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Research Article

A new approach to revolution surface with its focal surface in the Galilean 3-space $\mathbb{G}_3$

İlim Kişi

Department of Mathematics, Kocaeli University, Kocaeli, Turkey

Abstract

In this paper, we handle focal surfaces of surface of revolution in Galilean 3-space $\mathbb{G}_3$. We define the focal surfaces of surface of revolution and we obtain some results for these types of surfaces to become flat and minimal. Also, by giving some examples to these surfaces, we present the visualizations of each type of focal surface of surface of revolution in $\mathbb{G}_3$.

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1. Introduction

The concept of line congruences is first defined in the area of visualization by Hagen et al in 1991 [11]. Actually, line congruences are surfaces which are obtained from by transforming one surface to another by lines. Focal surface is one of these congruences. For a given surface M with the parametrization $X(u,v)$, the line congruence is defined as

$$C(u,v,z) = X(u,v) + zE(u,v). \tag{1.1}$$

Here $E(u,v)$ is the set of unit vectors and z is a distance. For each pair (u,v), the equation (1.1), expresses a line of the congruence and called as generatrix. On every generatrix of C, there are two points called as focal points and the focal surface is the locus of the focal points. If $E(u,v) = N(u,v)$, the unit normal vector field of the surface, then C is a normal congruence. In this case, the parametric equation of the focal surface $C = X^*(u,v)$ of $X(u,v)$ is given as

$$X^*(u,v) = C(u,v,z) = X(u,v) + {\kappa_i}^{-1} N(u,v); \quad i = 1,2$$

where κ_is; $(i = 1,2)$ are the principal curvature functions of $X(u,v)$ [10]. Focal surfaces are the subject of many studies such as [10, 15–17, 19, 23, 26].

Galilean geometry is a non-Euclidean geometry and associated with Galilei principle of relativity. This principle can be explained briefly as "in all inertial frames, all law of physics are the same." (Except for the Euclidean geometry in some cases), Galilean geometry is the easiest of all Klein geometries, and it is revelant to the theory of relativity of Galileo and Einstein. One can have a look at the studies [20, 24] for Galilean geometry. Recently, many works related to Galilean geometry have been done by several authors in [4, 6, 21].

Email address: ilim.ayvaz@kocaeli.edu.tr
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2. Preliminaries

In Galilean 3-space $\mathbb{G}_3$, we can give the following basic concepts.

The vector $a=(a_1,a_2,a_3)$ is isotropic if $a_1=0$ and non-isotropic otherwise. Thus, for the standard coordinates (x,y,z), the x-axis is non-isotropic while the others are isotropic. The yz-plane, i.e. $x=0$, is Euclidean and the xy-plane and xz-plane are isotropic. The scalar product of the vectors $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ and the length of the vector $a=(a_1,a_2,a_3)$ in $\mathbb{G}_3$ are respectively defined as

$$\langle a,b\rangle=\begin{cases} a_1b_1, & if\ a_1\neq 0\ \vee b_1\neq 0\\ a_2b_2+a_3b_3, & if\ a_1=0\ \wedge b_1=0,\end{cases}$$

$$\|a\|=\begin{cases} |a_1|, & if\ a_1\neq 0\\ a_2^2+a_3^2, & if\ a_1=0.\end{cases}$$

The cross product of the vectors $a=(a_1,a_2,a_3)$ and $b=(b_1,b_2,b_3)$ in $\mathbb{G}_3$ is also defined as

$$a\wedge b=\begin{vmatrix} 0 & e_2 & e_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

[18]. An admissible unit speed curve $\alpha: I\subset\mathbb{R}\to\mathbb{G}_3$ is given with the parametrization

$$\alpha(u)=(u,y(u),z(u)).$$

Let M be a surface parametrized with

$$X(u_1,u_2)=(x(u_1,u_2),y(u_1,u_2),z(u_1,u_2))$$

in $\mathbb{G}_3$. To represent the partial derivatives, we use

$$x_{,i}=\frac{\partial x}{\partial u_i}\ \text{and}\ x_{,ij}=\frac{\partial^2 x}{\partial u_i\partial u_j},\ \ 1\le i,j\le 2.$$

If $x_{,i}\neq 0$ for some $i=1,2$, then the surface is admissible (i.e. having not any Euclidean tangent planes). The first fundamental form I of the surface M is defined as

$$I=(g_1d_{u_1}+g_2d_{u_2})^2+\varepsilon(h_{11}d_{u_1}^2+2h_{12}d_{u_1}d_{u_2}+h_{22}d_{u_2}^2),$$

where $g_i=x_{,i}$, $h_{ij}=y_{,i}\,y_{,j}+z_{,i}\,z_{,j}$; $i,j=1,2$ and

$$\varepsilon=\begin{cases} 0, & if\ d_{u_1}:d_{u_2}\ is\ non-isotropic,\\ 1, & if\ d_{u_1}:d_{u_2}\ is\ isotropic.\end{cases}$$

Let a function W is given by

$$W=\sqrt{(x_{,1}\,z_{,2}-x_{,2}\,z_{,1})^2+(x_{,2}\,y_{,1}-x_{,1}\,y_{,2})^2}. \tag{2.1}$$

Then, the unit normal vector field is given as

$$N=\frac{1}{W}\,(0,-x_{,1}\,z_{,2}+x_{,2}\,z_{,1}\,,x_{,1}\,y_{,2}-x_{,2}\,y_{,1}). \tag{2.2}$$

Similarly, the second fundamental form II of the surface M is defined as

$$II=L_{11}d_{u_1}^2+2L_{12}d_{u_1}d_{u_2}+L_{22}d_{u_2}^2,$$

where

$$L_{ij}=\frac{1}{g_1}\left\langle g_1(0,y_{,ij}\,,z_{,ij}\,)-g_{i,j}(0,y_{,1}\,,z_{,1}\,),N\right\rangle,\quad g_1\neq 0$$

or

$$L_{ij}=\frac{1}{g_2}\left\langle g_2(0,y_{,ij}\,,z_{,ij}\,)-g_{i,j}(0,y_{,2}\,,z_{,2}\,),N\right\rangle,\quad g_2\neq 0.$$

The Gaussian and the mean curvatures of M are defined as

$$K=\frac{L_{11}L_{22}-L_{12}^2}{W^2}\quad\text{and}\quad H=\frac{g_2^2L_{11}-2g_1g_2L_{12}+g_1^2L_{22}}{2W^2}. \tag{2.3}$$

A surface is called as flat (resp. minimal) if its Gaussian (resp. mean) curvatures vanish [4,20]. The principal curvatures κ_1 and κ_2 of the surface M are given as

$$\kappa_1 = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{W^2} \text{ and } \kappa_2 = \frac{L_{11}L_{22} - L_{12}^2}{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}, \tag{2.4}$$

respectively [22].

