Mappings that transform helices from Euclidean space to Minkowski space

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Abstract

In this study, we introduce mappings that transform helices in Euclidean n-space to non-null helices in Minkowski n-space or Minkowski (n + 1)-space. Furthermore, we show that these mappings preserve the axes of the helices, and we also obtain the invariants of the mappings. Especially, by using these mappings, we give some examples of non-null helices which are constructed in Minkowski 3-space or Minkowski 4-space from some helices in Euclidean 3-space.

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1. Introduction

Helices are widely studied geometric objects that found relevancy in many fields, including but not limited to biology, computer aided design, architecture, or mechanical engineering. For example, the shape of the twisted-ladder structure of deoxyribonucleic acid (DNA) is a double helix [6,13,15].

In Euclidean 3-space, a helix is defined by the property that its tangent vector field makes a fixed angle with a fixed direction which is the axis of the helix. This well-known result was stated by M. A. Lancret in 1802 [9] and first proved by B. de Saint Venant in 1845. A necessary and sufficient condition for a curve to be a general helix is to have the ratio of its curvature to torsion constant. If both curvature and torsion are non-zero constants, then the curve is a circular helix [2,9,14]. We can adapt the helix notion to the Minkowski 3-space by using the angle notions in this space. Helix notion can similarly be extended to any n-dimensional (n > 3) Euclidean or Minkowski spaces see [4,10].

In [3], Altunkaya and Kula studied mappings that preserve helices in the n-dimensional Euclidean space, and in [1], Altunkaya studied mappings that preserve helices in the n-dimensional Minkowski space. These special mappings have been further characterized in these works.

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The papers mentioned above led us to study mappings from n-dimensional Euclidean space to n-dimensional Minkowski space that transform helices. We found some special mappings which can be associated with special relativity with a suitable change of the first coordinate to time (ct). Also, with these mappings, one can find the correlation of the helicoid motion of a particle in these spaces.

This paper is organized as follows. In section 2, we give basic theory of curves in Euclidean n-space and Minkowski n-space. Also, similar to well-known the notion of helix in Euclidean n-space, we give definition of non-null helix with non-null axis by using notion of angle between two non-null vector in Minkowski n-space.

In section 3, we define mappings that transform helices with the axis $e_1$ or $e_n$ from Euclidean n-space $\mathbb{R}^n$ to Minkowski n-space $\mathbb{R}^n_1$, and vice versa. While one mapping that transforms a helix with the axis $e_1$ in $\mathbb{R}^n$ to a non-null helix with the timelike axis $e_1$ in $\mathbb{R}^n_1$, the other mappings that transform a helix with axis $e_n$ in $\mathbb{R}^n$ to a non-null helix with the spacelike axis $e_n$ in $\mathbb{R}^n_1$. After, we give some examples about spacelike (or timelike) helix in Minkowski 3-space which is generated by a helix in Euclidean 3-space and illustrate them.

In section 4, we also introduce mappings that transform a helix with the axis $e_1$ (or $e_n$) in $\mathbb{R}^n$ to a non-null helix with the timelike axis $(e_1,0)$ (or the spacelike axis $(0,e_n)$) in $\mathbb{R}^{n+1}$. Finally, by using these mappings, we give some examples for non-null helices are constructed in Minkowski 4-space, which plays an important role in the theory of relativity, from some helices in Euclidean 3-space.

2. Preliminary

2.1. Euclidean space

Let $\mathbb{R}^n$ denote the Euclidean n-space, that is, the n-dimensional real vector space endowed with the standard inner product

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i,$$

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Also, the norm of a vector $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. Let $\{e_1, e_2, ..., e_n\}$ be the orthonormal basis where $e_j = (\delta_{1j}, \delta_{2j}, ..., \delta_{nj})$ is a unit vector in $\mathbb{R}^n$ for $j = 1, 2, ..., n$.

