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RIEMANNIAN SUBMERSIONS WITH QUARTER-SYMMETRIC NON-METRIC CONNECTION

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Abstract

In this paper, we study Riemannian submersions from a Riemannian manifold endowed with a quarter-symmetric non-metric connection onto a Riemannian manifold. We investigate O'Neill's tensor fields for quarter-symmetric non-metric connection and derive the covariant derivative of O'Neill's tensor fields. We obtain derivatives of those tensor fields and compare curvatures of the total manifold, the base manifold, and the fibers by computing curvatures.

Keywords: Riemannian submersions, quarter symmetric connection, O'Neill's tensors

1. Introduction

Friedmann et al defined the concept of the semi-symmetric non-metric connection in a differential manifold [1]. Later, Hayden studied metric connection with torsion a Riemannian manifold [2]. Afterward, Yano investigated a Riemannian manifold with a new connection called a semi-symmetric metric connection [3]. Many authors have studied semi-symmetric connections [4–6].

Let M be differentiable manifold with linear connection ∇ . Therefore for all $X, Y \in \Gamma(TM)$

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

where T is torsion tensor of ∇ . If $T = 0$, therefore the connection ∇ is said to be symmetric, otherwise it is called non-symmetric.

In 1975, Golab introduced quarter symmetric connection in a diferentiabile manifold [7]. A linear connection ∇ is said to be quarter symmetric connection if its torsion tensor T is given by

$$T(X, Y) = \eta(Y)\varphi(X) - \eta(X)\varphi(Y)$$

where φ is a tensor field of type (1,1) and η is a 1-form. Moreover, the quarter symmetric connection reduces to a semi-symmetric connection. Because quarter symmetric connection generalizes the semi symmetric connection. Many authors have studied the quarter symmetric connection [8-11].

One way to compare two manifolds is to define smooth maps from one manifold to another. One of these maps is submersion, the rank of the map is equal to the dimension of the target manifold. Riemannian submersion between Riemannian submanifolds was first introduced by O' Neill and Gray [12, 13].

The idea of semi-symmetric non-metric connections in a Riemannain submersion was introduced by Akyol and Beyendi [14]. They investigated O'Neill's tensor fields, obtain derivatives of those tensor fields and compare curvatures of the total manifold, the base manifold and the fibres by computing curvatures.

In this paper, we investigate Riemannian submersion with quarter symmetric non-metric connection. We investigate basic properties of Riemannian submersion with quarter symmetric non-metric connection.

2. Preliminaries

Let $\pi: (M^m, g_M) \rightarrow (B^b, g_B)$ be a submersion between two Riemannian manifolds. Then π said to be Riemannian submersion if

- i) π has maximal rank
- ii) The diferantial π_* preserves the lengths of horizontal vector.

On the other hand, $\pi^{-1}(k)$ is an $(m - b)$ dimensional submanifold of M , For each $k \in M$ The submanifolds $\pi^{-1}(k)$ are called fibers. Moreover, vector fields tangent to fibers are called vertical and vector fields orthogonal to fibers are horizontal. A vector field X on M is called basic if X is horizontal and $\pi_*X_p = X'_{\pi_*(q)}$ for all $q \in M$. We determine that V and H define projections $ker\pi_*$ and $(ker\pi_*)^\perp$, respectively.

On the other hand, a Riemannian submersion $\pi: M \rightarrow B$ determines tensor fields T and A on M such that,

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{1}$$

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{2}$$

for any $E, F \in \Gamma(TM)$. By virtue of (1) and (2), we have

$$\begin{aligned} \nabla_V W &= T_V W + \widehat{\nabla}_V W, \\ \nabla_V X &= T_V X + h(\nabla_V X), \end{aligned}$$

$$\begin{aligned}\nabla_X V &= v(\nabla_X V) + A_X V, \\ \nabla_X V &= A_X Y + h(\nabla_X Y)\end{aligned}$$

for any $X, Y \in \Gamma(\ker\pi_*)^\perp$ and $V, W \in \Gamma(\ker\pi_*)$. Moreover, if X is basic therefore

$$h(\nabla_V X) = A_X V.$$

$A_X Y = \frac{1}{2}v[X, Y]$ reflecting the complete integrability of the horizontal distribution H . It is known that A is alternating on the horizontal distribution, such that $A_X Y = -A_Y X$ for $X, Y \in \Gamma(\ker\pi_*)^\perp$ and T is symmetric on the vertical distribution.