3. Surface of revolution in $\mathbb{G}_3$

Surface of revolution is studied in different spaces by many authors in [1–3, 5, 7–9, 12–14, 25]. In Galilean 3-space, surface of revolution is studied in [7].

Definition 3.1. A surface of revolution in Galilean 3-space $\mathbb{G}_3$ is a surface formed by the rotation of a curve, a profile curve. The rotation is either an Euclidean rotation about an axis in the supporting plane of the profile curve, or an isotropic rotation for which a bundle of fixed planes is chosen [7].

Since there exists two kinds of planes (Euclidean and isotropic) in $\mathbb{G}_3$, the profile curve can lie on one of these two planes. An Euclidean plane contains only isotropic vectors, while an isotropic plane contains both isotropic and non-isotropic vectors. Thus, three types of surface of revolution can be defined in $\mathbb{G}_3$. An Euclidean rotation about the non-isotropic x-axis is given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

where θ is the Euclidean angle. An isotropic rotation about the fixed plane $z = constant$ is given by

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \theta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} c\theta \\ \frac{c}{2}\theta^2 \\ 0 \end{bmatrix},$$

where c is a constant.

Type I Surface of Revolution in $\mathbb{G}_3$: Let the unit speed profile curve α lies on the Euclidean yz-plane and be parametrized with $\alpha(v) = (0, f(v), g(v))$ for the real valued functions f and g. For this profile curve, an isotropic rotation about the z-axis is given

$$\begin{bmatrix} 1 & 0 & 0 \\ u & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ f(v) \\ g(v) \end{bmatrix} + \begin{bmatrix} cu \\ \frac{c}{2}u^2 \\ 0 \end{bmatrix}.$$

Then, parametrization of type I surface of revolution in $\mathbb{G}_3$ is given by

$$X(u,v) = \left(cu, f(v) + \frac{c}{2}u^2, g(v)\right) \tag{3.1}$$

[21].

Theorem 3.2 ([7]). *A type I surface of revolution in the Galilean 3-space is flat or, equivalently, minimal, if and only if it is either*

1) *a parabolic cylinder parameterized by*

$$X(u,v) = \left(cu, a + \frac{c}{2}u^2, g(v)\right),$$

2) *a part of an isotropic plane, consisting of a family of parabolas, parameterized by*

$$X(u,v) = \left(cu, f(v) + \frac{c}{2}u^2, a\right),$$

3) *or a parabolic cylinder parameterized by*

$$X(u,v) = \left(cu, f(v) + \frac{c}{2}u^2, af(v) + b\right). \tag{3.2}$$

Here $a, b, c \in \mathbb{R}$ with $c \neq 0$ and $a \neq 0$.

Type II Surface of Revolution in $\mathbb{G}_3$: In this case, let the unit speed profile curve α lies on the isotropic xy-plane and be parametrized with $\alpha(v) = (v, g(v), 0)$ for the real valued function g. For this profile curve, an isotropic rotation about the y-axis is given

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ u & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ g(v) \\ 0 \end{bmatrix} + \begin{bmatrix} cu \\ 0 \\ \frac{c}{2}u^2 \end{bmatrix}.$$

Then, parametrization of type II surface of revolution in $\mathbb{G}_3$ is given by

$$X(u,v) = \left(v + cu, g(v), uv + \frac{c}{2}u^2\right) \tag{3.3}$$

[7].

Theorem 3.3 ([7])**.** *A type II surface of revolution in the Galilean 3-space is flat or, equivalently, minimal, if and only if it is either*

1) *a part of an isotropic plane, consisting of a family of parabolas, parameterized by*

$$X(u,v) = \left(v + cu, a, uv + \frac{c}{2}u^2\right),$$

2) *a parabolic cylinder parameterized by*

$$X(u,v) = \left(a + cu, g(v), au + \frac{c}{2}u^2\right),$$

3) *or a cyclic surface (parabolic sphere) parameterized by*

$$X(u,v) = \left(v + cu, av^2 + b, uv + \frac{c}{2}u^2\right),$$

where $a, b, c \in \mathbb{R}$ with $c \neq 0$.

Type III Surface of Revolution in $\mathbb{G}_3$: Again, let the unit speed profile curve α lies on the isotropic xy-plane and be parametrized with $\alpha(v) = (v, g(v), 0)$ for the real valued function g. For this profile curve, an Euclidean rotation about the x-axis is given

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos u & \sin u \\ 0 & -\sin u & \cos u \end{bmatrix} \begin{bmatrix} v \\ g(v) \\ 0 \end{bmatrix}.$$

Then, parametrization of type III surface of revolution in $\mathbb{G}_3$ is given by

$$X(u,v) = (v, g(v)\cos u, -g(v)\sin u) \tag{3.4}$$

[21].

Theorem 3.4 ([7])**.** *A type III surface of revolution in the Galilean 3-space is flat if and only if it is either*

1) *a cylinder over an Euclidean circle parameterized by*

$$X(u,v) = (v, a\cos u, -a\sin u),$$

2) *or a circular cone with vertex $(b, 0, 0)$ parameterized by*

$$X(u,v) = (ag(v) + b, g(v)\cos u, -g(v)\sin u),$$

where $a, b \in \mathbb{R}$ with $a \neq 0$.

