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Inverse Scattering Problem for Sturm-Liouville Operator with Discontinuity Conditions on the Positive Half Line

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Abstract

In this paper, we consider the inverse scattering problem for Sturm-Liouville operator with discontinuity conditions at some point on the positive half line. The scattering data of this boundary value problem is examined. The resolvent operator is constructed and the expansion formula with respect to the eigenfunctions of this boundary value problem is obtained. The main equation or modified Marchenko equation of the inverse scattering problem is derived and an algorithm of the construction of the potential function according to scattering data of this boundary value problem is given.

Keywords: Sturm-Liouville equation, discontinuity conditions, inverse scattering problem, main equation

Pozitif Yarı Eksende Süreksizlik Koşuluna Sahip Sturm-Liouville Operatörünün Ters Saçılma Problemi

Öz

Bu çalışmada, pozitif yarı eksen üzerindeki bir noktada süreksizlik koşuluna sahip Sturm-Liouville operatörünün ters saçılma problemi ele alınmıştır. Ele alınan sınır değer probleminin saçılma verileri incelenmiştir. Rezolvent operatörü inşa edilmiş ve sınır değer probleminin özfonksiyonlarına göre ayrışım formülü elde edilmiştir. Ters saçılma probleminin temel denklemi veya modifiye edilmiş Marchenko denklemi elde edilmiş ve sınır değer probleminin saçılma verilerine göre potansiyel fonksiyonun inşa edilme algoritması verilmiştir.

Anahtar kelimeler: Sturm-Liouville denklemi, süreksizlik koşulları, ters saçılma problemi, temel denklem

INTRODUCTION

In physical and mathematical literature, there are numerous studies based on scattering theory because of its applications in the quantum mechanics (see Chadan and Sabatier, 1977; Faddeev and Takhtajan, 2007; Jaluent and Jean, 1976 and the references therein) and the investigations on this subject have been continued in detail. It is well known in quantum mechanics that the scattering of particles by a potential field is completely determined by the asymptotic form of the wave functions at infinity. Therefore, the following question arises: is it possible to reconstruct the potential from a knowledge of the asymptotic form of the wave functions at infinity? and then, if it is possible, to indicate a method for constructing the potential. This is known as the inverse problem of scattering theory. The mathematical side of this question is comprehensively studied and formalized in (Agranovich and Marchenko, 1963) and (Marchenko, 2011).

In this paper, we will solve the inverse scattering problem of Sturm-Liouville operator with discontinuity conditions at some point on positive half line by using the method of Marchenko. Then, consider the Sturm-Liouville equation (or equivalently the time-independent one-dimensional Schrödinger equation)

$$-y'' + q(x)y = \lambda^2, \quad 0 < x < \infty \tag{1}$$

with boundary conditions

$$y'(0) - hy(0) = 0 \tag{2}$$

and discontinuity conditions at the point $a \in (0, \infty)$

$$y(a-0) = \alpha y(a+0),$$
$$y'(a-0) = \alpha^{-1} y'(a+0) \tag{3}$$

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where $1 \neq \alpha > 0$, h is an arbitrary real number, $q(x)$ is a real function satisfying the condition

$$\int_0^\infty x|q(x)|dx < \infty. \tag{4}$$

In case of $\alpha = 1$ i.e. in classical case, the inverse scattering problem of the boundary value problem (1)-(3) was completely solved by Marchenko (1955; 2011) and Levitan (1975; 1987). The inverse scattering problem for the discontinuous case on the positive half line was firstly studied by Gasymov (1977) and Darwish (1994) and also Guseinov and Pashaev (2002) solved the inverse discontinuous scattering problem by using the new integral representation (non-triangular) which was obtained for the Jost solution of the Sturm-Liouville equation with discontinuous coefficient. Then, the direct and inverse scattering problems for discontinuous Sturm-Liouville equation under different boundary conditions were examined with the help of this integral representation in (Çöl, 2015; El-Raheem and Salama, 2015; Mamedov, 2010; Mamedov and Cetinkaya, 2015; Mızrak, Mamedov and Akhtyamov, 2017). Inverse problem for a wave equation with piecewise constant coefficient was worked in (Lavrent'ev Jr, 1992). In case of the Sturm-Liouville equation with discontinuity conditions (or transmission conditions) at a point on the positive half line, the direct and inverse scattering problem with various boundary conditions and discontinuity conditions were investigated in (Huseynov and Osmanova, 2007; Huseynov and Osmanli, 2009; Huseynov and Mammadova, 2013; Manafov and Kablan, 2013). Moreover, the direct and inverse scattering problem for Sturm-Liouville operator with nonlinear spectral parameter in the boundary conditions were studied in (Goktas and Mamedov, 2020; Mamedov, 2009; Mamedov and Kosar, 2010; Mamedov and Kosar, 2011).

