



## Spectral Properties of the Sturm-Liouville Operator Produced by the Unseparated Boundary Conditions with Spectral Parameter

RAUF AMIROV<sup>1</sup> , SELMA GÜLYAZ ÖZYURT<sup>1,\*</sup> 

<sup>1</sup>*Department of Mathematics, Faculty of Science, Sivas Cumhuriyet University, 58140 Sivas, Turkey.*

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**ABSTRACT.** In this study, firstly, the basic properties of the spectrum of the investigated problem were learned, sine and cosine type solutions were defined, their behaviors were examined and the properties of the solution of the given problem were learned with their help. Next, the characteristic equation of the studied problem was formed with the help of sine and cosine type solutions. Using the characteristic equation, the asymptotic behavior of the eigenvalues of the given problem and the ordering of the eigenvalues of the boundary value problems  $L(\alpha_j)$ ,  $j = 1, 2$  when  $\alpha_1 < \alpha_2$  are learned.

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### 1. INTRODUCTION

Let us consider the boundary value problem  $L(\alpha)$  :

$$-y'' + q(x)y = \lambda y, \quad 0 < x < \pi, \quad \lambda = \rho^2 \quad (1.1)$$

$$y'(0) + (\alpha\lambda + \beta)y(0) + \omega y(\pi) = 0, \quad (1.2)$$

$$y'(\pi) + \gamma y(\pi) - \bar{\omega}y(0) = 0,$$

where  $\alpha, \beta, \gamma$  are real numbers,  $\omega$  is a complex number and such that  $\alpha\omega \neq 0$ ,  $q(x)$  is a real-valued function in  $L_2 [0, \pi]$ .

The solution to many problems of applied sciences and mathematical physics is reduced to the problem of examining (1.1) - (1.2) boundary value problems.

When  $\omega = 0$ , (1.2) the boundary conditions become separated boundary conditions depending on the spectral parameter. Spectral properties of operators produced by Sturm-Liouville and diffusion equations given with separated boundary conditions depending on the spectral parameter were learned in [1, 5, 6, 10, 15] studies. In the studies of [2–4, 7–9, 11–14, 16], straight and inverse problems were investigated for diffusion equations given with different unseparated boundary conditions at  $p(x) \equiv 0$  and  $p(x) \neq 0$ .

In this study, important properties of the spectrum of the  $L(\alpha)$  boundary value problem was learned when  $\alpha \neq 0$ ,  $\omega \neq 0$ . In the case of one of the unseparated boundary conditions is linearly dependent on the spectral parameter, the basic properties of the spectrum of the  $L(\alpha)$  boundary value problem are given. In our study, the eigenvalues of the boundary value problem  $L(\alpha)$  are real, the absence of conditions that function, the asymptotic behavior of the

\*Corresponding Author

Email addresses: emirov@cumhuriyet.edu.tr (R. Amirov), sgulyaz@cumhuriyet.edu.tr (S. Gülyaz Özyurt)

eigenvalues and finally the ordering of the eigenvalues of the boundary value problems  $L(\alpha_1)$  and  $L(\alpha_2)$  when  $\alpha_1 < \alpha_2$  is examined.

## 2. MAIN RESULTS

In this section, the eigenvalues of the  $L(\alpha)$  boundary value problem and the properties of the eigenfunctions will be learned and the asymptotic behavior of the eigenvalues will be obtained.

$C(x, \lambda)$  and  $S(x, \lambda)$ , are linearly independent of equation (1.1) and

$$C(0, \lambda) = S'(0, \lambda) = 1, \quad C'(0, \lambda) = S(0, \lambda) = 0, \quad (2.1)$$

get solutions that meet the initial conditions.

To obtain the main results of the study, let's first give the Lemma as follow.

