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# Spectrum of Discrete Sturm-Liouville Equation with Self-adjoint Operator Coefficients on the Half-line

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**ABSTRACT:** We investigate the spectrum of the Sturm-Liouville difference equation on the half-line with self-adjoint operator coefficients in an infinite dimensional Hilbert space together with the Dirichlet boundary condition. We find the Jost solution and examine its analytical and asymptotical properties. Using these properties, we obtain the continuous and point spectrum of the discrete operator generated by the Sturm-Liouville difference equation with self-adjoint operator coefficients. We also show that this operator has a finite number of eigenvalues with finite multiplicities under a certain condition on the operator coefficients.

**Keywords:** Discrete operator, eigenvalues, self-adjoint operators, Sturm-Liouville difference equation

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## INTRODUCTION

Self-adjoint differential operators play a fundamental role in physics, especially in quantum mechanics. Discrete analogues of these operators which are generated by difference equations also have significant importance. Spectral analysis of difference equations provides a lot of information about these equations. As a result, there are many studies on the spectral analysis of discrete operators generated by difference equations recently (Yokus and Coskun, 2019; Yokus and Coskun, 2020).

Studies on Sturm-Liouville differential and difference equations with matrix or operator coefficients have grown extensively in recent years (Bairamov et al., 2017; Mutlu, 2020; Mutlu and Kir Arpat 2020a; Mutlu and Kir Arpat 2020b; Aktosun and Weder, 2020). In particular, spectral properties of Sturm-Liouville differential (Aktosun and Weder, 2020) and difference equations (Aygar and Bairamov, 2012; Bairamov et al., 2016) with hermitian matrix coefficients have been examined. In (Aygar and Bairamov, 2012) spectral analysis of the discrete operator $L_0$ generated by matrix difference equation of second order on the half-line

$$A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N} \tag{1}$$

and the boundary condition $y_0 = 0$ has been studied. Here $A_n$ $(n \in \mathbb{N} \cup \{0\})$ and $B_n$ $(n \in \mathbb{N})$ are $m \times m$ hermitian matrices $(m < \infty)$, $detA_n \neq 0$ $(n \in \mathbb{N} \cup \{0\})$ and $\lambda$ is a spectral parameter. Note that equation (1) can be written in the Sturm-Liouville form

$$\Delta(A_{n-1}\Delta y_{n-1}) + Q_n y_n = \lambda y_n,\ n \in \mathbb{N} \tag{2}$$

where $Q_n = A_{n-1} + A_n + B_n$ and "$\Delta$" denotes the forward difference operator. As a result, we can refer to equation equation (1) as Sturm-Liouville difference equation with matrix coefficients. It was proven that the continuous spectrum of $L_0$ is [-2,2] and the number of eigenvalues are finite if (Aygar and Bairamov, 2012)

$$\sum_{i=1}^{\infty} n(\|I_n - A_n\| + \|B_n\|) < \infty \tag{3}$$

holds where "$\|,\|$" denotes any of the equivalent matrix norms. These results have been extended to the whole axis (Bairamov et al., 2016) by considering equation (1) for $n \in \mathbb{Z}$.

There have been many efforts for transition to infinite dimension i.e. considering Sturm-Liouville differential or difference equations with operator coefficients in an infinite dimensional Hilbert space. First attempts were made to analyze Sturm-Liouville operator equation on the half-line (Gasymov et al., 1967). The authors considered the following operator equation with self-adjoint operator coefficients. Let $H$ be an infinite dimensional separable Hilbert space and $L_2(\mathbb{R}_+, H)$ denote the space of vector-valued functions $f(x)$ defined on $(0,\infty)$ which are strongly-integrable in each finite subinterval of $(0,\infty)$ and such that

$$\int_0^{\infty} \|f(x)\|_H^2 dx < \infty.$$

Consider the differential expression in $L_2(\mathbb{R}_+, H)$

$$l_0(y) = -y'' + V(x)y,\ x \epsilon \mathbb{R}_+, \tag{4}$$

where $V(x)$ is a self-adjoint, completely continuous operator in $H$ for each $x \in (0,\infty)$. The discrete spectrum of the operator generated by $l_0$ and the boundary condition $y_0 = 0$ has been studied (Gasymov et al., 1967). Recently, spectral properties of the non-self-adjoint operator which is generated by expression (4) with non-self-adjoint operator coefficients in an infinite dimensional Hilbert space and Dirichlet boundary condition has been studied (Bairamov et al., 2017). The discrete spectrum and the spectral singularities were obtained and the finiteness of eigenvalues and spectral singularities was proven (Bairamov et al., 2017). Further, these results have been extended to whole real line (Mutlu and Kir Arpat 2020a).