4. Focal surfaces of surface of revolution in $\mathbb{G}_3$

In this section, we respectively give the focal surfaces of surface of revolution which are mentioned in [7]. Furthermore, we obtain some results for these types surfaces to become flat and minimal.

4.1. Focal surface of type I surface of revolution

Let $\alpha(v) = (0, f(v), g(v))$ be a unit speed curve in $\mathbb{G}_3$. Then, taking $c = 1$ in (3.1), type I surface of revolution M can be written as in the following form

$$X(u, v) = \left(u, f(v) + \frac{u^2}{2}, g(v)\right). \tag{4.1}$$

Since α is a unit speed curve lying on the Euclidean plane, then we have $(f'(v))^2 + (g'(v))^2 = 1$. The tangent space of M at an arbitrary point is spanned by the vectors

$$X_u = (1, u, 0), \qquad X_v = (0, f'(v), g'(v)).$$

From (2.1) and (2.2), $W = ((f'(v))^2 + (g'(v))^2)^{\frac{1}{2}} = 1$ and the unit normal vector of M is

$$N(u, v) = (0, -g'(v), f'(v)).$$

Further, we get

$$g_1 = 1 \quad \text{and} \quad g_2 = 0.$$

Thus, the coefficients of the second fundamental form are obtained as

$$L_{11} = -g'(v), \quad , L_{12} = 0, \quad L_{22} = f'(v)g''(v) - f''(v)g'(v). \tag{4.2}$$

From (2.3), the Gaussian and the mean curvatures of M are

$$K = -g'(f'g'' - f''g'), \quad H = \frac{f'g'' - f''g'}{2}$$

[7].
By (2.4) and (4.2), we obtain the principal curvatures κ_1, κ_2 of M as

$$\kappa_1 = f'g'' - f''g' \text{ and } \kappa_2 = -g'. \tag{4.3}$$

From the definition of the focal surface of a given surface and using the equations (4.3), we obtain two focal surfaces M_1^* and M_2^* of M with the parametrizations

$$X_1^*(u, v) = \left(u, f(v) - \frac{g'(v)}{\kappa_1(v)} + \frac{u^2}{2}, g(v) + \frac{f'(v)}{\kappa_1(v)}\right), \tag{4.4}$$

$$X_2^*(u, v) = \left(u, f(v) + \frac{u^2}{2} + 1, g(v) - \frac{f'(v)}{g'(v)}\right), \tag{4.5}$$

respectively, which are type I surface of revolution, too.

From Theorem 3.2, we have the following results:

Proposition 4.1. *Let M be a type I surface of revolution with the parametrization (4.1). If M is a part of an isotropic plane, consisting of a family of parabolas with $g' = 0$, $f' \neq 0$, i.e. M is flat or, equivalently minimal, then we cannot construct the focal surfaces of M.*

Proposition 4.2. *Let M be a type I surface of revolution with the parametrization (4.1). If M is a parabolic cylinder with $f' = 0$, $g' \neq 0$, i.e. M is flat or, equivalently minimal, then we have only the focal surface M_2^* with the parametrization*

$$X_2^*(u, v) = \left(u, c + \frac{u^2}{2}, g(v)\right),$$

which means that M_2^* is a parabolic cylinder and it is flat or, equivalently, minimal, too. Here c is a constant.

Proposition 4.3. *Let M be a type I surface of revolution with the parametrization (4.1). If M is a a parabolic cylinder with $f'g'' - f''g' = 0$, $g' \neq 0$, $f' \neq 0$, i.e. M is flat or, equivalently minimal, then we have only the focal surface M_2^* with the parametrization*

$$X_2^*(u,v) = \left(u, f(v) + \frac{u^2}{2} + 1, af(v) + b\right),$$

which means that M_2^* is a parabolic cylinder and flat or, equivalently minimal, too. Here $a, b \in \mathbb{R}$ with $a \neq 0$.

Example 4.4. Let us consider the type I surface of revolution M given with the parametrization (4.1) and the focal surface M_1^* of M with the parametrization (4.4) in $\mathbb{G}_3$. For the functions $f(v) = v^2$ and $g(v) = \frac{1}{2}v\sqrt{1-4v^2} + \frac{1}{4}arcsin(2v)$, the surface and its focal surface have the following parametrizations, respectively,

$$\begin{aligned} X(u,v) &= \left(u, v^2 + \frac{u^2}{2}, \frac{1}{2}v\sqrt{1-4v^2} + \frac{1}{4}arcsin(2v)\right), \\ X_1^*(u,v) &= \left(u, -v^2 + \frac{u^2}{2} + \frac{1}{2}, -\frac{1}{2}v\sqrt{1-4v^2} + \frac{1}{4}arcsin(2v)\right). \end{aligned}$$

By using the maple programme, we plot the graph of the surface of revolution and its focal surface in $\mathbb{G}_3$.

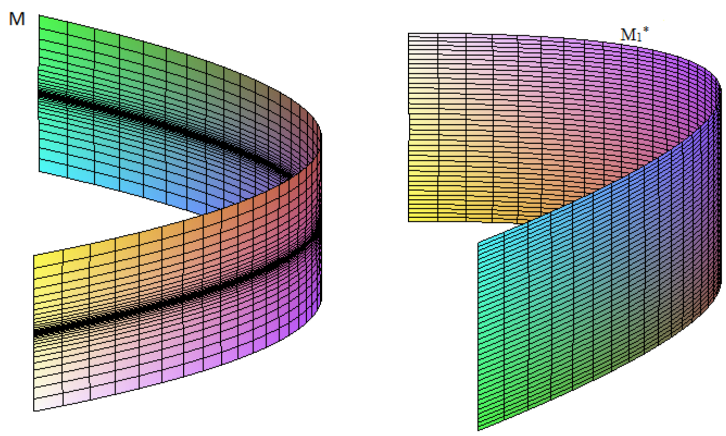

Figure 1. Surface of revolution M and the focal surface M_1^*

Example 4.5. Let us consider the type I surface of revolution M given with the parametrization (4.1) and the focal surface M_2^* of M with the parametrization (4.5) in $\mathbb{G}_3$. For the functions $f(v) = cosv$ and $g(v) = sinv$, the surface and its focal surface have the following parametrizations, respectively,

$$\begin{aligned} X(u,v) &= \left(u, cosv + \frac{u^2}{2}, sinv\right), \\ X_2^*(u,v) &= \left(u, cosv + \frac{u^2}{2} + 1, sinv + \frac{sinv}{cosv}\right). \end{aligned}$$

By using the maple programme, we plot the graph of the surface of revolution and its focal surface in $\mathbb{G}_3$.