**Lemma 2.1.** *The following representations are valid:*

$$\begin{aligned} C(\pi, \lambda) &= \cos \rho\pi + \pi c_0 \frac{\sin \rho\pi}{\rho} + \frac{1}{\rho} \int_{-\pi}^{\pi} \psi_1(t) e^{i\rho t} dt, \\ C'(\pi, \lambda) &= -\rho \sin \rho\pi + \pi c_0 \cos \rho\pi + \frac{1}{\rho} \int_{-\pi}^{\pi} \psi_2(t) e^{i\rho t} dt, \\ S(\pi, \lambda) &= \frac{\sin \rho\pi}{\rho} - \pi c_0 \frac{\cos \rho\pi}{\rho^2} + \frac{1}{\rho^2} \int_{-\pi}^{\pi} \psi_3(t) e^{i\rho t} dt, \\ S'(\pi, \lambda) &= \cos \rho\pi + \pi c_0 \frac{\sin \rho\pi}{\rho} + \frac{1}{\rho} \int_{-\pi}^{\pi} \psi_4(t) e^{i\rho t} dt, \end{aligned}$$

where  $c_0 = \frac{1}{2\pi} \int_0^{\pi} q(t) dt$ ,  $\psi_i(t) \in L_2[-\pi, \pi]$ ,  $i = \overline{1, 4}$ .

The proof of Lemma 2.1 is done similarly to the proof of Lemma 1.3.2 in [ [15]].

**Definition 2.2.** A complex number  $\lambda_0$  is called an eigenvalue of boundary value problem  $L(\alpha)$  if equation (1.1) with  $\lambda = \lambda_0$  has a nontrivial solution  $y_0(x)$  is called the eigenfunction of problem  $L(\alpha)$  corresponding to the eigenvalue  $\lambda_0$ . The set of eigenvalues is called the spectrum of  $L(\alpha)$ . The functions  $y_1(x), y_2(x), \dots, y_k(x)$  are called as the associated functions to the eigenfunction  $y_0(x)$ , if they absolutely continuous, then they satisfy the differential equations

$$-y_i''(x) + q(x)y_i(x) - y_{i-1}(x) = \lambda_0 y_i(x),$$

and boundary conditions

$$\begin{aligned} y_i'(0) + (\alpha\lambda + \beta) y_i(0) + \alpha y_{i-1}(0) + \omega y_i(\pi) &= 0, \\ y_i'(\pi) + \gamma y_i(\pi) - \varpi y_i(0) &= 0, \quad i = \overline{1, k}. \end{aligned} \quad (2.2)$$

**Theorem 2.3.** *If  $\alpha < 0$ , then the eigenvalues of the problem  $L(\alpha)$  are real.*

*Proof.* Let's assume the opposite, i.e.  $Im\lambda \neq 0$ . Let  $\lambda$  be an eigenvalue of the boundary problem  $L(\alpha)$ , and let  $y(x) = y(x, \lambda)$  be the corresponding eigenfunction. Multiply the equality (1.1) by  $\overline{y(x)}$ ,

$$-\overline{y''(x)} + q(x)\overline{y(x)} = \lambda \overline{y(x)}$$

by  $y(x)$  and subtract one product from the other. As a result, we get

$$(\lambda - \overline{\lambda}) |y(x)|^2 = \frac{d}{dx} [y(x)\overline{y'(x)} - y'(x)\overline{y(x)}].$$

Integrating this equality from zero to  $\pi$ , then we have

$$2iIm\lambda \int_0^{\pi} |y(x)|^2 dx = \overline{y(\pi)}y'(\pi) - y(\pi)\overline{y'(\pi)} - \overline{y(0)}y'(0) + y(0)\overline{y'(0)}. \quad (2.3)$$

It follows from the boundary conditions (1.2) there are

$$\begin{aligned} y'(0) &= -(\alpha\lambda + \beta)y(0) - \omega y(\pi), \\ y'(\pi) &= -\gamma y(\pi) + \varpi y(0). \end{aligned}$$

Let's consider these at the equality (2.3)

$$\begin{aligned} 2i\text{Im}\lambda \int_0^\pi |y(x)|^2 dx &= \overline{y(\pi)} [-\gamma y(\pi) + \varpi y(0)] - y(\pi) [-\gamma \overline{y(\pi)} + \omega \overline{y(0)}] \\ &\quad - \overline{y(0)} [-(\alpha\lambda + \beta)y(0) - \omega y(\pi)] + y(0) [-(\alpha\bar{\lambda} + \beta)\overline{y(0)} - \overline{\omega y(\pi)}] \\ &= 2i\alpha\text{Im}\lambda |y(0)|^2. \end{aligned}$$