Let the curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve of order $n$ (i.e. that $\langle \gamma'(t), \gamma''(t), ..., \gamma^{(n)}(t) \rangle$ is a linearly independent subset of $\mathbb{R}^n$ for any $t \in I$). Now, let $\{V_1, V_2, ..., V_n\}$ be the moving Frenet frame along the regular curve $\gamma$ where $V_i$ ($i = 1, 2, ..., n$) denotes the $i$th Frenet vector field. Then, the Frenet formulae are given by

$$V_1' = \nu \kappa_1 V_2,$$

$$V_i' = -\nu \kappa_{i-1} V_{i-1} + \nu \kappa_i V_{i+1}, \quad i = 2, 3, ..., n - 1,$$

$$V_n' = -\nu \kappa_{n-1} V_{n-1},$$

where $\nu = \|\gamma'\|$ and $\kappa_i$ is the curvature functions of $\gamma$ [5, 7].

**Definition 2.1.** The curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^n$ is a helix if its tangent vector field $V_1$ makes the fixed angle $\theta$ with a fixed direction $U$ which is the axis. That is, $\langle V_1, U \rangle = \cos \theta$ where $\theta \in (0, \pi) \setminus \frac{\pi}{2}$ is a constant [12].

**Definition 2.2.** The $(n + 1)$-helix mapping $G : \mathbb{R}^n \setminus N \to \mathbb{R}^{n+1}$ is defined by

$$G(x_1, x_2, ..., x_n) = \frac{c}{d^2 + x_1^2 + x_2^2 + ... + (1 - a^2)x_n^2}(d, x_1, x_2, ..., x_n),$$

where $N = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | \quad x_1^2 + x_2^2 + ... + x_{n-1}^2 - (a^2 - 1)x_n^2 + d^2 \neq 0\}, a > 1, c \neq 0$ and $d \neq 0$ [3].
Theorem 2.3. The curve \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset R \rightarrow \mathbb{R}^n \) is a helix in \( \mathbb{R}^n \) whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{c}) \) with axis \( e_n \) if \( c \) and \( \gamma_n \).

\[
\mathcal{H}(\gamma) = \frac{c}{d^2 + \gamma_1^2 + \gamma_2^2 + \ldots + (1 - a^2)\gamma_n^2} (d, \gamma_1, \gamma_2, \ldots, \gamma_n) \quad (2.1)
\]

is a helix in \( \mathbb{R}^{n+1} \) whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{c}) \) with axis \((0, e_n)\) where \( c \neq 0, d \neq 0, a > 1 \) and \( d^2 + \gamma_1^2 + \gamma_2^2 + \ldots + (1 - a^2)\gamma_n^2 \neq 0 \) \([3]\).

2.2. Minkowski space

Let \( \mathbb{R}^n_1 \) denote the Minkowski \( n \)-space, that is, the \( n \)-dimensional real vector space \( \mathbb{R}^n \) endowed with the scalar product

\[
\langle x, y \rangle_* = -x_1y_1 + \sum_{i=2}^{n} x_iy_i,
\]

for all \( x = (x_1, x_2, \ldots, x_n), \ y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \). Also, the norm of a vector \( x \in \mathbb{R}^n_1 \) is defined by \( |x|_* = \sqrt{\langle x, x \rangle_*} \). A vector \( x \in \mathbb{R}^n_1 \) is said to be spacelike (resp. timelike, null) if \( \langle x, x \rangle_* > 0 \) or \( x = 0 \) (resp. \( \langle x, x \rangle_* < 0 \), \( \langle x, x \rangle_* = 0 \)).

A curve \( \alpha : I \rightarrow \mathbb{R}^n_1 \) is said to be spacelike (resp. timelike, null) if \( \alpha' = \frac{dx}{dt} \) is a spacelike (resp. timelike, null) vector at any \( t \in I \) \([10]\).

Definition 2.4. Let \( U, W \) be any two non-null vectors in \( \mathbb{R}^n_1 \).

1. Assume that \( U \) and \( W \) are spacelike vectors, then
   a) if \( S\mathcal{P} \{ U, W \} \) is a spacelike plane, then there is a unique number \( 0 \leq \bar{\theta} \leq \pi \) such that \( \langle U, W \rangle_* = \|U\|_*\|W\|_*\cos \bar{\theta} \).
   b) if \( S\mathcal{P} \{ U, W \} \) is a timelike plane, then there is a unique number \( \bar{\theta} \geq 0 \) such that \( \langle U, W \rangle_* = \|U\|_*\|W\|_*\cosh \bar{\theta} \) where \( \varepsilon = 1 \) or \( \varepsilon = -1 \) according to \( \text{sgn}(U_2) = \text{sgn}(W_2) \) or \( \text{sgn}(U_2) \neq \text{sgn}(W_2) \), respectively.