For the operator coefficient case, transition to infinite dimension is not trivial. If the coefficients are finite dimensional matrices, then the Jost function is the determinant of the Jost matrix and therefore a scalar function. Eigenvalues and spectral singularities are obtained as zeros of this scalar analytic function and by using properties of analytic functions one can obtain several results about the structure of the spectrum. However, if the coefficients are infinite dimensional operators, then the Jost function is an operator-valued function. In this case eigenvalues and spectral singularities correspond to the singular points of an operator-valued function and therefore the methods and tools need to be changed. The new approach (see Bairamov et al., 2017) originates from Keldysh's famous results (Keldysh, 1971).

Motivated by the above studies we consider the following difference equation

$$A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N} \tag{5}$$

where $A_n$ $(n \in \mathbb{N} \cup \{0\})$ and $B_n$ $(n \in \mathbb{N})$ are self-adjoint, $A_n - I$ $(n \in \mathbb{N} \cup \{0\})$ and $B_n$ $(n \in \mathbb{N})$ are compact operators in an infinite dimensional separable Hilbert space $H$ where $I$ denotes the identity operator in $H$ and $\lambda$ is a spectral parameter. Also, we assume $A_n$ is invertible for each $(n \in \mathbb{N} \cup \{0\})$. Let $H_1 \coloneqq l_2(\mathbb{N}, H)$ denote the space of vector sequences $y = (y_n) \subset H$ such that

$$\|y\|_1 \coloneqq \sum_{n=1}^{\infty} \|y_n\|_H^2 < \infty.$$

$H_1$ is a Hilbert space with the inner product

$$(y, z)_1 \coloneqq \sum_{n=1}^{\infty} (y_n, z_n)_H.$$

Note that the sequence $y_n$ in equation (5) can be regarded both as a vector sequence in $H_1$ or an operator sequence i.e. $y_n$ is an operator in $H$ for each $n \in \mathbb{N}$. We consider the operator $L$ generated by equation (5) in $H_1$ and the boundary condition

$$y_0 = 0. \tag{6}$$

One can also define the operator $L$ by infinite Jacobi matrix

$$J = \begin{bmatrix} B_1 & A_1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ A_1 & B_2 & A_2 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & A_2 & B_3 & A_3 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{7}$$

From this definition it easily follows that $L$ is a self-adjoint operator in $H_1$.

In this paper, we investigate the spectral properties of Sturm-Liouville difference equation with self-adjoint operator coefficients. This equation is the most general form of a difference equation of order 2 and can be regarded as a discretization of Sturm-Liouville operator equation defined by (4). Although there are several studies on the spectral analysis of Sturm-Liouville differential equation with matrix or operator coefficients (Gasymov et al., 1967; Kir Arpat and Mutlu, 2015; Bairamov et al., 2017; Aktosun and Weder, 2020; Mutlu and Kir Arpat 2020a), Sturm-Liouville difference equation with operator coefficients awaits further investigation. We aim to extend the results in finite dimensional case i.e. Sturm-Liouville difference equation with $m \times m$ matrix coefficients to the infinite dimensional case by considering Sturm-Liouville difference equation with self-adjoint operator coefficients in an infinite dimensional Hilbert space. When the problem works in infinite dimension it creates some important diversity due the theory of infinite dimensional operators. The outline of this paper is as follows. Firstly, we present the Jost solution and its analytical and asymptotic properties in the next section. Later, we obtain the continuous and point spectrum and prove the finiteness of the eigenvalues in results and discussion section.

## MATERIALS AND METHODS

In this section we introduce the Jost solution of equation (5) and provide necessary properties of Jost solution which are required for our main results. Note that the proofs of the results in this section can be obtained similarly as in matrix coefficient case (Aygar and Bairamov, 2012). However we shall outline the proofs for the sake of completeness.