Consequently,

$$\text{Im}\lambda \left\{ \int_0^\pi |y(x)|^2 dx - \alpha |y(0)|^2 \right\} = 0.$$

Since  $\alpha < 0$ , the expression in braces is distinct from zero. Therefore,  $\text{Im}\lambda = 0$ . Theorem is proved. □

**Theorem 2.4.** *If  $\alpha < 0$ , then the eigenfunctions of the boundary value problem  $L(\alpha)$  have no associated functions.*

*Proof.* Let's assume the opposite. Suppose that the boundary value problem  $L(\alpha)$  has an eigenfunction of  $y_1(x)$  associated to the  $y_0(x)$  eigenfunction, corresponding to the eigenvalue  $\lambda_0$ . Then, by virtue of (1.1) and (2.2), the following equalities hold:

$$-y_0''(x) + q(x)\overline{y_0(x)} = \lambda_0 \overline{y_0(x)} \tag{2.4}$$

$$-y_1''(x) + q(x)y_1(x) - y_0(x) = \lambda_0 y_1(x). \tag{2.5}$$

Multiply the equality (2.4) by  $y_1(x)$  and equality (2.5) by  $\overline{y_0(x)}$  and subtract them side by side, we will get

$$|y_0(x)|^2 = \frac{d}{dx} [y_1'(x)\overline{y_0(x)} - y_1(x)\overline{y_0'(x)}].$$

Integrating this equality from zero to  $\pi$  for  $x$ , we obtain

$$\int_0^\pi |y_0(x)|^2 dx = y_1'(\pi)\overline{y_0(\pi)} - y_1(\pi)\overline{y_0'(\pi)} - y_1'(0)\overline{y_0(0)} + y_1(0)\overline{y_0'(0)}.$$

According to the boundary conditions (1.2) and (2.2), we have

$$\begin{aligned} y_0'(0) &= -(\alpha\lambda + \beta)y_0(0) - \omega y_0(\pi), \\ y_0'(\pi) &= -\gamma y_0(\pi) + \varpi y_0(0), \\ y_1'(0) &= -(\alpha\lambda + \beta)y_1(0) - \omega y_1(\pi) - \alpha y_0(0), \\ y_1'(\pi) &= -\gamma y_1(\pi) + \varpi y_1(0). \end{aligned}$$

$$\begin{aligned} \int_0^\pi |y_0(x)|^2 dx &= \overline{y_0(\pi)} [-\gamma y_1(\pi) + \varpi y_1(0)] - y_1(\pi) [-\gamma \overline{y_0(\pi)} + \omega \overline{y_0(0)}] \\ &\quad + \overline{y_0(0)} [(\alpha\lambda + \beta)y_1(0) - \omega y_1(\pi) + \alpha y_0(0)] - y_1(0) [(\alpha\bar{\lambda} + \beta)\overline{y_0(0)} + \overline{\omega y_0(\pi)}] \\ &= \alpha |y_0(0)|^2. \end{aligned}$$

Hence

$$\int_0^{\pi} |y_0(x)|^2 dx - \alpha |y_0(0)|^2 = 0.$$

This contradicts the fact that the left side of this relation is positive according to the inequality  $\alpha < 0$ . Theorem is proved.  $\square$

The general solution of the equation (1.1) has a form

$$y(x, \lambda) = AS(x, \lambda) + BC(x, \lambda),$$

where  $A$  and  $B$  are arbitrary constant. Considering the boundary conditions (1.2) and initial conditions (2.1) taking into account the identity

$$C(x, \lambda) [S'(x, \lambda) + \gamma S(x, \lambda)] - S(x, \lambda) [C'(x, \lambda) + \gamma C(x, \lambda)] = 1,$$

it is easy to verify that the characteristic function of the boundary value problem  $L(\alpha)$  will be

$$\begin{aligned} \Delta(\lambda) &= 2Re\omega - [C'(\pi, \lambda) + \gamma C(\pi, \lambda)] + |\omega|^2 S(\pi, \lambda) \\ &+ (\alpha\lambda + \beta) [S'(\pi, \lambda) + \gamma S(\pi, \lambda)]. \end{aligned} \quad (2.6)$$

The square of zeros of the characteristic function  $\Delta(\lambda)$  are the eigenvalues of the problem  $L(\alpha)$ .