In summary, the results obtained in this paper can be given as follows: firstly, the scattering data of the boundary value problem (1)-(3) are examined. Secondly, the resolvent operator is constructed and the eigenfunction expansion formula is obtained. Finally, we examine the inverse scattering problem that can be stated in the following way: determine a method of constructing the potential $q(x)$ from the scattering data of the boundary value problem (1)-(3). Therefore, the main equation of the inverse problem is derived and an algorithm for the construction of the potential function $q(x)$ according to scattering data is given.

MATERIAL AND METHODS

To solve this inverse scattering problem, we use the method of Marchenko, in this method, the transformation operator is used and the central role is played by the main equation with respect to the kernel of the transformation operator. However, due to the discontinuity conditions (3), the integral representation (not transformation operator) obtained in (Huseynov and Osmanova, 2007) is used and so, the main equation of the problem (1)-(3) is different from the classical main equation or Marchenko equation. Hereby, we must specify that the existence of discontinuity conditions (3) in the boundary value problem (1)-(3) strongly influences the structure of the representation of the Jost solution and the main equation of the inverse scattering problem.

Now, we give the integral representation of the Jost solution of the equation (1) with discontinuity conditions (3) obtained in (Huseynov and Osmanova, 2007):

For all λ from the upper half-plane, the equation (1) satisfying the conditions (3) and (4) has the Jost solution $e(x,\lambda)$ that can be represented in the form

$$e(x,\lambda) = e_0(x,\lambda) + \int_x^\infty K(x,t)e^{i\lambda t}dt \tag{5}$$

where

$$e_0(x,\lambda) = \begin{cases} e^{i\lambda x}, & x > a, \\ \alpha^+ e^{i\lambda x} + \alpha^- e^{i\lambda(2a-x)}, & 0 < x < a \end{cases}$$

$\alpha^\pm = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$, for each fixed $x \in (0,a) \cup (a,\infty)$ the kernel $K(x,.)$ belongs to the space $L_1(x,\infty)$ and satisfies the following properties:

$$K(x,x) = \frac{\alpha^+}{2}\int_x^\infty q(t)dt, \quad x \in (0,a) \tag{6}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)dt, \quad x \in (a,\infty) \tag{7}$$

$$K(x,2a-x+0) - K(x,2a-x-0) = \\ = \frac{\alpha^-}{2}\left(\int_a^\infty q(t)dt - \int_x^a q(t)dt\right), \ x \in (0,a). \tag{8}$$

It is seen from this representation that the triangular property of Jost solution representation is lost and the kernel function has a discontinuity along the line $t = 2a - x$ for $x \in (0,a)$.

The solution $e(x,\lambda)$ is regular with respect to λ in the upper half plane $Im\lambda > 0$ and continuous for $Im\lambda \geq 0$. For real $\lambda \neq 0$, the function $e(x,\lambda)$ and $\overline{e(x,\lambda)}$ form a fundamental system of solutions of equation (1) with discontinuity conditions (3) and their Wronskian is as follows:

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$$W\left\{e(x,\lambda),\overline{e(x,\lambda)}\right\} =$$
$$= e'(x,\lambda)\overline{e(x,\lambda)} - e(x,\lambda)\overline{e'(x,\lambda)} = 2i\lambda. \quad (9)$$

Let $\phi(x,\lambda)$ be the solution of the equation (1) with discontinuity conditions (3) under the initial conditions

$$\phi(0,\lambda) = 1, \;\; \phi'(0,\lambda) = h.$$

RESULTS AND DISCUSSION

Scattering Data

Lemma 1. The following identity holds for all real $\lambda \neq 0$:

$$\frac{2i\lambda\phi(x,\lambda)}{e'(0,\lambda)-he(0,\lambda)} = \overline{e(x,\lambda)} - S_h(\lambda)e(x,\lambda) \qquad (10)$$

where

$$S_h(\lambda) = \frac{\overline{e'(0,\lambda)}-h\overline{e(0,\lambda)}}{e'(0,\lambda)-he(0,\lambda)} \qquad (11)$$

and $S_h(\lambda) = \overline{S_h(-\lambda)} = [S_h(-\lambda)]^{-1}$.

Proof. Since the functions $e(x,\lambda)$ and $\overline{e(x,\lambda)}$ form a fundamental system of solutions of the equation (1) with the condition (3) for all real $\lambda \neq 0$, we obtain

$$\phi(x,\lambda) = \frac{1}{2i\lambda}\left\{[e'(0,\lambda) - he(0,\lambda)]\overline{e(x,\lambda)} - \right.$$
$$\left. - \left[\overline{e'(0,\lambda)} - h\overline{e(0,\lambda)}\right] e(x,\lambda)\right\}. \quad (12)$$

Now, let us show that $\omega(\lambda) := e'(0,\lambda) - he(0,\lambda) \neq 0$ for all real $\lambda \neq 0$. Assume that

$$\omega(\tilde{\lambda}) := e'(0,\tilde{\lambda}) - he(0,\tilde{\lambda}) = 0.$$

According to (9), we get

$$e'(0,\tilde{\lambda})\overline{e(0,\tilde{\lambda})} - e(0,\tilde{\lambda})\overline{e'(0,\tilde{\lambda})} = 2i\tilde{\lambda}.$$

Then, it follows from the last two equality that $\tilde{\lambda} = 0$, but this contradicts $\tilde{\lambda} \neq 0$. Thus, we have $\omega(\lambda) = e'(0,\lambda) - he(0,\lambda) \neq 0$ for real $\lambda \neq 0$. Taking into account this in the equality (12), we find (10) and (11) as claimed. The lemma is proved.