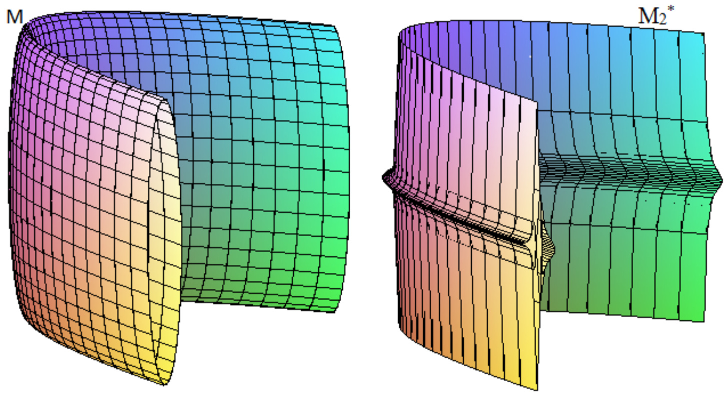

Figure 2. Surface of revolution M and the focal surface M_2^*

For the first focal surface M_1^*, the tangent space is spanned by the vectors

$$(X_1^*)_u = (1, u, 0), \quad (X_1^*)_v = (0, \lambda_1(v), \lambda_2(v)),$$

where

$$\begin{aligned} \lambda_1(v) &= f'(v) - \frac{g''(v)\kappa_1(v) - g'(v)\kappa_1'(v)}{(\kappa_1(v))^2}, \\ \lambda_2(v) &= g'(v) + \frac{f''(v)\kappa_1(v) - f'(v)\kappa_1'(v)}{(\kappa_1(v))^2}. \end{aligned}$$

Thus, from (2.1) and (2.2), $W^* = ((\lambda_1(v))^2 + (\lambda_2(v))^2)^{\frac{1}{2}}$ and the unit normal vector field N^* of M_1^* is

$$N^* = \frac{1}{W^*}(0, -\lambda_2(v), \lambda_1(v)). \tag{4.6}$$

Further, we get

$$g_1^* = 1, \quad g_2^* = 0. \tag{4.7}$$

The second partial derivatives of X_1^* are

$$(X_1^*)_{uu} = (0,1,0), \quad (X_1^*)_{uv} = (0,0,0), \quad (X_1^*)_{vv} = (0, \lambda_1'(v), \lambda_2'(v)). \tag{4.8}$$

Thus from the equations (4.6)-(4.8), the coefficients of the second fundamental form become

$$L_{11}^* = \frac{-\lambda_2(v)}{W^*}, \quad , L_{12}^* = 0, \quad L_{22}^* = \frac{-\lambda_1'(v)\lambda_2(v) + \lambda_1(v)\lambda_2'(v)}{W^*}. \tag{4.9}$$

By using the equations (4.7) and (4.9), we give the following theorems:

Theorem 4.6. *Let M be a type I surface of revolution given with the parametrization (4.1) and M_1^* be the focal surface of M with the parametrization (4.4) in $\mathbb{G}_3$. Then, the Gaussian and the mean curvatures of M_1^* are*

$$\begin{aligned} K^* &= \frac{-\lambda_2(\lambda_1\lambda_2' - \lambda_1'\lambda_2)}{(W^*)^4}, \\ H^* &= \frac{\lambda_1\lambda_2' - \lambda_1'\lambda_2}{2(W^*)^3}. \end{aligned}$$

Theorem 4.7. *Let M be a type I surface of revolution given with the parametrization (4.1) and M_1^* be the focal surface of M with the parametrization (4.4) in $\mathbb{G}_3$. The focal surface M_1^* is flat if and only if one of the following differential equations is hold:*

$$g'(v)(\kappa_1(v))^2 + f''(v)\kappa_1(v) - f'(v)\kappa_1'(v) = 0,$$

or

$$g'(v)(\kappa_1(v))^2 + f''(v)\kappa_1(v) - f'(v)\kappa_1'(v) = (f'(v)(\kappa_1(v))^2 - g''(v)\kappa_1(v) + g'(v)\kappa_1'(v))c_1,$$

where c_1 *is an integral constant.*

Proof. Let the focal surface M_1^* be flat. Then by the expression of the Gaussian curvature, either $\lambda_2(v) = 0$ or $\lambda_1(v)\lambda_2'(v) - \lambda_1'(v)\lambda_2(v) = 0$. If $\lambda_2(v) = 0$, then the first differential equation holds. If $\lambda_1(v)\lambda_2'(v) - \lambda_1'(v)\lambda_2(v) = 0$, we have $\frac{\lambda_2'(v)}{\lambda_2(v)} = \frac{\lambda_1'(v)}{\lambda_1(v)}$. Inregrating both sides of the last equation, we get $\lambda_2(v) = \lambda_1(v)c_1$, which corresponds to the second differential equation. □