Now, we define a linear connection on a Riemannian manifold M by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\varphi Y \quad (3)$$

where $X, Y \in \Gamma(TM)$, ∇ is Levi-Civita connection on M , η is a 1-form associated with the vector field P on M by $\eta(Y) = g_M(Y, P)$ and φ is (1,1)- type tensor field.

On the other hand, the torsion tensor \tilde{T} with respect to the connection $\tilde{\nabla}$ is

$$\tilde{T}_X Y = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y]$$

for all $X, Y \in \Gamma(TM)$.

Then, using (3) and ∇ is Levi-Civita connection, we have

$$\tilde{T}_X Y = \eta(Y)\varphi X - \eta(X)\varphi Y. \quad (4)$$

Moreover, for Riemannian metric g , from (3) we get,

$$(\tilde{\nabla}_X g)(Y, Z) = \eta(Y)g(\varphi X, Z) + \eta(Z)g(Y, \varphi X) \quad (5)$$

for all $X, Y \in \Gamma(TM)$. A linear connection $\tilde{\nabla}$ defined by (3) satisfies (4) and (5) and hence we call $\tilde{\nabla}$ is quarter-symmetric non-metric connection.

On the other hand, let $\pi: M \rightarrow B$ be a Riemannian submersion between two Riemannian manifolds and $\tilde{\nabla}$ is a quarter symmetric non-metric connection. Therefore, tensors \tilde{T} and \tilde{A} with type (1,2) on M with respect to $\tilde{\nabla}$ is given by

$$\tilde{T}_E F = h\tilde{\nabla}_{vE} vF + v\tilde{\nabla}_{vE} hF \quad (6)$$

and

$$\tilde{A}_E F = v\tilde{\nabla}_{hE} hF + h\tilde{\nabla}_{hE} vF. \quad (7)$$

3. Riemannian submersions with a quarter-symmetric non-metric connection

We note that, for brevity we use a abbreviation ” π is a Riemannian submersion with QSNMC” for $\pi: M \rightarrow B$ is a Riemannian submersion between two Riemannian manifolds with quarter-symmetric non-metric connection $\tilde{\nabla}$ on M .

Theorem 3.1. Let π is a Riemannian submersion with QSNMC. Then the tensor T and A with type (1.2) on M with respect to $\tilde{\nabla}$ is given by

$$\tilde{T}_E F = T_E F + \eta(vF)h\varphi(vE) + \eta(hF)v\varphi(vE) \quad (8)$$

and

$$\tilde{A}_E F = A_E F + \eta(hF)v\varphi(hE) + \eta(vF)h\varphi(hE) \quad (9)$$

for all $E, F \in \Gamma(TM)$.

Proof. For all $E, F \in \Gamma(TM)$ using (6) and (3) we get,

$$\tilde{T}_E F = h\nabla_{vE} vF + \eta(vF)h\varphi(vE) + v\nabla_{vE} hF + h(hF)v\varphi(vE).$$

Then we using (1), which gives (8).

On the other hand for all $E, F \in \Gamma(TM)$ using (2) and (3)

$$\tilde{A}_E F = v\nabla_{hE} hF + (hF)v\varphi(hE) + h\nabla_{hE} vF + h(vF)h\varphi(hE).$$

After some calculate we have proof the theorem.