Definition 2. The function $S_h(\lambda)$ expressed by the formula (11) is called the ***scattering function*** of the problem (1)-(3).

Now, we will examine the zeros of the function $\omega(\lambda)$.

Lemma 3. The function $\omega(\lambda)$ may have only a finite number of zeros in the half plane $Im\lambda > 0$ and these zeros lie on the imaginary axis.

Proof. Since $\omega(\lambda) \neq 0$ for all real $\lambda \neq 0$, the point $\lambda = 0$ can be the only possible real zero of the function $\omega(\lambda)$. The function $\omega(\lambda)$ is analytic in the upper half plane. Therefore, taking into account this fact and the representation of the solution (5), it is obtained that the zeros of the function $\omega(\lambda)$ form bounded and at most countable set whose unique limit point may be only a zero.

Now, let us prove that all zeros of the function $\omega(\lambda)$ lie on the imaginary axis. Assume that τ_1 and τ_2 are two zeros of the function $\omega(\lambda)$. Then,

$$\omega(\tau_i) = e'(0,\tau_i) - he(0,\tau_i) = 0, \;\; i = 1,2. \quad (13)$$

Since the functions $e(x,\tau_1)$ and $\overline{e(x,\tau_2)}$ satisfy the equation (1), we can write

$$-e''(x,\tau_1) + q(x)e(x,\tau_1) = \tau_1^2 e(x,\tau_1),$$

$$-\overline{e''(x,\tau_2)} + q(x)\overline{e(x,\tau_2)} = \overline{\tau_2}^2\overline{e(x,\tau_2)},$$

and it follows from these equalities that

$$\frac{d}{dx}W\left\{\overline{e(x,\tau_2)}, e(x,\tau_1)\right\}$$
$$= \left(\tau_1^2 - \overline{\tau_2}^2\right)e(x,\tau_1)\overline{e(x,\tau_2)}.$$

Integrating this equality over the interval $(0,\infty)$ and then using the discontinuous conditions (3), we find

$$\left(\tau_1^2 - \overline{\tau_2}^2\right)\int_0^\infty e(x,\tau_1)\overline{e(x,\tau_2)}\,dx -$$
$$-W\left\{\overline{e(x,\tau_2)}, e(x,\tau_1)\right\}_{x=0} = 0. \quad (14)$$

In the second expression of the left hand side of (14), using the relation (13), we have

$$W\left\{e(x,\tau_1),\overline{e(x,\tau_2)}\right\}_{x=0} =$$
$$= e'(0,\tau_1)\overline{e(0,\tau_2)} - e(0,\tau_1)\overline{e'(0,\tau_2)}=0.$$

Thus, we obtain

$$\left(\tau_1^2 - \overline{\tau_2}^2\right)\int_0^\infty e(x,\tau_1)\overline{e(x,\tau_2)}\,dx = 0 \quad (15)$$

In particular, when $\tau_2 = \tau_1$ is chosen, $\tau_1^2 - \overline{\tau_1}^2 = 0$ or $\tau_1 = i\lambda_1$, here $\lambda_1 \geq 0$. Consequently, the zeros of the function $\omega(\lambda)$ can lie on the imaginary axis.

Moreover, the number of the zeros of the functions $\omega(\lambda)$ is finite and this fact is similarly proved by using the method in (Marchenko, 2011 see Lemma 3.1.6., pp. 186). The lemma is proved.

Lemma 4. The zeros of the function $\omega(\lambda)$ are simple.

Proof. Denote

$$\dot{e}(x,\lambda) = \frac{d}{d\lambda}e(x,\lambda), \;\; e'(x,\lambda) = \frac{d}{dx}e(x,\lambda).$$

Consider the following equation

$$-e''(x,\lambda) + q(x)e(x,\lambda) = \lambda^2 e(x,\lambda).$$

Differentiating this equation with respect to λ, we get

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$$-\dot{e}''(x,\lambda)+q(x)\dot{e}(x,\lambda)=\lambda^2\dot{e}(x,\lambda)+2\lambda e(x,\lambda).$$

It follows from these two equalities that

$$\frac{d}{dx}W\{e(x,\lambda),\dot{e}(x,\lambda)\}=2\lambda[e(x,\lambda)]^2.$$

Integrating this over the interval $(0,\infty)$ and then using the discontinuity conditions (3) and the function $\omega(\lambda)=e'(0,\lambda)-he(0,\lambda)$, we calculate

$$\omega(\lambda)\dot{e}(0,\lambda)-\dot{\omega}(\lambda)e(0,\lambda)+2\lambda\int_0^\infty[e(x,\lambda)]^2dx=0.$$