2. Assume that \( U \) and \( W \) are timelike vectors, then there is a unique number \( \bar{\theta} \geq 0 \) such that \( \langle U, W \rangle_* = \|U\|_*\|W\|_*\cosh \bar{\theta} \) where \( \varepsilon = 1 \) or \( \varepsilon = -1 \) according to \( U \) and \( W \) have different time-orientation or same time-orientation, respectively.

3. Assume that \( U \) is spacelike and \( W \) is timelike, then there is a unique number \( \bar{\theta} \geq 0 \) such that \( \langle U, W \rangle_* = \|U\|_*\|W\|_*\sinh \bar{\theta} \) where \( \varepsilon = 1 \) or \( \varepsilon = -1 \) according to \( \text{sgn}(U_2) = \text{sgn}(W_1) \) or \( \text{sgn}(U_2) \neq \text{sgn}(W_1) \), respectively.

where \( \bar{\theta} \) is angle between \( U \) and \( W \) \([11]\).

Let \( \alpha : I \rightarrow \mathbb{R}^n_1 \) be a non-null (spacelike or timelike) curve. We assume that \( \{\alpha'(t), \alpha''(t), \ldots, \alpha^{(m)}(t)\} \) are linearly independent at any \( t \in I \). Now, let \( \{\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_n\} \) be the moving Frenet frame along the regular curve \( \alpha \) where \( \mathbf{V}_i \ (i = 1, 2, \ldots, n) \) denotes the \( i \)th Frenet vector field. Then, the Frenet formulae are given by

\[
\begin{align*}
\mathbf{V}_1' &= \nu_*\epsilon_2k_1\mathbf{V}_2, \\
\mathbf{V}_i' &= -\nu_*\epsilon_{i-1}k_{i-1}\mathbf{V}_{i-1} + \nu_*\epsilon_{i+1}k_i\mathbf{V}_{i+1}, \quad i = 2, 3, \ldots, n - 1 \\
\mathbf{V}_n' &= -\nu_*\epsilon_{n-1}k_{n-1}\mathbf{V}_{n-1}
\end{align*}
\]

where, \( k_i \ (i = 1, 2, \ldots, n - 1) \) denotes the \( i \)th curvature, \( \nu_* = \|\alpha'\|_* \) and \( \epsilon_i = \langle \mathbf{V}_i, \mathbf{V}_i \rangle_* \) for \( 1 \leq i \leq n \) \([8]\).

By means of Definition 2.4, we can give the following two definitions of non-null helices with non-null axis in \( \mathbb{R}^n_1 \).

Definition 2.5. A regular curve \( \alpha : I \rightarrow \mathbb{R}^n_1 \) is a spacelike helix if its tangent vector field \( V_1 \) makes the fixed angle \( \bar{\theta} \) with a non-null unit vector \( U \in \mathbb{R}^n_1 \) which is the axis. That is, \( \varepsilon(V_1, U)_* = f(\bar{\theta}) \) is a constant where,
If $U$ is a timelike vector, then $f(\bar{\theta}) = \sinh \bar{\theta}$, $\bar{\theta} \geq 0$,

If $U$ is a spacelike vector and $Sp \{V_1, U\}$ is a spacelike plane in $\mathbb{R}^n_1$, then $f(\bar{\theta}) = \cos \bar{\theta}$, $\bar{\theta} \in (0, \pi) \setminus \frac{\pi}{2}$,

If $U$ is a spacelike vector and $Sp \{V_1, U\}$ is a timelike plane in $\mathbb{R}^n_1$, then $f(\bar{\theta}) = \cosh \bar{\theta}$, $\bar{\theta} \geq 0$.