Theorem 3.2 Let π is a Riemannian submersion with QSNMC. Then we have

$$g(\tilde{T}_E F, G) = g(\tilde{T}_E G, F) + 2g(T_E F, G) + \eta(vF)g(h\varphi(vE), G) + \eta(hF)g(v\varphi(vE), G) - \eta(vG)g(h\varphi(vE), F) - \eta(hG)g(v\varphi(vE), vF) \quad (10)$$

and

$$g(\tilde{A}_E F, G) = g(A_E G, F) + 2g(A_E F, G) + \eta(hF)g(v\varphi(hE), vG) - \eta(hG)g(v\varphi(hE), vF) + \eta(vF)g(h\varphi(hE), hG) - \eta(vG)g(h\varphi(hE), hF) \quad (11)$$

for all $E, F, G \in \Gamma(TM)$.

Proof. For all $E, F, G \in \Gamma(TM)$, by using (8) we get,

$$g(\tilde{T}_E F, G) = g(T_E G, F) + \eta(vF)g(h\varphi(vE), G) + (hF)g(v\varphi(vE), G).$$

We know that,

$$G = vG + hG.$$

Then, we have

$$g(\tilde{T}_E F, G) = g(T_E F, G) + \eta(vF)g(h\varphi(vE), G) + \eta(hF)g(v\varphi(vE), G) \quad (12)$$

In similiary, we arrive

$$g(\tilde{T}_E G, F) = g(T_E G, F) + \eta(vG)g(h\varphi(vE), F) - \eta(hG)g(V\varphi(vE), vF) \quad (13)$$

Then substracting (12) from (13) we get equation (10). Similar to (10), we can prove (11).

Theorem 3.3 Let π is a Riemannian submersion with QSNMC. Including basic tensors \tilde{T} and \tilde{A}

$$\tilde{T}_U W = \tilde{T}_W U + \eta(vW)h\varphi(vU) - \eta(vU)h\varphi(vW)$$

and

$$\tilde{A}_X Y = \tilde{A}_Y X + \eta(hY)v\varphi(hX) - \eta(hX)v\varphi(hY)$$

for all $U, W \in \Gamma(V)$ and $X, Y \in \Gamma(H)$.

Proof. For all $U, W \in \Gamma(V)$, by using (8), we get

$$\tilde{T}_U W = T_U W + \eta(vW)h\varphi(vU) + \eta(hW)v\varphi(vU) \quad (14)$$

and

$$\tilde{T}_W U = T_W U + \eta(vU)h\varphi(vW) + \eta(hU)v\varphi(vW) \quad (15)$$

Considering that $T_U W = T_W U$ and $hW = hU = 0$ is subtracted from (14) and (15), the desired is obtained.

Secondly for all $X, Y \in \Gamma(H)$ by using (9), we have

$$\tilde{A}_X Y = A_X Y + \eta(hY)v\varphi(hX) + \eta(vY)h\varphi(hX) \quad (16)$$

And

$$\tilde{A}_Y X = A_Y X + \eta(hX)v\varphi(hY) + \eta(vX)h\varphi(hY) \quad (17)$$

Similarly, considering that $\eta(vY) = 0$; $\eta(vX) = 0$, $A_X Y = -A_Y X$, using (16) and (17) is added, the desired is obtained.

Theorem 3.4 Let π is a Riemannian submersion with QSNMC. Then we have

$$\tilde{\nabla}_V W = \hat{\nabla}_V W + \tilde{T}_V W - \eta(W)\varphi(V), \quad (18)$$

$$\tilde{\nabla}_V X = h\tilde{\nabla}_V X + T_V X + \eta(X)\varphi(V), \quad (19)$$

$$\tilde{\nabla}_X Y = A_X Y + h\tilde{\nabla}_X Y, \quad (20)$$

$$\tilde{\nabla}_X V = A_X V + v\tilde{\nabla}_X V + \eta(V)h\varphi(X) \quad (21)$$

for all $U, W \in \Gamma(V)$ and $X, Y \in \Gamma(H)$.