Let $\lambda=i\tau$ $(\tau>0)$ be a zero of the function $\omega(\lambda)$. Then, we have

$$2i\tau\int_0^\infty|e(x,i\tau)|^2dx=\dot{\omega}(i\tau)e(0,i\tau).$$

Since $\int_0^\infty|e(x,i\tau)|^2dx>0$, it is obtained that $\dot{\omega}(i\tau)\neq 0$ i.e., the zeros of the function $\omega(\lambda)$ are all simple. The lemma is proved.

Now, let $i\lambda_k$, $(\lambda_k>0,\ k=1,2,\dots,n)$ be the zeros of the function $\omega(\lambda)$ and denote

$$m_k^{-2}:=\int_0^\infty|e(x,i\lambda_k)|^2\,dx=\frac{\dot{\omega}(i\lambda_k)e(0,i\lambda_k)}{2i\lambda_k}. \tag{16}$$

The numbers m_k are called the ***normalized numbers*** of the problem (1)-(3).

Definition 5. A collection

$$\{S_h(\lambda),(-\infty<\lambda<\infty);\lambda_k;m_k,(k=1,2,\dots,n)\}$$

is called the ***scattering data*** of the boundary value problem (1)-(3).

Eigenfunction Expansion

The functions

$$u(x,\lambda)=\overline{e(x,\lambda)}-S_h(\lambda)e(x,\lambda),\ \ (-\infty<\lambda<\infty), \tag{17}$$

$$u(x,i\lambda_k)=m_ke(x,i\lambda_k),\ \ (k=1,2,\dots,n) \tag{18}$$

are bounded solutions of the boundary value problem (1)-(3). They form a complete set of normalized eigenfunctions of this problem.

Consider the operator L with the domain

$D(L)=\{f(x)\in L_2(0,\infty)$:
$f(x),f'(x)\in AC[0,a]\cap AC[a,\infty),l(f)\in L_2(0,\infty)$
$f'(0)-hf(0)=0,f(a-0)=\alpha f(a+0)$
$f'(a-0)=\alpha^{-1}f(a+0)\}$

where

$$l(f)=-f''(x)+q(x)f(x).$$

Assume that λ^2 is not a spectrum point of the operator L. Then, the resolvent operator $R_{\lambda^2}(L)=(L-\lambda^2I)^{-1}$ exists.

Lemma 6. The resolvent operator is an integral operator formed by

$$y(x,\lambda):=R_{\lambda^2}(L)f=\int_0^\infty g(x,t,\lambda)f(t)dt \tag{19}$$

with the kernel

$$g(x,t,\lambda)=-\frac{1}{\omega(\lambda)}\begin{cases}e(x,\lambda)\phi(t,\lambda), & t\le x,\\ e(t,\lambda)\phi(x,\lambda), & x\le t.\end{cases} \tag{20}$$

Proof. Let $f(x)\in D(L)$ be a finite function at infinity. To obtain the resolvent operator of L, consider the following boundary value problem

$$-y''+q(x)y=\lambda^2y+f(x),$$
$$y'(0)-hy(0)=0,$$
$$y(a-0)=\alpha y(a+0),y'(a-0)=\alpha^{-1}y'(a+0).$$

Seek the solution of this problem as in the form:

$$y(x,\lambda)=c_1(x,\lambda)\phi(x,\lambda)+c_2(x,\lambda)e(x,\lambda),$$

where the functions $\phi(x,\lambda)$ and $e(x,\lambda)$ are the solutions of homogeneous problem for $Im\lambda>0$. Consequently, applying the method of variation of parameters, we find (19) and (20).

Theorem 7. The eigenfunctions expansion formula of the boundary value problem (1)-(3) is as follows:

$$\delta(t-x)=\sum_{k=1}^n u(x,i\lambda_k)u(t,i\lambda_k)+\frac{1}{2\pi}\int_0^\infty u(x,\lambda)\overline{u(t,\lambda)}\,d\lambda \tag{21}$$

where $\delta(x)$ is a Dirac delta function.