**Assumption 1.** We assume that the coefficients of equation (5) satisfy

$$\sum_{i=1}^{\infty}\|I_n - A_n\| + \|B_n\| < \infty$$

where "$\|,\|$" denotes the operator norm.

**Definition 1:** Let us denote the operator solution of the equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = (z+z^{-1})Y_n,\ n \in \mathbb{N},\ z \in D_0 := \{z \in \mathbb{C}: |z| = 1\} \quad (8)$$

which satisfies the condition

$$\lim_{n\to\infty} Y_n(z)\, z^{-n} = I,$$

by $F(z) := F_n(z)$ $(n \in \mathbb{N} \cup \{0\})$. $F(z)$ is called the Jost solution of Equation (5).

**Theorem 1.** The Jost solution exists and has the representation

$$F_n(z) = z^nI + \sum_{k=n+1}^{\infty}\frac{z^{k-n}-z^{n-k}}{z-z^{-1}}\left[(I-A_{k-1})F_{k-1}(z) - B_kF_k(z) + (I-A_k)F_{k+1}(z)\right]. \quad (9)$$

**Proof.** Let $U_n := (I-A_{n-1})Y_{n-1} - B_nY_n + (I-A_n)Y_{n+1}$. Equation (8) can be written

$$Y_{n-1} + Y_{n+1} - (z+z^{-1})Y_n = U_n. \quad (10)$$

The general solution of the homogenous part of this equation is

$$Y_n = cz^nI + dz^{-n}I,$$

where $c$ and $d$ are constants. Applying the variation of parameters method, we look for the general solution of (10) in the form

$$Y_n = C_nz^n + D_nz^{-n},$$

where $C_n$ and $D_n$ are operators in $H$. We obtain

$$C_n = T - \sum_{k=n+1}^{\infty}\frac{z^{-k}U_k}{z-z^{-1}},\quad D_n = S + \sum_{k=n+1}^{\infty}\frac{z^{k}U_k}{z-z^{-1}},$$

where $T$ and $S$ are operators in $H$. Note that the series on the right hand sides of last equations are convergent in the operator norm under Assumption 1. Substituting $C_n$ and $D_n$ we obtain the general solution of (10)

$$Y_n(z) = z^nT + z^{-n}S + \sum_{k=n+1}^{\infty}\frac{z^{k-n}-z^{n-k}}{z-z^{-1}}U_k.$$

Imposing the boundary condition $\lim_{n\to\infty} Y_n(z)\, z^{-n} = I$ we reach (9).

**Assumption 2.** We make a stronger assumption than the Assumption 1. We assume

$$\sum_{n=1}^{\infty} n(\|I-A_n\| + \|B_n\|) < \infty.$$

**Theorem 2.** The Jost solution can be represented

$$F_n(z) = T_nz^n\left[I + \sum_{m=1}^{\infty}K_{n,m}z^m\right], n \in \mathbb{N} \cup \{0\}, \quad (11)$$

where $T_n$ and $K_{n,m}$ can be represented by using $A_n$ and $B_n$. Further,

$$\|K_{n,m}\| \le c\sum_{p=n+\left[\!\left[\frac{m}{2}\right]\!\right]}^{\infty}\left(\|I-A_p\| + \|B_p\|\right), \quad (12)$$

holds where $c > 0$ is a constant. Moreover, $F_n(z)$ $(n \in \mathbb{N} \cup \{0\})$ has an analytic continuation from $D_0$ to $D_1 := \{z \in \mathbb{C}: |z| < 1\}\backslash\{0\}$.

**Proof.** Plugging $F_n(z)$ defined by (11) into (8) we have

$$A_{n-1}\left\{z^{n-1}T_{n-1}\left[I + \sum_{m=1}^{\infty}K_{n-1,m}z^m\right]\right\} + B_n\left\{z^nT_n\left[I + \sum_{m=1}^{\infty}K_{n,m}z^m\right]\right\}$$
$$+A_n\left\{z^{n+1}T_{n+1}\left[I + \sum_{m=1}^{\infty}K_{n+1,m}z^m\right]\right\} = z^{n+1}T_n\left[I + \sum_{m=1}^{\infty}K_{n,m}z^m\right] + z^{n-1}T_n\left[I + \sum_{m=1}^{\infty}K_{n,m}z^m\right].$$

This equation easily implies

$T_n = \prod_{p=n}^{\infty} A_p^{-1},$ (13)
$K_{n,1} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p,$ (14)
$K_{n,2} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,1} + \sum_{p=n+1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p,$ (15)
$K_{n,m+2} = \sum_{p=n+1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p K_{p+1,m} - \sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{n+1,m},$ (16)
for $n \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$. Note that Assumption 2 guarantees the strong convergence of the infinite series and product in equations (13)-(16). Moreover, equations (14)-(16) yield the estimate (12). From (11) and (12) it follows $F_n(z)$ $(n \in \mathbb{N} \cup \{0\})$ has an analytic continuation from $D_0$ to $D_1$.