Using the representation in the Lemma 1, we transform the characteristic function (2.6) to the form

$$\begin{aligned} \Delta(\lambda) &= \alpha\rho^2 \cos \rho\pi + (1 + \alpha\pi c_0) \rho \sin \rho\pi + \alpha\gamma \sin \rho\pi \\ &+ (\beta - \alpha\gamma\pi c_0 - \pi c_0 - \gamma) \cos \rho\pi + 2Re\omega + \psi_{\xi}(\lambda) \end{aligned}$$

where  $\psi_{\xi}(\lambda) = \int_{-\pi}^{\pi} \tilde{\psi}_{\xi}(t) e^{i\lambda t} dt$ ,  $\tilde{\psi}_{\xi}(t) \in L_2[-\pi, \pi]$ .

Let  $\Delta_0(\lambda) = \alpha\rho^2 \cos \rho\pi + \rho \sin \rho\pi + \alpha\gamma \sin \rho\pi + (\beta - \gamma) \cos \rho\pi + 2Re\omega$ .

Using Lemma 1 which was proven in study [7], we can easily prove the following proposition.

**Theorem 2.5.** *The following statements hold.*

a) Let  $\lambda_n$  be the  $n$  th eigenvalue of the problem (1.1)-(1.2). Let  $\{\mu_k\}_{k \geq 0}$  be the zeros of the entire function  $\Delta_0(\lambda)$ . If  $\mu_p$  is the closest point to  $\lambda_n$  then as  $n \rightarrow \infty$

$$\rho_n = \sqrt{\mu_p} + O\left(\frac{1}{\sqrt{n}}\right). \quad (2.7)$$

In addition, if there exists a positive number  $\sigma_0$  such that  $\{\mu_k\}_{k \geq 0}$  satisfy

$$|\sqrt{\mu_k} - \sqrt{\mu_{k'}}| > \sigma_0 \text{ as } \mu_k \neq \mu_{k'}$$

then (2.7) holds for  $n = p$ .

b) Fix  $\delta > 0$ . Then there exists a constants  $C_{\delta}$  such that

$$|\Delta(\lambda)| > C_{\delta} |\rho|^2, \quad \rho \in G_{\delta}, \quad |\rho| \geq \rho^*$$

for sufficiently large  $\rho^*$ , where  $G_{\delta} = \{\rho \mid |\rho - \sqrt{\mu_n}| \geq \delta, n \geq 0\}$ .

It should be noted that  $\{\mu_n\}_{n \geq 0}$  are independent of the potential  $q(x)$ .

Furthermore, it is easy to see that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\mu_p}}{n} = a_1, \quad \lim_{n \rightarrow \infty} \frac{\rho_n}{\sqrt{\mu_p}} = a_2,$$

where both  $a_1$  and  $a_2$  are positive constans.

For the  $j = 1, 2$  denote the characteristic functions of the boundary value problems  $L(\alpha_j)$  by

$$\begin{aligned} \Delta_j(\lambda) &= 2Re\omega - [C'(\pi, \lambda) + \gamma C(\pi, \lambda)] + |\omega|^2 S(\pi, \lambda) \\ &+ (\alpha_j\lambda + \beta) [S'(\pi, \lambda) + \gamma S(\pi, \lambda)] \end{aligned} \quad (2.8)$$

and the negative eigenvalues arranged in increasing order by

$$\lambda_0^-(j), \lambda_{-1}^+(j), \lambda_{-1}^-(j), \lambda_{-2}^+(j), \lambda_{-2}^-(j), \dots$$

the positive eigenvalues arranged in increasing order by

$$\lambda_0^+(j), \lambda_1^-(j), \lambda_1^+(j), \lambda_2^-(j), \lambda_2^+(j), \dots$$