Proof. Let $f(x)\in D(L)$ be a twice continuously differential function and be finite at infinity. Then, for $Im\lambda>0$ we can write from (19) and (20) that

$$y(x,\lambda)=\int_0^\infty g(x,t,\lambda)f(t)dt=$$
$$=-\frac{1}{\omega(\lambda)}\int_0^x e(x,\lambda)\,\phi(t,\lambda)f(t)dt$$
$$-\frac{1}{\omega(\lambda)}\int_x^\infty e(t,\lambda)\phi(x,\lambda)f(t)dt$$
$$=-\frac{e(x,\lambda)}{\lambda^2\omega(\lambda)}\int_0^x[-\phi''(t,\lambda)+q(t)\phi(t,\lambda)]f(t)dt$$
$$-\frac{\phi(x,\lambda)}{\lambda^2\omega(\lambda)}\int_x^\infty[-e''(t,\lambda)+q(t)e(t,\lambda)]f(t)dt.$$

Integrating by parts for both cases $x<a$ and $x>a$, we calculate

$$y(x,\lambda)=-\frac{f(x)}{\lambda^2}+\frac{z(x,\lambda)}{\lambda^2} \tag{22}$$

where

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$$z(x,\lambda) := \int_0^\infty g(x,t,\lambda)[-f''(t)+q(t)f(t)]dt.$$

Assume that Γ_R denote the positively oriented contour formed by the circle of radius R and center at zero. Consider $D_1 = \{z: |z| \le R, |Imz| \ge \epsilon\}$ and $D_2 = \{z: |z| \le R, |Imz| \le \epsilon\}$. Denote by $\Gamma'_{R,\epsilon}$ the positive oriented boundary contour of D_1 and $\Gamma''_{R,\epsilon}$ the negative oriented boundary contour of D_2. Then, we can use the properties of the integration as follows:

$$\int_{\Gamma'_{R,\epsilon}} = \int_{\Gamma_R} + \int_{\Gamma''_{R,\epsilon}} \tag{23}$$

Now, multiplying both sides of the expression (22) by $\frac{\lambda}{2\pi i}$ and then integrating along the contour Γ_R with respect to λ, we get

$$\frac{1}{2\pi i}\int_{\Gamma_R} \lambda y(x,\lambda)d\lambda = -\frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(x)}{\lambda}d\lambda + Z_R(x), \tag{24}$$

where

$$Z_R(x) := \frac{1}{2\pi i}\int_{\Gamma_R} \frac{z(x,\lambda)}{\lambda}d\lambda,$$

and since $\lim_{|\lambda|\to\infty} sup_{x\ge 0}|z(x,\lambda)| = 0$ which is obtained from the expressions of the functions $e(x,\lambda)$, $\phi(x,\lambda)$ and $\omega(\lambda)$, we have $Z_R(x) \to 0$ uniformly to x as $R \to \infty$. According to (23), we can write

$$\frac{1}{2\pi i}\int_{\Gamma'_{R,\epsilon}} \lambda y(x,\lambda)d\lambda =$$

$$= \frac{1}{2\pi i}\int_{\Gamma_R} \lambda y(x,\lambda)d\lambda + \frac{1}{2\pi i}\int_{\Gamma''_{R,\epsilon}} \lambda y(x,\lambda)d\lambda. \tag{25}$$

It follows from (24) and (25) that as $R \to \infty$ and $\epsilon \to 0$

$$\frac{1}{2\pi i}\int_{\Gamma'_{R,\epsilon}} \lambda y(x,\lambda)d\lambda =$$

$$= -f(x) + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \lambda[y(x,\lambda+i0) - y(x,\lambda-i0)]d\lambda.$$

On the other hand, using the residue theorem, we have

$$\frac{1}{2\pi i}\int_{\Gamma'_{R,\epsilon}} \lambda y(x,\lambda)d\lambda =$$

$$= \sum_{k=1}^{n} Res_{\lambda=i\lambda_k}\lambda y(x,\lambda) + \sum_{k=1}^{n} Res_{\lambda=-i\lambda_k}\overline{\lambda y(x,\lambda)}.$$

It is found from the last two relations that

$$f(x) = -\sum_{k=1}^{n} Res_{\lambda=i\lambda_k}\lambda y(x,\lambda) - \sum_{k=1}^{n} Res_{\lambda=-i\lambda_k}\overline{\lambda y(x,\lambda)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \lambda[y(x,\lambda+i0) - y(x,\lambda-i0)]d\lambda. \tag{26}$$

Next, let us examine the right hand side of the equation (26). Denote by $\psi(x,\lambda)$ the solution of equation (1) with the discontinuity conditions (3) satisfying the initial conditions

$$\psi(0,\lambda) = 0, \quad \psi'(0,\lambda) = -1.$$

Moreover, $W\{\phi(x,\lambda),\psi(x,\lambda)\} = 1$. Therefore, we have

$$e(x,\lambda) = e(0,\lambda)\phi(x,\lambda) - \omega(\lambda)\psi(x,\lambda). \tag{27}$$

Using the expression (20) and (27), we can write for $t \le x$

$$g(x,t,\lambda) = -\frac{e(0,\lambda)}{\omega(\lambda)}\phi(x,\lambda)\phi(t,\lambda) + \psi(x,\lambda)\phi(t,\lambda),$$

for $x \le t$

$$g(x,t,\lambda) = -\frac{e(0,\lambda)}{\omega(\lambda)}\phi(x,\lambda)\phi(t,\lambda) + \phi(x,\lambda)\psi(t,\lambda)$$

and for $Im\lambda > 0$, we have

$$y(x,\lambda) = -\frac{e(0,\lambda)}{\omega(\lambda)}\int_0^\infty \phi(x,\lambda)\phi(t,\lambda)f(t)dt + \\ + \int_0^x \psi(x,\lambda)\phi(t,\lambda)f(t)dt + \\ + \int_x^\infty \phi(x,\lambda)\psi(t,\lambda)f(t)dt.$$