**Theorem 3.** The following asymptotic relation holds:

$F_n(z) = z^n[I + o(1)], \; z \in D := \{z \in \mathbb{C}: |z| \le 1\}\backslash\{0\}, \quad n \to \infty.$ (17)

**Proof.** From (13) it follows $T_n \to I, n \to \infty$. Equation (12) implies

$$\left\|\sum_{m=1}^{\infty} K_{n,m} z^m\right\| \le c \sum_{p=n}^{\infty} p\left(\|I - A_p\| + \|B_p\|\right).$$

Taking $n \to \infty$ in this inequality and using Assumption 2 yield

$$\sum_{m=1}^{\infty} K_{n,m} z^m = o(1), z \in D, n \to \infty.$$

As a result, we have the asymptotic relation (17).

## RESULTS AND DISCUSSION

In this section we find the continuous and discrete spectrum of $L$ denoted by $\sigma_c(L)$ and $\sigma_d(L)$ by using the analytical and asymptotic properties of the Jost solution $F_n(z)$ that we obtained in previous section. We show that the point spectrum is bounded, countable and its only limit point (if exists) can be $\lambda = -2$ or $\lambda = 2$. Moreover, we prove that $L$ has a finite number of eigenvalues with finite multiplicities under Assumption 2.

**Theorem 4.** The continuous spectrum of $L$ is $\sigma_c(L) = [-2, 2]$.

**Proof.** Let $L_0$ and $L_1$ denote the operators defined in $H_1$

$$L_0(y)_n = y_{n-1} + y_{n+1}, n \in \mathbb{N},$$
$$L_1(y)_n = (A_{n-1} - I) y_{n-1} + B_n y_n + (A_n - I) y_{n+1}, n \in \mathbb{N},$$

with the boundary condition $y_0 = 0$, respectively.

We can also define the operators $L_0$ and $L_1$ by using Jacobi matrices

$$J_0 = \begin{bmatrix} 0 & I & 0 & 0 & \cdots \\ I & 0 & I & 0 & \cdots \\ 0 & I & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix}$$

and

$$J_1 = \begin{bmatrix} B_1 & A_1 - I & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ A_1 - I & B_2 & A_2 - I & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & A_2 - I & B_3 & A_3 - I & 0 & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

respectively. We have $L_0 = L_0^*$ since the matrix $J_0$ is symmetric and also $\sigma_c(L_0) = [-2, 2]$ (Serebrjakov, 1980).

In order to show that $L_1$ is a compact operator we will show that $L_1$ is bounded and the set

$$R = \{L_1 y: \; \|y\|_1 \le 1\}$$

is compact in $H_1$. It is obvious that $L_1$ is bounded. Moreover, if we use the compactness criteria in $l_p$ spaces (Lusternik and Sobolev, 1974), we obtain the compactness of $R$. Indeed, let $y \in H_1$ such that $\|y\|_1 \le 1$. Then, Assumption 2 implies that for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$\sum_{i=n+1}^{\infty}(\|A_i - I\| + \|B_i\|) < \frac{\varepsilon}{C},$$

holds and also

$\sum_{i=n+1}^{\infty}\|(L_1 y)_i\|_H^2 = \sum_{i=n+1}^{\infty}\|(A_{i-1} - I)y_{i-1} + B_i y_i + (A_i - I)y_{i+1}\|_H^2$

$\leq \sum_{i=n+1}^{\infty}\|A_{i-1} - I\|^2\|y_{i-1}\|_H^2 + \|B_i\|^2\|y_i\|_H^2 + \|A_i - I\|^2\|y_{i+1}\|_H^2$