**Theorem 2.6.** *The relative position of the eigenvalues  $\lambda_k^\pm(1)$  and  $\lambda_k^\pm(2)$  ( $k = 0, \pm 1, \pm 2, \dots$ ) of the boundary value problems  $L(\alpha_1)$  and  $L(\alpha_2)$  with  $\alpha_1 < \alpha_2$  is determined for  $Im\omega \neq 0$  by the inequalities*

$$0 < \lambda_0^+(1) < \lambda_0^+(2) < \lambda_1^-(1) < \lambda_1^-(2) < \lambda_1^+(1) < \lambda_1^+(2) < \lambda_2^-(1) < \lambda_2^-(2) < \dots,$$

$$0 > \lambda_0^-(1) > \lambda_0^-(2) > \lambda_{-1}^+(1) > \lambda_{-1}^+(2) > \lambda_{-1}^-(1) > \lambda_{-1}^-(2) > \lambda_{-2}^+(1) > \lambda_{-2}^+(2) > \dots,$$

for real  $\omega < 0$  by the inequalities

$$0 < \lambda_0^+(1) < \lambda_0^+(2) < \lambda_1^-(1) < \lambda_1^-(2) < \lambda_1^+(1) \leq \lambda_1^+(2) < \lambda_2^-(1) \leq \lambda_2^-(2) \leq \dots,$$

$$0 > \lambda_0^-(1) > \lambda_0^-(2) > \lambda_{-1}^+(1) > \lambda_{-1}^+(2) \geq \lambda_{-1}^-(1) \geq \lambda_{-1}^-(2) > \lambda_{-2}^+(1) \geq \lambda_{-2}^+(2) \geq \dots,$$

and for real  $\omega > 0$  by the inequalities

$$0 < \lambda_0^+(1) \leq \lambda_0^+(2) \leq \lambda_1^-(1) \leq \lambda_1^-(2) < \lambda_1^+(1) \leq \lambda_1^+(2) \leq \lambda_2^-(1) \leq \lambda_2^-(2) < \dots,$$

$$0 > \lambda_0^-(1) \geq \lambda_0^-(2) \geq \lambda_{-1}^+(1) \geq \lambda_{-1}^+(2) > \lambda_{-1}^-(1) \geq \lambda_{-1}^-(2) \geq \lambda_{-2}^+(1) \geq \lambda_{-2}^+(2) > \dots$$

*Proof.* It is clear that function

$$U(x, \lambda) = [S'(\pi, \lambda) + \gamma S(\pi, \lambda)] C(x, \lambda) + [\omega - C'(\pi, \lambda) - \gamma C(\pi, \lambda)] S(x, \lambda)$$

is the solution of the equation (1.1) that satisfies the initial conditions

$$U(0, \lambda) = S'(\pi, \lambda) + \gamma S(\pi, \lambda), \quad U'(0, \lambda) = \omega - C'(\pi, \lambda) - \gamma C(\pi, \lambda). \tag{2.9}$$

Also,

$$U(\pi, \lambda) = \omega S(\pi, \lambda) + 1, \quad U'(0\pi, \lambda) = \omega S'(\pi, \lambda) - \gamma \tag{2.10}$$

conditions are met at the point  $x = \pi$ .

By differentiating the equation

$$U''(x, \lambda) + (\lambda - q(x)) U(x, \lambda) = 0 \tag{2.11}$$

with respect to  $\lambda$  and by passing to the complex conjugate in the resulting equation, for reel  $\lambda$  we obtain

$$\overline{U''(x, \lambda)} + (\lambda - q(x)) \overline{U(x, \lambda)} + \overline{U(x, \lambda)} = 0.$$