Then, it is obtained from this expression that

$$\sum_{k=1}^{n} Res_{\lambda=i\lambda_k}\lambda y(x,\lambda) + \sum_{k=1}^{n} Res_{\lambda=-i\lambda_k}\overline{\lambda y(x,\lambda)} = \\ = \sum_{k=1}^{n} \frac{-2i\lambda_k e(0,i\lambda_k)}{\dot{\omega}(i\lambda_k)}\int_0^\infty \phi(x,i\lambda_k)\phi(t,i\lambda_k)f(t)dt. \tag{28}$$

Now, taking into account the relation $y(x,\lambda-i0) = \overline{y(x,\lambda+i0)}$, we find

$$y(x,\lambda+i0) - \overline{y(x,\lambda+i0)} =$$

$$= \int_0^x \left[-\frac{e(x,\lambda)}{\omega(\lambda)} + \overline{\frac{e(x,\lambda)}{\omega(\lambda)}}\right]\phi(t,\lambda)f(t)dt +$$

$$+ \int_x^\infty \phi(x,\lambda)\left[-\frac{e(t,\lambda)}{\omega(\lambda)} + \overline{\frac{e(t,\lambda)}{\omega(\lambda)}}\right]f(t)dt$$

and using (27) in this expression, we get

$$y(x,\lambda+i0) - \overline{y(x,\lambda+i0)} =$$

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$$=\frac{2i\lambda}{|\omega(\lambda)|^2}\int_0^\infty \phi(x,\lambda)\phi(t,\lambda)f(t)dt.$$

Thus, it follows from the last equality that

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\lambda\,[y(x,\lambda+i0)-y(x,\lambda-i0)]d\lambda=$$
$$=\frac{2}{\pi}\int_0^\infty \frac{\lambda^2}{|\omega(\lambda)|^2}\phi(x,\lambda)\int_0^\infty \phi(t,\lambda)\,f(t)dtd\lambda. \tag{29}$$

Consequently, substituting (28) and (29) into (26), we calculate

$$f(x)=$$
$$=\sum_{k=1}^{n}\frac{2i\lambda_k e(0,i\lambda_k)}{\dot{\omega}(i\lambda_k)}\int_0^\infty \phi(x,i\lambda_k)\phi(t,i\lambda_k)f(t)dt$$
$$+\frac{2}{\pi}\int_0^\infty \frac{\lambda^2}{|\omega(\lambda)|^2}\phi(x,\lambda)\int_0^\infty \phi(t,\lambda)f(t)dtd\lambda.$$

Moreover, in the last expression considering $e(x,\lambda)=e(0,\lambda)\phi(x,\lambda)$, (16), (17) and (18), the eigenfunction expansion formula (21) is obtained. The theorem is proved.

Inverse Scattering Problem

In this section, we will reconstruct the potential $q(x)$ by scattering data of the boundary value problem (1)-(3). The problem (1)-(3) has bounded solutions (17) and (18) and as $x\to\infty$, the following asymptotic formulas are satisfied:

$u(x,\lambda)=e^{-i\lambda x}-S_h(\lambda)e^{i\lambda x}+o(1),(-\infty<\lambda<\infty)$
$u(x,i\lambda_k)=m_k e^{-\lambda_k x}(1+o(1)),\ \ (k=1,2,\dots,n).$

Then, it must be specified that the scattering data

$\{S_h(\lambda),(-\infty<\lambda<\infty);\lambda_k;m_k,(k=1,2,\dots,n)\}$

provides a complete description of the behavior at infinity of the normed eigenfunctions $u(x,\lambda)$ of the problem (1)-(3).

When $q(x)\equiv 0$ in the equation (1), the following relation which is similar to (10) is valid:

$$\frac{2i\lambda\tilde{\phi}(x,\lambda)}{e_0'(0,\lambda)}=\overline{e_0(x,\lambda)}-S_0(\lambda)e_0(x,\lambda), \tag{30}$$

where $\tilde{\phi}(x,\lambda)$ is a solution of the equation (1) with (3) under the initial conditions

$$\tilde{\phi}(0,\lambda)=1,\ \ \tilde{\phi}'(0,\lambda)=0,$$

moreover,

$$S_0(\lambda)=\frac{\overline{e_0'(0,\lambda)}}{e_0'(0,\lambda)}=\frac{-\alpha^+ +\alpha^- e^{-2i\lambda a}}{\alpha^+ -\alpha^- e^{2i\lambda a}} \tag{31}$$

and we have $S_0(\lambda)-S_h(\lambda)=O\left(\frac{1}{\lambda}\right)$ as $|\lambda|\to\infty$. Thus, the function $S_0(\lambda)-S_h(\lambda)\in L_2(-\infty,\infty)$ is the Fourier transform of the function

$$F_S(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda)-S_h(\lambda)]\,e^{i\lambda x}d\lambda \tag{32}$$

which belongs to the space $L_2(-\infty,\infty)$.