$\leq \|y\|_1^2 \sum_{i=n+1}^{\infty}\|A_{i-1} - I\|^2 + \|B_i\|^2 + \|A_i - I\|^2$

$\leq \sum_{i=n+1}^{\infty} 2\|A_i - I\|^2 + \|B_i\|^2$

$\leq \sum_{i=n+1}^{\infty} C_1\|A_i - I\| + C_2\|B_i\|$

$\leq \sum_{i=n+1}^{\infty} C(\|A_i - I\| + \|B_i\|)$

$< \varepsilon,$

where

$$C_1 = 2sup_{i\geq n_0}\|A_i - I\|, \qquad C_2 = sup_{i\geq n_0}\|B_i\|, \qquad C = C_1 + C_2.$$

Therefore, we have proved that $L_1$ is compact operator in $H_1$. Weyl's theorem of compact perturbation (Glazman, 1965) implies

$$\sigma_c(L) = \sigma_c(L_0) = [-2, 2].$$

**Remark 1.** Since the operator $L$ is self-adjoint, the eigenvalues of $L$ are real. The point spectrum of $L$ satisfies

$$\sigma_d(L) \subset (-\infty, -2] \cup [2, \infty). \qquad (18)$$

Moreover, it easily follows

$$\sigma_d(L) = \{\lambda: \lambda = z + z^{-1},\ z\epsilon(-1,0) \cup (0,1),\ F_0(z) \text{ is not invertible}\}.$$

$F_0(z)$ is called the Jost function of $L$. Note that, this function is an infinite dimensional operator-valued function on the contrary to finite dimensional matrix coefficients case. Hence we need a different approach in order to locate the singular points of this operator-valued function. Luckily, the theory of infinite dimensional operators in Hilbert spaces provides the necessary tools. We summarize below our main tool which is due to (Keldysh, 1971) for the sake of completeness.

**Theorem 5.** (Keldysh, 1971) Let $A(z)$ be an operator-valued function defined on a region $D$ such that $A(z)$ is an operator in a Hilbert space $H$ for each $z$. Suppose $A(z)$ is analytic in $D$ and $A(z)$ is a compact operator for each $z \in D$. Then the following statements hold:

i. The operator $I - A(z)$ has an inverse except for a countable set of points in $D$.
ii. If the set of points where $I - A(z)$ doesn't have an inverse is infinite, then the only limit point of this set lies on the boundary of $D$.
iii. If $(I - A(z))^{-1}$ exists for $z = z_0 \in D$ then it exists over the whole of $D$, except possibly for a set of isolated points, and is a meromorphic function of $z$. Moreover $(I - A(z_0))^{-1} = I + B(z_0)$ where $B(z_0)$ is a compact operator.

**Theorem 6.** $\sigma_d(L)$ is bounded and countable. Furthermore, its only limit point (if exists) can be $\lambda = -2$ or $\lambda = 2$.

**Proof.** We try to adapt Theorem 5 into our case. Let us define

$$M = \{z: z\epsilon(-1,0) \cup (0,1), F_0(z) \text{ is not invertible}\}.$$

Then

$$\sigma_d(L) = \{\lambda: \lambda = z + z^{-1}, z \in M\}.$$

The Jost function can be represented (see equation (11))

$$F_0(z) = T_0\left[I + \sum_{m=1}^{\infty} K_{0,m} z^m\right], \qquad (19)$$

where

$$T_0 = \prod_{p=0}^{\infty} A_p^{-1}$$

is invertible (see equation (13)). The last two equations imply that $F_0(z)$ is invertible iff

$$G(z) := I + \sum_{m=1}^{\infty} K_{0,m} z^m$$

is invertible. From Theorem 2 we have

$$K_{0,1} = -\sum_{p=1}^{\infty} T_p^{-1} B_p T_p, \tag{20}$$

$$K_{0,2} = -\sum_{p=1}^{\infty} T_p^{-1} B_p T_p K_{p,1} + \sum_{p=1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p, \tag{21}$$

$$K_{0,m+2} = \sum_{p=1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p K_{p+1,m} - \sum_{p=1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{1,m}, \tag{22}$$

where $m \in \mathbb{N}$. It follows $K_{0,m}$ is compact for each $m \in \mathbb{N}$ from (20)-(22) and the conditions that $A_n - I$ $(n \in \mathbb{N} \cup \{0\})$ and $B_n$ $(n \in \mathbb{N})$ are compact operators. As a result,