Theorem 4.8. *Let* M *be a type I surface of revolution given with the parametrization (4.1) and* M_1^* *be the focal surface of* M *with the parametrization (4.4) in* $\mathbb{G}_3$*. The focal surface* M_1^* *is minimal if and only if the following differential equation is hold:*

$$g'(v)(\kappa_1(v))^2 + f''(v)\kappa_1(v) - f'(v)\kappa_1'(v) = (f'(v)(\kappa_1(v))^2 - g''(v)\kappa_1(v) + g'(v)\kappa_1'(v))c_1,$$

where c_1 *is an integral constant.*

Corollary 4.9. *If the focal surface* M_1^* *is minimal, then it is flat.*

Now, we consider the focal surface M_2^* given with the parametrization (4.5) in $\mathbb{G}_3$. The tangent space of the focal surface M_2^* is spanned by the vectors

$$(X_2^*)_u = (1, u, 0), \quad (X_2^*)_v = (0, f'(v), \lambda_3(v)),$$

where

$$\lambda_3(v) = g'(v) + \frac{\kappa_1(v)}{(g'(v))^2}, \quad W^* = ((f'(v))^2 + (\lambda_3(v))^2)^{\frac{1}{2}}.$$

From (2.2), the unit normal vector field N^* of M_2^* is

$$N^* = \frac{1}{W^*}(0, -\lambda_3(v), f'(v)). \tag{4.10}$$

Further, we get

$$g_1^* = 1, \quad g_2^* = 0. \tag{4.11}$$

The second partial derivatives of X_2^* are

$$(X_2^*)_{uu} = (0, 1, 0), \quad (X_2^*)_{uv} = (0, 0, 0), \quad (X_2^*)_{vv} = (0, f''(v), \lambda_3'(v)). \tag{4.12}$$

Thus from the equations (4.10)-(4.12), the coefficients of the second fundamental form become

$$L_{11}^* = \frac{-\lambda_3(v)}{W^*}, \quad , L_{12}^* = 0, \quad L_{22}^* = \frac{f'(v)\lambda_3'(v) - f''(v)\lambda_3(v)}{W^*}. \tag{4.13}$$

By using the equations (4.11) and (4.13), we give the following theorems:

Theorem 4.10. *Let* M *be a type I surface of revolution given with the parametrization (4.1) and* M_2^* *be the focal surface of* M *with the parametrization (4.5) in* $\mathbb{G}_3$*. Then, the Gaussian and the mean curvatures of* M_2^* *are*

$$K^* = \frac{-\lambda_3(f'\lambda_3' - f''\lambda_3)}{(W^*)^4},$$
$$H^* = \frac{f'\lambda_3' - f''\lambda_3}{2(W^*)^3},$$

respectively.

Theorem 4.11. *Let M be a type I surface of revolution given with the parametrization (4.1) and M_2^* be the focal surface of M with the parametrization (4.5) in $\mathbb{G}_3$. The focal surface M_2^* is flat if and only if one of the following systems is hold:*

$$\begin{aligned}(g'(v))^3 - f''(v)g'(v) + f'(v)g''(v) &= 0,\\ (f'(v))^2 + (g'(v))^2 &= 1, \quad g'(v) \neq 0,\end{aligned}$$

or

$$\begin{aligned}(g'(v))^3 - f''(v)g'(v) + f'(v)g''(v) &= f'(v)c_2,\\ (f'(v))^2 + (g'(v))^2 &= 1, \quad g'(v) \neq 0,\end{aligned}$$

where c_2 is an integral constant.

Proof. Let the focal surface M_2^* be flat. Then by the expression of the Gaussian curvature, either $\lambda_3(v) = 0$ or $f'(v)\lambda_3'(v) - f''(v)\lambda_3(v) = 0$. If $\lambda_3(v) = 0$, then the first differential equation system holds. If $f'(v)\lambda_3'(v) - f''(v)\lambda_3(v) = 0$, we have $\frac{\lambda_3'(v)}{\lambda_3(v)} = \frac{f''(v)}{f'(v)}$. Inregrating both sides of the last equation, we get $\lambda_3(v) = f'(v)c_2$, which corresponds to the second differential equation system. □

Theorem 4.12. *Let M be a type I surface of revolution given with the parametrization (4.1) and M_2^* be the focal surface of M with the parametrization (4.5) in $\mathbb{G}_3$. The focal surface M_2^* is mimimal if and only if the following system is hold:*

$$\begin{aligned}(g'(v))^3 - f''(v)g'(v) + f'(v)g''(v) &= f'(v)c_2,\\ (f'(v))^2 + (g'(v))^2 &= 1, \quad g'(v) \neq 0,\end{aligned}$$

where c_2 is an integral constant.

Corollary 4.13. *If the focal surface M_2^* is minimal, then it is flat.*

4.2. Focal surface of type II surface of revolution

Let $\alpha(v) = (v, g(v), 0)$ be a unit speed curve in $\mathbb{G}_3$. Then, taking $c = 1$ in (3.3), type II surface of revolution M can be written as in the following form

$$X(u, v) = \left(u + v, g(v), uv + \frac{u^2}{2}\right). \tag{4.14}$$

The tangent space of M at an arbitrary point is spanned by the vectors

$$X_u = (1, 0, u + v), \quad X_v = (1, g'(v), u).$$

From (2.1) and (2.2), $W = (v^2 + (g'(v))^2)^{\frac{1}{2}}$ and the unit normal vector field of M is

$$N(u, v) = \frac{1}{W}(0, v, g'(v)).$$

Further, we get

$$g_1 = 1 \quad \text{and} \quad g_2 = 1.$$

Thus, the coefficients of the second fundamental form are obtained

$$L_{11} = \frac{g'(v)}{W}, \quad , L_{12} = \frac{g'(v)}{W}, \quad L_{22} = \frac{vg''(v)}{W}. \tag{4.15}$$

The Gaussian and the mean curvatures of M are

$$K = \frac{g'(vg'' - g')}{W^4}, \quad H = \frac{vg'' - g'}{2W^3}$$

[7].

