A \alpha_2(y) + l_{*_B}(r_{\cdot_A}(\alpha_1(x))\beta_2(a))\alpha_2(y) \\
= \ & (l_{*_B}(\beta_2(a))\alpha_1(y)) \succ_A \alpha_2(x) + l_{\succ_B}(r_{*_A}(\alpha_1(y))\beta_2(a))\alpha_2(x) \\
& +l_{\prec_B}(\beta_1\beta_2(a))(\alpha_1(x) *_A y), \qquad (3.84) \\
& l_{*_B}(r_{\cdot_B}(\beta_1(a))\alpha_2(x))\alpha_2(y) + (r_{\cdot_B}(\beta_1(a))\alpha_2(x)) *_A \alpha_2(y) \\
= \ & r_{\succ_B}(\beta_2(a))(\alpha_2(x) *_A \alpha_1(y)) + \alpha_1\alpha_2(x) \prec_A (l_{*_B}(\beta_1(a))y) \\
& +r_{\prec_B}(r_{*_A}(y)\beta_1(a))\alpha_1\alpha_2(x), \qquad (3.85) \\
& r_{*_B}(\beta_2(a))(\alpha_2(x) \cdot_A \alpha_1(y)) = (r_{*_B}(\beta_1(a))\alpha_2(x)) \succ_A \alpha_2(y) \\
& +l_{\succ_B}(l_{*_A}(\alpha_2(x))\beta_1(a))\alpha_2(y) + \alpha_1\alpha_2(x) \prec_A (r_{*_B}(a)\alpha_1(y)) \\
& +r_{\prec_B}(l_{*_A}(\alpha_2(x))\beta_1(a))\alpha_2(y) + \alpha_1\alpha_2(x) \prec_A (r_{*_B}(a)\alpha_1(y)) \\
& +r_{\prec_B}(l_{*_A}(\alpha_1(y))a)\alpha_1\alpha_2(a), \qquad (3.86)
\end{aligned}$$

where

$$\begin{aligned}
& x \cdot_A y = x \prec_A y + x \succ_A y, \ l_{\cdot_A} = l_{\prec_A} + l_{\succ_A}, \ r_{\cdot_A} = r_{\prec_A} + r_{\succ_A}, \\
& a \cdot_B b = a \prec_B b + a \succ_B b, \ l_{\cdot_B} = l_{\prec_B} + l_{\succ_B}, \ r_{\cdot_B} = r_{\prec_B} + r_{\succ_B}, \\
& \{\alpha_2(x),\alpha_1(y)\}_A = \alpha_2(x) *_A \alpha_1(y) - \alpha_2(y) *_A \alpha_1(x), \\
& \{\beta_2(a),\beta_1(b)\}_B = \beta_2(a) *_B \beta_1(b) - \beta_2(b) *_B \beta_1(a), \\
& (\rho_A \circ \alpha_2)\beta_1 = (l_{*_A} \circ \alpha_2)\beta_1 - (r_{*_A} \circ \alpha_1)\beta_2, \\
& (\rho_B \circ \beta_2)\alpha_1 = (l_{*_B} \circ \beta_2)\alpha_1 - (r_{*_B} \circ \beta_1)\alpha_2.
\end{aligned}$$

Then $(A, B, l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$ *is called a matched pair of noncommutative BiHom-pre-Poisson algebras. In this case, there exists a noncommutative BiHom-pre-Poisson algebra structure on the direct sum* $A \oplus B$ *of the underlying vector spaces of* A *and* B *given by*

$$\begin{aligned}
(x+a) \prec (y+b) &:= (x \prec_A y + r_{\prec_B}(b)x + l_{\prec_B}(a)y) + (l_{\prec_A}(x)b + r_{\prec_A}(y)a + a \prec_B b), \\
(x+a) \succ (y+b) &:= (x \succ_A y + r_{\succ_B}(b)x + l_{\succ_B}(a)y) + (l_{\succ_A}(x)b + r_{\succ_A}(y)a + a \succ_B b), \\
(x+a) * (y+b) &:= (x *_A y + r_{*_B}(b)x + l_{*_B}(a)y) + (l_{*_A}(x)b + r_{*_A}(y)a + a *_B b), \\
(\alpha_1 \oplus \beta_1)(x+a) &:= \alpha_1(x) + \beta_1(a), \\
(\alpha_2 \oplus \beta_2)(x+a) &:= \alpha_2(x) + \beta_2(a),
\end{aligned}$$

for any $x, y \in A, \ a, b \in B$.