$$A(z) := \sum_{m=1}^{\infty} K_{0,m} z^m \tag{23}$$

is compact for every $z \epsilon (-1,0) \cup (0,1)$. Furthermore, $A(z)$ is analytic in $(-1,0) \cup (0,1)$ since $F_0(z)$ is analytic in $D_1$. Now $-A(z)$ where $A(z)$ is defined by (23) satisfies the conditions of Theorem 5 where $D = (-1,0) \cup (0,1)$. From Theorem 5 (i) it follows $I + A(z) = G(z)$ and equivalently $F_0(z)$ has an inverse except for a countable set of points in $D$. Hence $\sigma_d(L)$ is countable.

From (19) we have

$$F_0(z) = T_0 + \circ(1), |z| \to 0$$

which implies $F_0(z)$ is invertible for sufficiently small $z$ and hence $\lambda = z + z^{-1}$ can not be an eigenvalue when $|z| \to 0$. Therefore $\sigma_d(L)$ is bounded.

Finally, if $\sigma_d(L)$ is finite then it doesn't have any limit point. Suppose $\sigma_d(L)$ has infinitely many elements. Then $M$ also has infinitely many elements. Theorem 5 (ii) implies that the only limit point of $M$ can lie on the boundary of $D$. Since $\sigma_d(L)$ is bounded this limit point of $M$ can not be $z = 0$. Hence the limit point of $M$ can be $z = \pm 1$ meaning that the only limit point of $\sigma_d(L)$ can be $\lambda = -2$ or $\lambda = 2$.

**Theorem 7.** $L$ has a finite number of eigenvalues with finite multiplicities.

**Proof.** From Theorem 6 the only limit point (if exists) of $\sigma_d(L)$ can be $\lambda = -2$ or $\lambda = 2$. If $\lambda = 2$ is a limit point of $\sigma_d(L)$ then there exist an eigenvalue in the neighborhood $[2 - \varepsilon, 2)$ for sufficiently small $\varepsilon > 0$. This contradicts with (18). Therefore $\lambda = 2$ can not be a limit point of $\sigma_d(L)$. Similarly we can show that $\lambda = -2$ can not be a limit point of $\sigma_d(L)$. Since $\sigma_d(L)$ is bounded (see Theorem 6) and has no limit point, it is finite by well-known Bolzano-Weierstrass Theorem. Therefore $L$ has a finite number of eigenvalues.

We have at least one point $z = z_0$ in $D$ such that $(I + A(z))^{-1}$ exists. Theorem 5 (iii) implies that $(I + A(z))^{-1} = (G(z))^{-1}$ exists over the whole of $D$, except for a set of isolated points. Obviously these isolated points are eigenvalues of $L$. Further, $(G(z))^{-1}$ is meromorphic in $D$. Therefore $(G(z))^{-1}$ can be written

$$(G(z))^{-1} = \frac{U(z)}{v(z)},$$

where $U(z)$ is an operator-valued function and $v(z)$ is a scalar function both analytic on $D$. It is clear from the last equation that $\lambda = z + z^{-1}$ is an eigenvalue of $L$ with multiplicity $m_z$ if and only if $z$ is a zero of the function $v(z)$ with multiplicity $m_z$. Since $(G(z))^{-1}$ is meromorphic on $D$ all zeros of $v(z)$ have finite multiplicities. Therefore $L$ has a finite number of eigenvalues with finite multiplicities.

## CONCLUSION

In this paper, spectrum of discrete analogue of Sturm-Liouville equation with operator coefficient on the half-line is investigated. Here, operator coefficients are defined in an infinite dimensional Hilbert space unlike the matrix coefficient case which has been studied in (Aygar and Bairamov, 2012). This makes the problem under investigation different since in infinite dimension one needs to deal with operator-valued Jost function. In order to overcome this difficulty, we use (Keldysh, 1971) in which the singular points of an analytic, compact operator-valued function are characterized. It is shown that the eigenvalues of this boundary value problem corresponds to singular points of operator-valued Jost function. Further, it is proven that the spectrum is discrete under Assumption 2.

## Conflict of Interest

The author declares that there is no conflict of interest.

## Author's Contributions

I hereby declare that the planning, execution and writing of the article was done by me as the sole author of the article.

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