From (2.4) and (4.15), we obtain the principal curvatures κ_1, κ_2 of M as

$$\kappa_1 = \frac{1}{W^3}\left(vg'' - g'\right) \quad \text{and} \quad \kappa_2 = \frac{g'}{W}. \tag{4.16}$$

From the definition of the focal surface of a given surface and using the equations (4.16), we obtain two focal surfaces M_1^* and M_2^* of M as

$$X_1^*(u,v) = \left(u+v, g(v)+\frac{v}{W\kappa_1(v)}, uv+\frac{u^2}{2}+\frac{g'(v)}{W\kappa_1(v)}\right), \tag{4.17}$$

$$X_2^*(u,v) = \left(u+v, g(v)+\frac{v}{g'(v)}, uv+\frac{u^2}{2}+1\right), \tag{4.18}$$

respectively.

From Theorem 3.3., we obtain the following results:

Proposition 4.14. *Let M be a type II surface of revolution with the parametrization (4.14). If $g'=0$, g is a constant function, then we cannot construct the focal surfaces of M.*

Proposition 4.15. *Let M be a type II surface of revolution with the parametrization (4.14). If M is a cyclic surface (parabolic sphere), i.e. flat or, equivalently minimal, then we have only the focal surface M_2^* with the parametrization*

$$X_2^*(u,v) = (u+v, cv^2+d, uv+\frac{u^2}{2}+1), \tag{4.19}$$

which means that M_2^* is a cyclic surface (parabolic sphere) and flat or, equivalently minimal, too.

Example 4.16. Let us consider the type II surface of revolution M given with the parametrization (4.14) and the focal surface M_1^* of M with the parametrization (4.17) in $\mathbb{G}_3$. For the function $g(v)=e^v$, the surface and its focal surface have the following parametrizations, respectively

$$X(u,v) = \left(u+v, e^v, uv+\frac{u^2}{2}\right),$$

$$X_1^*(u,v) = \left(u+v, e^v+\frac{(v^2+e^{2v})v}{e^v(v-1)}, uv+\frac{u^2}{2}+\frac{(v^2+e^{2v})e^v}{e^v(v-1)}\right).$$

By using the maple programme, we plot the graph of the surface of revolution and its focal surface in $\mathbb{G}_3$.

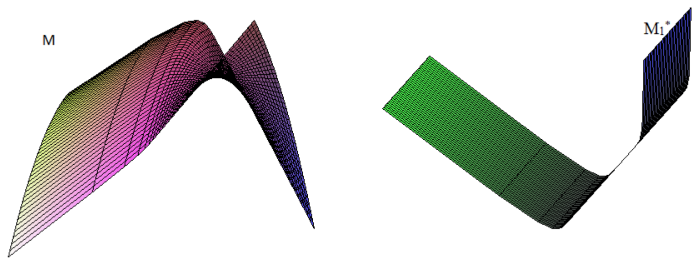

Figure 3. Surface of revolution M and the focal surface M_1^*

Example 4.17. Let us consider the type II surface of revolution M given with the parametrization (4.14) and the focal surface M_2^* of M with the parametrization (4.18) in $\mathbb{G}_3$. For the function $g(v)=e^v$, the surface and its focal surface have the following parametrizations, respectively

$$\begin{aligned} X(u,v) &= \left(u+v, e^v, uv+\frac{u^2}{2}\right), \\ X_2^*(u,v) &= \left(u+v, e^v+\frac{v}{e^v}, uv+\frac{u^2}{2}+1\right). \end{aligned}$$

By using the maple programme, we plot the graph of the surface of revolution and its focal surface in $\mathbb{G}_3$.

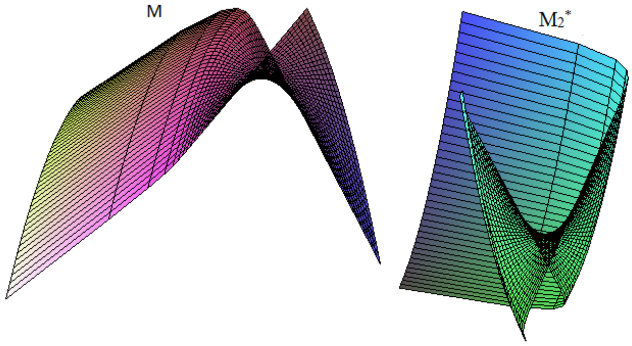

Figure 4. Surface of revolution M and the focal surface M_2^*

For the first focal surface M_1^*, the tangent space is spanned by the vectors

$$(X_1^*)_u = (1, 0, u+v), \quad (X_1^*)_v = (1, \lambda_4(v), \lambda_5(v)),$$

where

$$\begin{aligned} \lambda_4(v) &= g'(v) + \frac{W\kappa_1(v) - v(W\kappa_1(v))'}{(W\kappa_1(v))^2}, \\ \lambda_5(v) &= u + \frac{g''(v)W\kappa_1(v) - g'(v)(W\kappa_1(v))'}{(W\kappa_1(v))^2}. \end{aligned}$$

Thus, from (2.1) and (2.2), $W^* = ((-\lambda_5(v)+u+v)^2 + (\lambda_4(v))^2)^{\frac{1}{2}}$ and the unit normal vector field N^* of M_1^* is

$$N^* = \frac{1}{W^*}(0, -\lambda_5(v)+u+v, \lambda_4(v)). \tag{4.20}$$

Further, we get

$$g_1^* = 1, \quad g_2^* = 1. \tag{4.21}$$

The second partial derivatives of X_1^* are

$$(X_1^*)_{uu} = (0,0,1), \quad (X_1^*)_{uv} = (0,0,1), \quad (X_1^*)_{vv} = (0, \lambda_4'(v), \lambda_5'(v)). \tag{4.22}$$

Thus from the equations (4.20)-(4.22), the coefficients of the second fundamental form become

$$L_{11}^* = \frac{\lambda_4(v)}{W^*}, \quad , L_{12}^* = \frac{\lambda_4(v)}{W^*}, \quad L_{22}^* = \frac{\lambda_4'(v)(-\lambda_5(v)+u+v) + \lambda_4(v)\lambda_5'(v)}{W^*}. \tag{4.23}$$