Definition 2.6. A regular curve $\alpha : I \to \mathbb{R}^n$ is a timelike helix if its tangent vector field $V_1$ makes the fixed angle $\bar{\theta}$ with a non-null unit vector $U \in \mathbb{R}^n_1$ which is the axis. That is, $\varepsilon(V_1, U)_* = g(\bar{\theta})$ is a constant where,

1. If $U$ is a timelike vector, then $g(\bar{\theta}) = \cosh \bar{\theta}$, $\bar{\theta} \geq 0$,
2. If $U$ is a spacelike vector, then $g(\bar{\theta}) = \sinh \bar{\theta}$, $\bar{\theta} \geq 0$.

Remark 2.7. Throughout this study, all curves are regular and the mappings are built for helices with the axes $e_1$ or $e_n$. Moreover, by using similar method, the mappings can be constructed by helices with different axis.

3. Mappings that transform helices from $\mathbb{R}^n$ to $\mathbb{R}^n_1$

In this section, we introduce mappings that transform helices with axes $e_1$ or $e_n$ from Euclidean $n$-space to Minkowski $n$-space, and vice versa.

3.1. A mapping for helices with axis $e_1$

In this subsection, we introduce a mapping that transforms a helix with axis $e_1$ in $\mathbb{R}^n$ to a non-null helix with the timelike axis $e_1$ in $\mathbb{R}^n_1$.

Now, let us define the mapping $\Psi : \mathbb{R}^n \setminus \Gamma \to \mathbb{R}^n_1 \setminus \Gamma$

$$\Psi (x) = \lambda \bigg( -a^2 x_1^2 + \| x \|^2 \bigg) x,$$

where $\Gamma = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \| x \|^2 - a^2 x_1^2 \neq 0 \}$ and $a \in (1, \sqrt{2}) \cup (\sqrt{2}, \infty)$, $\lambda \neq 0$.

We get easily the following corollary for the mapping $\Psi$.

Corollary 3.1. $\Psi$ is an involution. That is, $\Psi = \Psi^{-1}$.

Lemma 3.2. The hypercone $C = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=2}^n x_i^2 = b^2 x_1^2, \quad b^2 \neq a^2 - 1 \}$ is invariant under the mapping $\Psi$ where $a > 1$ is a constant.

Proof. Let $x = (x_1, x_2, \ldots, x_n) \in C$ and $\Psi (x) = y$. So,

$$y_i = \frac{\lambda}{-a^2 x_1^2 + \| x \|^2} x_i, \quad 1 \leq i \leq n,$$

and

$$\sum_{i=2}^n y_i^2 = \left( \frac{\lambda}{-a^2 x_1^2 + \| x \|^2} \right)^2 \sum_{i=2}^n x_i^2$$

$$= b^2 \left( \frac{\lambda}{-a^2 x_1^2 + \| x \|^2} \right)^2 x_1^2$$

$$= b^2 y_1^2.$$

Thus, $y = (y_1, y_2, \ldots, y_n) \in C$. \qed
Lemma 3.3. \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n \) is a helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{a}) \) with axis \( e_1 \) iff \( \|\gamma\|^2 = a^2 (\gamma')^2 \) or equivalently, 
\[ (1 - a^2) (\gamma')^2 + \sum_{i=2}^{n} (\gamma_i')^2 = 0 \] where \( a > 1 \) [3].

Lemma 3.4. \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : J \subset \mathbb{R} \rightarrow \mathbb{R}^n \) is a timelike (spacelike) helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cosh^{-1}(1/b) \), \( \theta = \sinh^{-1}(1/b) \) with the timelike axis \( e_1 \) iff
\[ \langle \alpha', \alpha' \rangle = eb^2 (\alpha_1')^2, \quad (3.2) \]
or equivalently,
\[ -(1 + eb^2) (\alpha_1')^2 + \sum_{i=2}^{n} (\alpha_i')^2 = 0, \quad (3.3) \]
where \( b \) is a nonzero constant and \( \epsilon = \pm 1 \).

Proof. The proof can be obtained easily by using Lemma 2.1 and Lemma 2.2 in [1].

By the following theorem, we say that the mapping \( \Psi \) transforms a helix with axis \( e_1 \) in \( \mathbb{R}^n \) to a non-null helix with the timelike axis \( e_1 \) in \( \mathbb{R}^n \).