Next, we multiply this equation by  $U(x, \lambda)$  and equation (2.11) by  $\overline{U(x, \lambda)}$ , subtract the second result from the first, and integrate with respect to  $x$  over  $[0, \pi]$ ; by (2.9) and (2.10) and since the functions  $C(\pi, \lambda)$ ,  $C'(\pi, \lambda)$ ,  $S(\pi, \lambda)$  and  $S'(\pi, \lambda)$  are real, we obtain

$$\int_0^\pi |U(x, \lambda)|^2 dx = \overline{U(\pi, \lambda)} U'(\pi, \lambda) - \overline{U'(\pi, \lambda)} U(\pi, \lambda) - \overline{U(0, \lambda)} U'(0, \lambda) + \overline{U'(0, \lambda)} U(0, \lambda)$$

$$= [\omega S'(\pi, \lambda) - \gamma] \overline{\omega S(\pi, \lambda)} - \overline{\omega S'(\pi, \lambda)} [\omega S(\pi, \lambda) + 1] - [\omega - C'(\pi, \lambda) - \gamma C(\pi, \lambda)]$$

$$\times [S'(\pi, \lambda) + \gamma S(\pi, \lambda)] - [C'(\pi, \lambda) + \gamma C(\pi, \lambda)] [S'(\pi, \lambda) + \gamma S(\pi, \lambda)]$$

$$= |\omega|^2 S'(\pi, \lambda) \overline{S(\pi, \lambda)} - \gamma \overline{\omega S(\pi, \lambda)} + |\omega|^2 S(\pi, \lambda) \overline{S'(\pi, \lambda)} - \overline{\omega S'(\pi, \lambda)}$$

$$- \omega [S'(\pi, \lambda) + \gamma S(\pi, \lambda)] + [C'(\pi, \lambda) + \gamma C(\pi, \lambda)] [S'(\pi, \lambda) + \gamma S(\pi, \lambda)]$$

$$- [C'(\pi, \lambda) + \gamma C(\pi, \lambda)] [S'(\pi, \lambda) + \gamma S(\pi, \lambda)]$$

$$= - [S'(\pi, \lambda) + \gamma S(\pi, \lambda)] [2Re\omega - C'(\pi, \lambda) - \gamma C(\pi, \lambda) + |\omega|^2 S(\pi, \lambda) + \beta(S'(\pi, \lambda) + \gamma S(\pi, \lambda))]$$

$$+ [S'(\pi, \lambda) + \gamma S(\pi, \lambda)] [|\omega|^2 \overline{S(\pi, \lambda)} - C'(\pi, \lambda) - \gamma C(\pi, \lambda) + \beta(S'(\pi, \lambda) + \gamma S(\pi, \lambda))].$$

With regard for the equations

$$S'(\pi, \lambda) + \gamma S(\pi, \lambda) = \frac{\Delta_1(\lambda) - \Delta_2(\lambda)}{(\alpha_1 - \alpha_2)\lambda},$$

$$2Re\omega - [C'(\pi, \lambda) + \gamma C(\pi, \lambda)] + \beta[S'(\pi, \lambda) + \gamma S(\pi, \lambda)] + |\omega|^2 S(\pi, \lambda) = \frac{\alpha_2 \Delta_1(\lambda) - \alpha_1 \Delta_2(\lambda)}{\alpha_2 - \alpha_1},$$

which follows from (2.8), we find that

$$(\alpha_2 - \alpha_1) \left\{ \int_0^\pi |U(x, \lambda)|^2 dx + \alpha_j |U(0, \lambda)|^2 \right\} = \frac{\Delta_1(\lambda)\Delta_2(\lambda) - \Delta_1(\lambda)\Delta_2(\lambda)}{\lambda} + \frac{[\Delta_2(\lambda) - \Delta_1(\lambda)]\Delta_j(\lambda)}{\lambda^2}, \quad (j = 1, 2)$$

or

$$\lambda^2 (\alpha_2 - \alpha_1) \left\{ \int_0^\pi |U(x, \lambda)|^2 dx + \alpha_j |U(0, \lambda)|^2 \right\} = \lambda [\Delta_2(\lambda)]^2 \frac{d}{d\lambda} \left[ \frac{\Delta_1(\lambda)}{\Delta_2(\lambda)} \right] + [\Delta_2(\lambda) - \Delta_1(\lambda)] \Delta_j(\lambda), \quad (j = 1, 2).$$

If we make use of [12], we get the inequalities in the ruling of the theorem.  $\square$

#### CONFLICTS OF INTEREST

The authors declare that there no conflicts of interest regarding the publication of this paper.

#### AUTHORS CONTRIBUTION STATEMENT

The authors contributed equally to this study. All authors have read and agreed to the published version of the manuscript.

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