By using the equations (4.21) and (4.23), we give the following theorems:

Theorem 4.18. *Let M be a type II surface of revolution given with the parametrization (4.14) and M_1^* be the focal surface of M with the parametrization (4.17) in $\mathbb{G}_3$. Then, the Gaussian and the mean curvatures of M_1^* are*

$$\begin{aligned} K^* &= \frac{\lambda_4}{(W^*)^4}\left(\lambda_4'(-\lambda_5+u+v)+\lambda_4(\lambda_5'-1)\right), \\ H^* &= \frac{\lambda_4'(-\lambda_5+u+v)+\lambda_4(\lambda_5'-1)}{2(W^*)^3}. \end{aligned}$$

Theorem 4.19. *Let M be a type II surface of revolution given with the parametrization (4.14) and M_1^* be the focal surface of M with the parametrization (4.17) in $\mathbb{G}_3$. The focal surface M_1^* is flat if and only if one of the following differential equations is hold:*

$$g'\frac{(vg''-g')^2}{(v^2+(g')^2)^2}+\frac{vg''-g'}{v^2+(g')^2}-v\left(\frac{vg''-g'}{v^2+(g')^2}\right)'=0,$$

or

$$(g'+vh)\frac{(vg''-g')^2}{(v^2+(g')^2)^2}+(1-g''h)\frac{vg''-g'}{v^2+(g')^2}+(g'h-v)\left(\frac{vg''-g'}{v^2+(g')^2}\right)'=0,$$

where $h=h(u)$ is a function of the variable u.

Proof. Let the focal surface M_1^* be flat. Then by the expression of the Gaussian curvature, either $\lambda_4(v)=0$ or $\lambda_4'(v)(-\lambda_5(v)+u+v)+\lambda_4(v)(\lambda_5'(v)-1)=0$. If $\lambda_4(v)=0$, then the first differential equation holds. If $\lambda_4'(v)(-\lambda_5(v)+u+v)+\lambda_4(v)(\lambda_5'(v)-1)=0$, we have $\frac{\lambda_4'(v)}{\lambda_4(v)}=\frac{\lambda_5'(v)-1}{\lambda_5(v)-u-v}$. Inregrating both sides of the last equation, we get $\lambda_4(v)=(\lambda_5(v)-u-v)h(u)$, which corresponds to the second differential equation. □

Theorem 4.20. *Let M be a type II surface of revolution given with the parametrization (4.14) and M_1^* be the focal surface of M with the parametrization (4.17) in $\mathbb{G}_3$. The focal surface M_1^* is minimal if and only if the following differential equation is hold:*

$$(g'+vh)\frac{(vg''-g')^2}{(v^2+(g')^2)^2}+(1-g''h)\frac{vg''-g'}{v^2+(g')^2}+(g'h-v)\left(\frac{vg''-g'}{v^2+(g')^2}\right)'=0,$$

where $h=h(u)$ is a function of the variable u.

Corollary 4.21. *If the focal surface M_1^* is minimal, then it is flat.*

Now, we consider the focal surface M_2^* given with the parametrization (4.18). The tangent space of the focal surface M_2^* is spanned by the vectors

$$(X_2^*)_u=(1,0,u+v),\quad (X_2^*)_v=(1,\lambda_6(v),u),$$

where

$$\lambda_6(v)=g'(v)+\frac{g'(v)-vg''(v)}{(g'(v))^2},\quad W^*=((\lambda_6(v))^2+v^2)^{\frac{1}{2}}.$$

Thus, from (2.2), the unit normal vector field N^* of M_2^* is

$$N^*=\frac{1}{W^*}(0,v,\lambda_6(v)). \tag{4.24}$$

Further, we get

$$g_1^*=1,\quad g_2^*=1. \tag{4.25}$$

The second partial derivatives of X_2^* are

$$(X_2^*)_{uu}=(0,0,1),\quad (X_2^*)_{uv}=(0,0,1),\quad (X_2^*)_{vv}=(0,\lambda_6'(v),0). \tag{4.26}$$

Thus from the equations (4.24)-(4.26), the coefficients of the second fundamental form become

$$L_{11}^*=\frac{\lambda_6(v)}{W^*},\quad ,L_{12}^*=\frac{\lambda_6(v)}{W^*},\quad L_{22}^*=\frac{v\lambda_6'(v)}{W^*}. \tag{4.27}$$

By using the equations (4.25) and (4.27), we give the following theorems:

Theorem 4.22. *Let M be a type II surface of revolution given with the parametrization (4.14) and M_2^* be the focal surface of M with the parametrization (4.18) in $\mathbb{G}_3$. Then, the Gaussian and the mean curvatures of M_2^* are*

$$\begin{aligned} K^* &= \frac{\lambda_6(v\lambda_6' - \lambda_6)}{(W^*)^4}, \\ H^* &= \frac{v\lambda_6' - \lambda_6}{2(W^*)^3}. \end{aligned}$$

Theorem 4.23. *Let M be a type II surface of revolution given with the parametrization (4.14) and M_2^* be the focal surface of M with the parametrization (4.18) in $\mathbb{G}_3$. The focal surface M_2^* is flat if and only if either*

$$g(v) = \pm\sqrt{-v^2 + c_1} + c_2,$$

where c_1 and c_2 are integral contants or

$$(g'(v))^3 + g'(v) - vg''(v) - v(g'(v))^2 c_3 = 0,$$

where c_3 is an integral contant.

Proof. Let the focal surface M_2^* be flat. Then by the expression of the Gaussian curvature, either $\lambda_6(v) = 0$ or $v\lambda_6'(v) - \lambda_6(v) = 0$. If $\lambda_6(v) = 0$, then $g(v) = \pm\sqrt{-v^2 + c_1} + c_2$. If $v\lambda_6'(v) - \lambda_6(v) = 0$, we have $\frac{\lambda_6'(v)}{\lambda_6(v)} = \frac{1}{v}$. Inregrating both sides of the last equation, we get $\lambda_6(v) = c_3 v$, which corresponds to the second differential equation. □

Theorem 4.24. *Let M be a type II surface of revolution given with the parametrization (4.14) and M_2^* be the focal surface of M with the parametrization (4.18) in $\mathbb{G}_3$. The focal surface M_2^* is minimal if and only if*

$$(g'(v))^3 + g'(v) - vg''(v) - v(g'(v))^2 c_3 = 0,$$

where c_3 is an integral contant.