Theorem 8. For each $x\geq 0$, $x\neq a$ the kernel $K(x,y)$ of the integral representation (5) satisfies the following equation

$$K(x,y)-\frac{\alpha^-}{\alpha^+}K(x,2a-y)+$$
$$+\hat{F}(x,y)+\int_x^\infty K(x,t)F(y+t)\,dt=0,\ \ x<y, \tag{33}$$

where

$$F(x)=F_S(x)+\textstyle\sum_{k=1}^{n}m_k^2 e^{-\lambda_k x} \tag{34}$$

and

$$\hat{F}(x,y)=\begin{cases}\alpha^+F(x+y)+\\ +\alpha^-F(2a-x+y), & 0<x<a\\ F(x+y), & x>a.\end{cases} \tag{35}$$

Proof. It is obtained from (10) and (30) that

$$\frac{2i\lambda\phi(x,\lambda)}{e'(0,\lambda)-he(0,\lambda)}-\frac{2i\lambda\tilde{\phi}(x,\lambda)}{e_0'(0,\lambda)}=$$
$$=\int_x^\infty K(x,t)e^{-i\lambda t}dt+[S_0(\lambda)-S_h(\lambda)]e_0(x,\lambda)$$
$$+\int_x^\infty K(x,t)[S_0(\lambda)-S_h(\lambda)]e^{i\lambda t}dt-$$
$$-S_0(\lambda)\int_x^\infty K(x,t)e^{i\lambda t}dt$$

or equivalently,

$$2i\lambda\phi(x,\lambda)\left\{\frac{1}{e'(0,\lambda)-he(0,\lambda)}-\right.$$
$$\left.-\frac{1}{e_0'(0,\lambda)-he_0(0,\lambda)}\right\}+$$
$$+\frac{2i\lambda\left(\phi(x,\lambda)-\tilde{\phi}(x,\lambda)\right)}{e_0'(0,\lambda)-he_0(0,\lambda)}+$$
$$+2i\lambda\tilde{\phi}(x,\lambda)\left\{\frac{1}{e_0'(0,\lambda)-he_0(0,\lambda)}-\frac{1}{e_0'(0,\lambda)}\right\}$$
$$=\int_x^\infty K(x,t)e^{-i\lambda t}dt+[S_0(\lambda)-S_h(\lambda)]e_0(x,\lambda)$$
$$+\int_x^\infty K(x,t)[S_0(\lambda)-S_h(\lambda)]\,e^{i\lambda t}dt-$$
$$-S_0(\lambda)\int_x^\infty K(x,t)e^{i\lambda t}dt. \tag{36}$$

Now, let us multiply the both hand side of the equality (36) by $\frac{1}{2\pi}e^{i\lambda y}$ and integrate with respect to

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$\lambda \in (-\infty, \infty)$. Then, the right hand side of the new equality is as follows:

$$K(x,y) + \frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda) - S_h(\lambda)]e_0(0,\lambda)e^{i\lambda y}d\lambda + \\ + \int_x^{\infty}K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}[S_0(\lambda) - S_h(\lambda)]e^{i\lambda(y+t)}d\lambda\right\}dt \\ - \int_x^{\infty}K(x,t)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)e^{i\lambda(y+t)}d\lambda\right\}dt. \tag{37}$$

Let us calculate:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)e^{i\lambda(y+t)}d\lambda.$$

It follows from (31) that

$$S_0(\lambda) = \sum_{n=0}^{\infty}\left(\frac{\alpha^-}{\alpha^+}\right)^n\left\{\left(\frac{\alpha^-}{\alpha^+}\right)e^{2i\lambda a(n-1)} - e^{2i\lambda an}\right\}$$

Using this expression, we find

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}S_0(\lambda)e^{i\lambda(y+t)}d\lambda = \\ = \sum_{n=0}^{\infty}\left(\frac{\alpha^-}{\alpha^+}\right)^n\left\{\left(\frac{\alpha^-}{\alpha^+}\right)\delta(2a(n-1)+y+t) \\ -\delta(2an+y+t)\right\} \tag{38}$$

Thus, substituting (32) and (38) into (37) and taking into account the relation $K(x,y) = 0$ for $x > y$ (see Huseynov and Osmanova, 2007), it is obtained for $y > x$ that the equality (37) is as follows:

$$K(x,y) - \frac{\alpha^-}{\alpha^+}K(x,2a-y) + \hat{F}_S(x,y) + \\ + \int_x^{\infty}K(x,t)F_S(y+t)dt, \tag{39}$$

where

$$\hat{F}_S(x,y) = \begin{cases} \alpha^+F_S(x+y) + \\ +\alpha^-F_S(2a-x+y), & 0 < x < a \\ F_S(x+y), & x > a. \end{cases}$$