Proof. It is obtained in a similar way as for Theorem 2.8.

□

Let $A \bowtie^{l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, l_{*_A}, r_{*_A}, \beta_1, \beta_2}_{l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2} B$ denote this noncommutative BiHom-pre-Poisson algebra.

Corollary 3.30. *Let* $(A, B, l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$ *be a matched pair of regular noncommutative BiHom-pre-Poisson algebras* $(A, \prec_A, \succ_A, *_A, \alpha_1, \alpha_2)$ *and* $(B, \prec_B, \succ_B, *_B, \beta_1, \beta_2)$. *Then,* $(A, B, l_{\prec_A} + l_{\succ_A}, r_{\prec_A} + r_{\succ_A}, l_{*_A} - (r_{*_A} \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \beta_1, \beta_2, l_{\prec_B} + l_{\succ_B}, r_{\prec_B} + r_{\succ_B}, l_{*_B} - (r_{*_B} \circ \beta_1\beta_2^{-1})\alpha_1^{-1}\alpha_2, \alpha_1, \alpha_2)$ *is a matched* *pair of the associated noncommutative BiHom-Poisson algebras* $(A, \cdot_A, \{\cdot,\cdot\}_A, \alpha_1, \alpha_2)$ *and* $(B, \cdot_B, \{\cdot,\cdot\}_B, \beta_1, \beta_2)$.

Proof. Let $(A, B, l_{\prec_A}, r_{\prec_A}, l_{\succ_A}, r_{\succ_A}, l_{*_A}, r_{*_A}, \beta_1, \beta_2, l_{\prec_B}, r_{\prec_B}, l_{\succ_B}, r_{\succ_B}, l_{*_B}, r_{*_B}, \alpha_1, \alpha_2)$ be a
matched pair of regular noncommutative BiHom-pre-Poisson algebras $(A, \prec_A, \succ_A, *_A, \alpha_1, \alpha_2)$ and $(B, \prec_B, \succ_B, *_B, \beta_1, \beta_2)$. Then by Proposition 3.7 and Proposition 3.14, $(A, B, l_{\prec_A} + l_{\succ_A}, r_{\prec_A} + r_{\succ_A}, \beta_1, \beta_2, l_{\prec_B} + l_{\succ_B}, r_{\prec_B} + r_{\succ_B}, \alpha_1, \alpha_2)$ is a matched pair of the associated BiHom-associative algebras $(A, \cdot_A, \alpha_1, \alpha_2)$ and $(B, \cdot_B, \beta_1, \beta_2)$ and $(A, B, l_{*_A} - (r_{*_A} \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2, \beta_1, \beta_2, l_{*_B} - (r_{*_B} \circ \beta_1\beta_2^{-1})\alpha_1^{-1}\alpha_2, \alpha_1, \alpha_2)$ is a matched pair of the associated BiHom-Lie algebras $(A, \{\cdot,\cdot\}_A, \alpha_1, \alpha_2)$ and $(B, \{\cdot,\cdot\}_B, \beta_1, \beta_2)$. Besides, in view of Corollary 3.24, the linear maps $l_{\prec_A} + l_{\succ_A}, r_{\prec_A} + r_{\succ_A}, l_{*_A} - (r_{*_A} \circ \alpha_1\alpha_2^{-1})\beta_1^{-1}\beta_2 : A \to gl(B)$ and $l_{\prec_B} + l_{\succ_B}, r_{\prec_B} + r_{\succ_B}, l_{*_B} - (r_{*_B} \circ \beta_1\beta_2^{-1})\alpha_1^{-1}\alpha_2 : B \to gl(A)$ are a representations of the underlying noncommutative BiHom-Poisson algebras $(A, \cdot_A, \{\cdot,\cdot\}_A, \alpha_1, \alpha_2)$ and $(B, \cdot_B, \{\cdot,\cdot\}_B, \beta_1, \beta_2)$, respectively. Therefore, (2.27)-(2.28) are equivalents to (3.69)-(3.77) and (2.29)-(2.30) are equivalents to (3.78)-(3.86). □

4. $\mathcal{O}$-operators of noncommutative BiHom-Poisson algebras

In this section we introduce the notions of an $\mathcal{O}$-operator of noncommutative BiHom-Poisson algebras and we give some related properties.