Proof. Firstly, for all $U, W \in \Gamma(V)$, we have

$$\tilde{\nabla}_V W = \nabla_V W + \eta(V)\varphi W \quad (22)$$

or

$$\nabla_V W = \tilde{\nabla}_V W + \eta(V)\varphi W.$$

We know that,

$$\nabla_V W = v\nabla_V W + h\nabla_V W.$$

Then using (3), we arrive

$$\nabla_V W = v\tilde{\nabla}_V W + \eta(V)v\varphi W + h\tilde{\nabla}_V W + \eta(V)h\varphi W \quad (23)$$

Hence proof can be completed placing (23) in (22) considering

$$v\tilde{\nabla}_V W = \hat{\nabla}_V W \text{ and } h\tilde{\nabla}_V W = T_V W.$$

Similarly, for all $V \in \Gamma(V)$ and $X \in \Gamma(H)$ by using (3), we have,

$$\nabla_V X = v\tilde{\nabla}_V X - \eta(X)v\varphi V + h\tilde{\nabla}_V X - \eta(X)h\varphi V.$$

Hence, we consider $v\tilde{\nabla}_V X = \tilde{T}_V X$ and $h\tilde{\nabla}_V X = \tilde{A}_V X$. Then, (19) is achieved.

On the other hand, for all $X, Y \in \Gamma(H)$ from (3) we arrive

$$\nabla_V X = v\tilde{\nabla}_X Y - \eta(Y)v\varphi X + h\tilde{\nabla}_X Y - \eta(Y)h\varphi X \quad (24)$$

If the expression (24), including $v\tilde{\nabla}_X Y = A_X Y$, which gives (20).

Finally $V \in \Gamma(V)$ and $X \in \Gamma(H)$ by using (3), we conclude,

$$\nabla_X V = v\tilde{\nabla}_X V - \eta(V)v\varphi X + h\tilde{\nabla}_X V - \eta(V)h\varphi X$$

Then, including which gives (21).

4. Curvature relations with respect to quarter-symmetric non-metric connection

Finally, curvature properties of vertical and horizontal spaces of a Riemannian submersion defined by a quarter symmetric non-metric connection between two Riemannian manifolds are investigated.

Theorem 4.1 Let π is a Riemannian submersion with QSNMC. We denote Riemannian curvatures of M and any fibre $\pi^{-1}(x), \hat{g}_x$ by \tilde{R}, \hat{R} respectively. Then we have

$$g(\tilde{R}(U, V)W, F) = g(\hat{R}(U, V)W, F) - \eta(\hat{\nabla}_V W)g(\varphi(U), F) + \eta(\hat{\nabla}_U W)g(\varphi(V), F) \\ + \eta(\tilde{T}_V W)g(v\varphi(U), F) - \eta(\tilde{T}_U W)g(v\varphi(V), F) - g(\tilde{\nabla}_U \eta(W)\varphi(V), F)$$

$$+g(\tilde{\nabla}_V \eta(W)\varphi(U), F) - \eta(W)g(\varphi([U, V]), F) \quad (25)$$

And

$$\begin{aligned} g(\tilde{R}(U, V)W, F) &= g(\tilde{T}_U \tilde{\nabla}_V W, X) - g(\tilde{T}_V \tilde{\nabla}_U W, X) - \eta(\tilde{\nabla}_V W)g(\varphi(U), X) \\ &\quad + (\tilde{\nabla}_U W)g(\varphi(V), X) + g(h\tilde{\nabla}_U T_V W, X) - g(h\tilde{\nabla}_V T_U W, X) + g(T_U \tilde{T}_V W, X) \\ &\quad - g(T_V \tilde{T}_U W, X) - g(\tilde{\nabla}_U \eta(W)\varphi(V), X) + g(\tilde{\nabla}_V \eta(W)\varphi(U), X) \\ &\quad - g(\tilde{T}_{[U, V]} W, X) + \eta(W)g(\varphi([U, V]), X) \end{aligned} \quad (26)$$

for all $U, V, W, F \in \Gamma(V)$ and $X \in \Gamma(H)$.