Now, it remains to examine the integral of the product of the left hand side of the equality (36) and $\frac{1}{2\pi}e^{i\lambda y}$, taken over the real line $-\infty < \lambda < \infty$. Then, applying Jordan's lemma, for $y > x$ we find

$$\sum_{k=1}^{n}\frac{2i\lambda_k\phi(x,i\lambda_k)e^{-\lambda_k y}}{\dot{\omega}(i\lambda_k)} = \sum_{k=1}^{n}m_k^2 e(x,i\lambda_k)\,e^{-\lambda_k y} \\ = \sum_{k=1}^{n}m_k^2\left\{e_0(x,i\lambda_k)e^{-\lambda_k y} + \int_x^{\infty}K(x,t)e^{-\lambda_k(t+y)}dt\right\} \tag{40}$$

Consequently, using (39) and (40), for $y > x$ we derive the equation (33). The theorem is proved.

Definition 9. The equation (33) is called the ***main equation*** of the inverse problem of the scattering theory for the problem (1)-(3).

Note that this equation is different from the classical Marchenko equation and we define the equation (33) as the ***modified Marchenko equation.***

Lemma 10. For each $x \geq 0$, $x \neq a$ the main equation (33) has a unique solution $K(x,.) \in L_2(x,\infty)$.

Proof. In case of $x > a$, the main equation (33) is in the form of classical Marchenko equation, so for $x > a$ the proof of this theorem is as in (Marchenko, 2011). Now, suppose that $x < a$. The equation (33) can be written as follows:

$$T_xK(x,.) + F_xK(x,.) = -\hat{F}(x,.),$$

where

$$(T_xf)(y) = \begin{cases} f(y), & x > a, \\ f(y) - \frac{\alpha^-}{\alpha^+}f(2a-y), & x < a, \end{cases}$$

$$(F_xf) = \int_x^{\infty}F(t+y)f(t)dt, \qquad y > x.$$

The operator T_x is invertible in the space $L_2(x,\infty)$ and the operator F_x is completely continuous in the space $L_2(x,\infty)$ (see Marchenko, 2011, Lemma 3.3.1). Then, the main equation (33) can be expressed as

$$K(x,.) + T_x^{-1}F_xK(x,.) = -T_x^{-1}\hat{F}(x,.),$$

where the operator $T_x^{-1}F_x$ is completely continuous operator in $L_2(x,\infty)$. Thus, to prove the theorem, it is sufficient to show that the homogeneous equation

$$f_x(y) - \frac{\alpha^-}{\alpha^+}f_x(2a-y) + \\ + \int_x^{\infty}F(t+y)f_x(t)dt = 0, \qquad y > x$$

has only trivial solution $f_x(y) = 0$ in $L_2(x,\infty)$ and this fact is similarly obtained as in (Huseynov and Osmanli, 2009).

Theorem 11. The potential function $q(x)$ is uniquely determined by scattering data $\{S_h(\lambda), (-\infty < \lambda < \infty);\ \lambda_k;\ m_k\ (k = 1,2,\dots,n)\}$.

Proof. Using the scattering data $\{S_h(\lambda), (-\infty < \lambda < \infty);\ \lambda_k;\ m_k\ (k = 1,2,\dots,n)\}$ we construct the functions $F(x)$ and $\hat{F}(x,y)$ via the formulas (34) and (35). Then, with the help of constructed functions, we can write the main equation (33). From Lemma 10, the main equation has a unique solution $K(x,y)$ which is the kernel of the integral

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representation (5) for every $x \geq 0$, $x \neq a$. Hence, the potential $q(x)$ is uniquely constructed according to the formulas (6), (7) and (8).

CONCLUSION

In this paper, we deal with Sturm-Liouville operator with discontinuity conditions at the point $x = a \in (0, +\infty)$. Firstly, we examine the scattering data of the boundary value problem (1)-(3). Then, the expansion formula with respect to the eigenfunctions of this problem is obtained. Finally, we solve the inverse scattering problem by using the method of Marchenko. In this method, the main equation (or Marchenko equation) with respect to the kernel of the transformation operator plays the central role. However, the existence of the discontinuity conditions (3) strongly influences the structure of the representation of the Jost solution, so Jost solution is not in the form of transformation operator, is in the form of integral representation. Therefore, we use this integral representation of the Jost solution when solving the inverse scattering problem. Consequently, we derive the main equation of the inverse scattering problem and we give an algorithm for the construction of the potential function $q(x)$ according to scattering data.

CONFLICT OF INTEREST

The Author report no conflict of interest relevant to this article

RESEARCH AND PUBLICATION ETHICS STATEMENT

The author declares that this study complies with research and publication ethics.

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