Definition 4.1. Let $(A, \cdot, \alpha_1, \alpha_2)$ be a BiHom-associative algebra and $(l, r, \beta_1, \beta_2, V)$ be a bimodule of A. Then, a linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to $(l, r, \beta_1, \beta_2, V)$, if T satisfies

$$\alpha_1 T = T\beta_1,\ \alpha_2 T = T\beta_2 \text{ and } T(u) \cdot T(v) = T(l(T(u))v + r(T(v))u) \text{ for all } u, v \in V.$$

Lemma 4.2 ([23]). *Let* $(A, \cdot, \alpha_1, \alpha_2)$ *be a BiHom-associative algebra, and let* $(l, r, \beta_1, \beta_2, V)$ *be a bimodule. Let* $T : V \to A$ *be an* $\mathcal{O}$*-operator associated to* $(l, r, \beta_1, \beta_2, V)$*. Then, there exists a BiHom-dendriform algebra structure on* V *given by*

$$u \succ v = l(T(u))v,\ u \prec v = r(T(v))u$$

for all $u, v \in V$.

Now we recall the definition of an $\mathcal{O}$-operator on a BiHom-Lie algebra associated to a given representation, which generalize the Rota-Baxter operator of weight 0 introduced in [31].

Definition 4.3. Let $(A, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ be a BiHom-Lie algebra, and let $(\rho, \beta_1, \beta_2, V)$ be a representation of A. Then, a linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to $(\rho, \beta_1, \beta_2, V)$, if T satisfies

$$\alpha_1 T = T\beta_1,\ \alpha_2 T = T\beta_2 \text{ and } \{T(u), T(v)\} = T(\rho(T(u))v - \rho(T(\beta_1^{-1}\beta_2(v)))\beta_1\beta_2^{-1}(u)),$$

for all $u, v \in V$.

Example 4.4. An $\mathcal{O}$-operator on a BiHom-Lie algebra $(A, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ with respect to the adjoint representation is called a Rota-Baxter operator on A.

Lemma 4.5. *Let* $T : V \to A$ *be an* $\mathcal{O}$*-operator on a BiHom-Lie algebra* $(A, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ *with respect to a representation* $(\rho, \beta_1, \beta_2, V)$*. Define a multiplication* $*$ *on* V *by*

$$u * v = \rho(T(u))v,\ \forall u, v \in V. \tag{4.1}$$

Then $(V, *, \alpha_1, \alpha_2)$ *is a BiHom-pre-Lie algebra.*

Definition 4.6. Let $(A, \cdot, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$ be a noncommutative BiHom-Poisson algebra, and let $(l, r, \rho, \beta_1, \beta_2, V)$ be a representation of A. A linear operator $T : V \to A$ is called an $\mathcal{O}$-operator on A if T is both an $\mathcal{O}$-operator on the BiHom-associative algebra $(A, \cdot, \alpha_1, \alpha_2)$ and an $\mathcal{O}$-operator on the BiHom-Lie algebra $(A, \{\cdot,\cdot\}, \alpha_1, \alpha_2)$.

Example 4.7. An $\mathcal{O}$-operator on a noncommutative BiHom-Poisson algebra $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ with respect the regular representation is called a Rota-Baxter operator on A.

Theorem 4.8. *Let $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ be a noncommutative BiHom-Poisson algebra and $T: V\to A$ an $\mathcal{O}$-operator on A with respect to the representation $(l,r,\rho,\beta_1,\beta_2,V)$. Define new operations $\prec,\succ$ and $*$ on V by*

$$u\prec v=l(T(u))v,\ u\succ v=r(T(v))u,\ u*v=\rho(T(u))v. \tag{4.2}$$

Then $(V,\prec,\succ,*,\alpha_1,\alpha_2)$ is a noncommutative BiHom-pre-Poisson algebra. Moreover, $T(V)=\{T(v);\ v\in V\}\subset A$ is a subalgebra of A and there is an induced noncommutative BiHom-pre-Poisson algebra structure on $T(V)$ given by

$$T(u)\prec T(v)=T(u\prec v),\ T(u)\succ T(v)=T(u\succ v),\ T(u)*T(v)=T(u*v), \tag{4.3}$$

for all $u,v\in V$.