Corollary 4.25. *If the focal surface M_2^* is minimal, then it is flat.*

4.3. Focal surface of type III surface of revolution

Let $\alpha(v) = (v, g(v), 0)$ be a unit speed curve in $\mathbb{G}_3$. Then, from (3.4), type III surface of revolution M is given as in the following:

$$X(u, v) = (v, g(v)cosu, -g(v)sinu). \tag{4.28}$$

The tangent space of M at an arbitrary point is spanned by the vectors

$$X_u = (0, -g(v)sinu, -g(v)cosu), \quad X_v = (1, g'(v)cosu, -g'(v)sinu).$$

Thus from (2.1) and (2.2), $W = |g(v)|$ and the unit normal vector field of M is

$$N(u, v) = (0, -cosu, sinu). \tag{4.29}$$

Further, we get

$$g_1 = 0 \quad \text{and} \quad g_2 = 1. \tag{4.30}$$

Thus, the coefficients of the second fundamental form are obtained

$$L_{11} = g(v), \quad , L_{12} = 0, \quad L_{22} = -g''(v). \tag{4.31}$$

The Gaussian and the mean curvatures of M are

$$K = \frac{-g''}{g}, \quad H = \frac{1}{2g}$$

[7]. Then from (2.4) and (4.31), we get the principal curvature functions as

$$\kappa_1 = \frac{1}{g}, \quad \text{and} \quad \kappa_2 = -g''.$$

For the function $\kappa_1 = \frac{1}{g}$, the focal surface degenerates to a curve. Thus, we obtain the focal surface M^* of M for the function $\kappa_2 = -g''$ as

$$X^*(u,v) = \left(v, \left(g(v) + \frac{1}{g''(v)}\right) cosu, -\left(g(v) + \frac{1}{g''(v)}\right) sinu\right), \tag{4.32}$$

where $g'' \neq 0$.

Example 4.26. Let us consider the type III surface of revolution M given with the parametrization (4.28) and the focal surface M^* of M with the parametrization (4.32) in $\mathbb{G}_3$. For the function $g(v) = lnv$, the surface and its focal surface have the following parametrizations, respectively

$$\begin{aligned} X(u,v) &= (v, ln(v)cosu, -ln(v)sinu), \\ X^*(u,v) &= (v, (ln(v) - v^2)cosu, -(ln(v) - v^2)sinu). \end{aligned}$$

By using the maple programme, we plot the graph of the surface of revolution and its focal surface in $\mathbb{G}_3$.

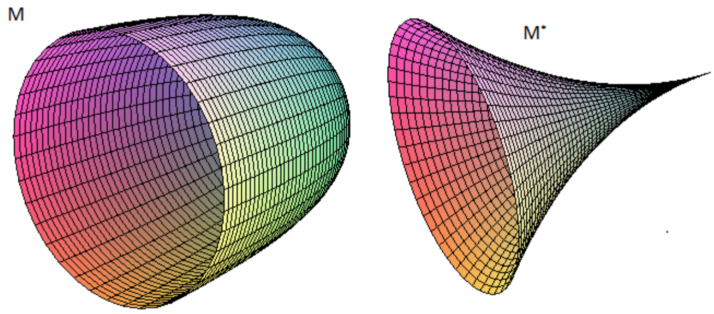

Figure 5. Surface of revolution M and the focal surface M^*

The tangent space of the focal surface M^* is spanned by the vectors

$$\begin{aligned} (X^*)_u &= (0, -\lambda_7(v)sinu, -\lambda_7(v)cosu), \\ (X^*)_v &= (1, \lambda_7'(v)cosu, -\lambda_7'(v)sinu), \end{aligned}$$

where $\lambda_7(v) = g(v) + \frac{1}{g''(v)}$ and $W^* = |\lambda_7(v)|$. Thus, from (2.2) the unit normal vector field N^* of M^* is

$$N^* = (0, -cosu, sinu). \tag{4.33}$$

Further, we get

$$g_1^* = 0, \quad g_2^* = 1. \tag{4.34}$$

The second partial derivatives of X^* are

$$\begin{aligned} (X^*)_{uu} &= (0, -\lambda_7(v)cosu, \lambda_7(v)sinu), \\ (X^*)_{uv} &= (0, -\lambda_7'(v)sinu, -\lambda_7'(v)cosu), \\ (X^*)_{vv} &= (0, \lambda_7''(v)cosu, -\lambda_7''(v)sinu). \end{aligned} \tag{4.35}$$

Thus from the equations (4.33)-(4.35), the coefficients of the second fundamental forms become

$$L^*_{11} = \lambda_7(v), \quad , L^*_{12} = 0, \quad L^*_{22} = -\lambda''_7(v). \tag{4.36}$$

By using the equations (4.34) and (4.36), we give the following theorems:

Theorem 4.27. *Let M be a type III surface of revolution given with the parametrization (4.28) and M^* be the focal surface of M with the parametrization (4.32) in $\mathbb{G}_3$. Then, the Gaussian and the mean curvatures of M^* are*

$$K^* = \frac{-\lambda''_7}{\lambda_7}, \quad H^* = \frac{1}{2\lambda_7}.$$

Theorem 4.28. *Let M be a type III surface of revolution given with the parametrization (4.28) and M^* be the focal surface of M with the parametrization (4.32) in $\mathbb{G}_3$. The focal surface M^* is flat if and only if*

$$\left(g(v) + \frac{1}{g''(v)}\right)'' = 0. \tag{4.37}$$

Corollary 4.29. *The focal surface M^* cannot be minimal.*

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