Proof. With respect to the quarter symmetric non-metric connection M is the Riemann curvature tensor \tilde{R} by

$$\tilde{R}(U, V)W = \tilde{\nabla}_U \tilde{\nabla}_V W - \tilde{\nabla}_V \tilde{\nabla}_U W - \tilde{\nabla}_{[U, V]} W \quad (27)$$

Using (18) and (19), we can reach the following equation by direct operations

$$\begin{aligned} \tilde{R}(U, V)W &= \hat{R}(U, V)W + \tilde{T}_U \tilde{\nabla}_V W - \tilde{T}_V \tilde{\nabla}_U W - \eta(\tilde{\nabla}_V W)\varphi(U) + \eta(\tilde{\nabla}_U W)\varphi(V) \\ &\quad + h\tilde{\nabla}_U T_V W - h\tilde{\nabla}_V T_U W + T_U \tilde{T}_V W - T_V \tilde{T}_U W + \eta(\tilde{T}_V W)v\varphi(U) - \eta(\tilde{T}_U W)v\varphi(V) \\ &\quad - \tilde{\nabla}_U \eta(W)\varphi(V) + \tilde{\nabla}_V \eta(W)\varphi(U) - \tilde{T}_{[U, V]} W + \eta(W)\varphi([U, V]) \end{aligned} \quad (28)$$

If equation (28) is inner product with $F \in \Gamma(V)$, the equation (25) is obtained.

Similarly, if (28) equation is made with $X \in \Gamma(H)$ inner product (26) is obtained.

Theorem 4.2 Let π is a Riemannian submersion with QSNMC. We denote Riemannian curvatures M and B by \tilde{R} , R' respectively. Then we have

$$\begin{aligned} g(\tilde{R}(X, Y)Z, K) &= g(R'(X, Y)Z, K) - g(A_X K, A_Y Z) + g(A_Y K, A_X Z) \\ &\quad + \eta(A_Y Z)g(h\varphi(X), K) - \eta(A_X Z)g(h\varphi(Y), K) \end{aligned} \quad (29)$$

and

$$\begin{aligned} g(\tilde{R}(X, Y)Z, U) &= g(v\tilde{\nabla}_X A_Y Z, U) - g(v\tilde{\nabla}_Y A_X Z, U) + g(A_X h\tilde{\nabla}_Y Z, U) \\ &\quad - g(A_Y h\tilde{\nabla}_X Z, U) - g(T_U Z, A_X Y) - \eta(Z)g(v\varphi([X, Y]), U) \end{aligned} \quad (30)$$

for all $X, Y, Z, K \in \Gamma(H)$ and $U \in \Gamma(V)$.

Proof. For all $X, Y, Z, K \in \Gamma(H)$, using (27), (20) and (21) we have

$$\begin{aligned} \tilde{R}(X, Y)Z &= R'(X, Y)Z + A_X A_Y Z - A_Y A_X Z + v\tilde{\nabla}_X A_Y Z - v\tilde{\nabla}_Y A_X Z + \eta(A_Y Z)h\varphi(X) \\ &\quad - \eta(A_X Z)h\varphi(Y) + A_X h\tilde{\nabla}_Y Z - A_Y h\tilde{\nabla}_X Z - T_{[X, Y]} Z - \eta(Z)v\varphi([X, Y]). \end{aligned} \quad (31)$$

Taking the inner product in (31) with K we obtain (29). In a similar way taking the inner product in (31) with U we obtain

$$g(\tilde{R}(X, Y)Z, V) = g(v\tilde{\nabla}_X A_Y Z, V) - g(v\tilde{\nabla}_Y A_X Z, V) + g(A_X h\tilde{\nabla}_Y Z, V)$$

$$-g(A_Y h \tilde{\nabla}_X Z, V) - g(T_{[X,Y]} Z, V) - \eta(Z)g(v\varphi([X, Y]), V)$$

Since $[X, Y]$ is vertical, we have $\tilde{\nabla}_X Y = \tilde{\nabla}_Y X$. On the other hand $T_{[X,Y]} Z = -2T_{A_X Y} Z$ and from the property of metric $g(T_{[X,Y]} Z, U) = -g(2T_{A_X Y} Z, U) = 2g(T_U Z, A_X Y)$ are obtained. So we arrive at the equation (30).

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