Proof. By Lemma 4.2 and Lemma 4.5, we deduce that $(A,\prec,\succ,\alpha_1,\alpha_2)$ is a BiHom-dendriform algebra and $(A,*,\alpha_1,\alpha_2)$ is a BiHom-pre-Lie algebra. Now, we prove only the axiom (3.48). The other being proved similarly, for any $x,y,z\in V$ we have

$$\begin{aligned}
&(\beta_2(x)*\beta_1(y)-\beta_2(y)*\beta_1(x))\prec\beta_2(z)\\
&\quad-\beta_1\beta_2(x)*(\beta_1(y)\prec z)+\beta_1\beta_2(y)\prec(\beta_1(x)*z)\\
&=(\rho(T(\beta_2(x))\beta_2(y)-\rho(T(\beta_2(y))\beta_1(x))\prec\beta_2(z)\\
&\quad-\rho(T(\beta_1\beta_2(x)))(\beta_1(y)\prec z)+l(T(\beta_1\beta_2(y)))(\beta_1(x)*z)\\
&=l(T(\rho(T(\beta_2(x))\beta_1(y)-\rho(T(\beta_2(y)))\beta_1(x))\beta_2(z)\\
&\quad-\rho(T(\beta_1\beta_2(x)))l(\beta_1(y))z+l(T(\beta_1\beta_2(y)))\rho(\beta_1(x))z\\
&=l(\{T(\beta_2(x),T(\beta_1(y))\})\beta_2(z)-\rho(T(\beta_1\beta_2(x)))l(\beta_1(y))z\\
&\quad+l(T(\beta_1\beta_2(y)))\rho(\beta_1(x))z=0\ (by\ (2.24)).
\end{aligned}$$

Therefore, $(V,\prec,\succ,*,\alpha_1,\alpha_2)$ is a BiHom-pre-Posson algebra. □

Corollary 4.9. *Let $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ be a noncommutative BiHom-Poisson algebra. Then there is a noncommutative BiHom-pre-Poisson algebra structure on A such that its sub-adjacent noncommutative BiHom-Poisson algebra is exactly $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ if and only if there exists an invertible $\mathcal{O}$-operator on $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$.*

Proof. Suppose that there exists an invertible $\mathcal{O}$-operator $T: V\to A$ associated to the representation $(l,r,\rho,\beta_1,\beta_2,V)$, then the compatible noncommutative BiHom-pre-Poisson algebra structure on A, for all $x,y\in A$ is given by

$$x\prec y=T(l(x)T^{-1}(y)),\ \ x\succ y=T(r(y)T^{-1}(x)),\ \ x*y=T(\rho(x)T^{-1}(y))\ \forall x,y\in A.$$

Conversely, let $(A,\prec,\succ,*,\alpha_1,\alpha_2)$ be a noncommutative BiHHom-pre-Poisson algebra and $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ the sub-adjacent noncommutative BiHom-Poisson algebra. Then the identity map id is an $\mathcal{O}$-operator on A with respect to the regular representation $(L_\prec,R_\succ,ad,\alpha_1,$
$\alpha_2,A)$.

□

Example 4.10. Let $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$ be a noncommutative BiHom-Poisson algebra and $R: A\longrightarrow A$ a Rota-Baxter operator. Define new operations on A by

$$x\prec y=R(x)\cdot y,\quad x\succ y=x\cdot R(y),\quad x*y=\{R(x),y\}.$$

Then $(A,\prec,\succ,*,\alpha_1,\alpha_2)$ is a noncommutative BiHom-pre-Poisson algebra and R is a homomorphism from the sub-adjacent noncommutative BiHom-Poisson algebra $(A,\cdot',\{\cdot,\cdot\}',\alpha_1,\alpha_2)$ to $(A,\cdot,\{\cdot,\cdot\},\alpha_1,\alpha_2)$, where $x\cdot' y=x\prec y+x\succ y$ and $\{x,y\}'=x*y-\alpha_1^{-1}\alpha_2(y)*$

$\alpha_1\alpha_2^{-1}(x)$.

The inverse relation existing between a noncommutative BiHom-pre-Poisson algebra and noncommutative BiHom-Poisson algebra, as illustrated by the following diagram:

$$\begin{array}{ccc}
\text{BiHom-associative alg+BiHom-Lie alg} & \longrightarrow & \text{BiHom-Poisson alg} \\
{\scriptstyle R(x)\cdot y,\ x\cdot R(y)}\Big\downarrow{\scriptstyle \{R(x),y\}} & & {\scriptstyle R(x)\cdot y,\ x\cdot R(y)}\Big\downarrow{\scriptstyle \{R(x),y\}} \\
\text{BiHom-dendriform alg+ BiHom-pre-Lie alg} & \longrightarrow & \text{BiHom-pre-Poisson alg.}
\end{array}$$

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