Theorem 3.5. Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n \) be a regular curve. Then, the curve \( \gamma \) is a helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{a}) \), \( a \in (1, \infty) \setminus \sqrt{2} \) with axis \( e_1 \) iff the curve,
\[ \alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}^n, \quad \alpha = \Psi(\gamma) = \frac{\lambda}{-a^2 \gamma_1^2 + \|\gamma\|^2} \quad (3.4) \]
is a timelike (spacelike) helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cosh^{-1}(\frac{\epsilon}{\sqrt{2-a^2}}) \) \( (\theta = \sinh^{-1}(\frac{\epsilon}{\sqrt{a^2-2}})) \) in \( \mathbb{R}^n \) with the timelike axis \( e_1 \) where \( \lambda \neq 0 \) and \( 1 < a < \sqrt{2} \) \( (a > \sqrt{2}). \)

Proof. Suppose that the curve \( \gamma \) is a helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{a}) \) with axis \( e_1 \) where \( a \in (1, \infty) \setminus \sqrt{2} \). Now, let the non-null curve \( \alpha \) be given by \( \alpha = \Psi(\gamma) \) in \( \mathbb{R}^n \). Then, we have,
\[ \alpha_i = h \gamma_i \quad \text{for} \quad i = 1, 2, \ldots, n \quad (3.5) \]
where
\[ h = \frac{\lambda}{-a^2 \gamma_1^2 + \|\gamma\|^2}. \quad (3.6) \]

So,
\[ \langle \alpha', \alpha' \rangle = -(h \gamma_1)^2 + \sum_{i=2}^{n} (h \gamma_i)^2 \quad (3.7) \]
and after a straightforward calculation, we obtain
\[ \sum_{i=2}^{n} (h \gamma_i)^2 = -\left(1 - a^2\right) (h \gamma_1)^2. \quad (3.8) \]

Furthermore, since \( \alpha \) is a non-null curve, by using (3.5),(3.7) and (3.8), we get
\[ \langle \alpha', \alpha' \rangle = \left(a^2 - 2\right) (\alpha_1')^2. \quad (3.9) \]

Therefore, from (3.2), the curve \( \alpha \) is a non-null helix with the timelike axis \( e_1 \) where
\[ eb^2 = a^2 - 2. \quad (3.10) \]

Thus, by using Lemma 3.4 and (3.10), we have two cases below.
Case 1: If $1 < a < \sqrt{2}$ then $\alpha$ is timelike helix ($\epsilon = -1$) whose tangent vector field $V_1$ makes the fixed angle $\vartheta = \cosh^{-1}\left(\frac{a}{\sqrt{2-a^2}}\right)$ with timelike axis $e_1$.

Case 2: If $a > \sqrt{2}$ then $\alpha$ is spacelike helix ($\epsilon = 1$) whose tangent vector field $V_1$ makes the fixed angle $\vartheta = \sinh^{-1}\left(\frac{a}{\sqrt{a^2-2}}\right)$ with timelike axis $e_1$.

Conversely, let us take the non-null helix $\alpha = \Psi(\gamma)$ that satisfies Case 1 or Case 2. Then, by Lemma 3.4, (3.5), (3.6) and (3.10), we get the following differential equation

$$\left(1 - a^2\right)(h\gamma_1)'^2 + \sum_{i=2}^{n} (h\gamma_i)'^2 = 0.$$  \hspace{1cm} (3.11)

After a straightforward calculation, we obtain

$$h^2 \left(-a^2(\gamma_1')^2 + \|\gamma'\|^2\right) = 0.$$  \hspace{1cm} (3.12)

Since $h \neq 0$, we have $\|\gamma'\|^2 = a^2(\gamma_1')^2$. From Lemma 3.3, $\gamma$ is a helix in $\mathbb{R}^n$. \hfill \Box

As a result of Theorem 3.5 and Corollary 3.1, $\Psi^{-1}(\alpha) = \gamma$ is also a helix in $\mathbb{R}^n$.

**Example 3.6.** Let us take the helix

$$\gamma(t) = \left(\frac{\sqrt{2}t^3}{3} + \sqrt{2}t, \frac{1}{3} \left(t^2 + 2\right)^{3/2}, t\right)$$

and its tangent vector

$$V_1(t) = \left(\frac{\sqrt{2}}{3}, t\sqrt{t^2 + 2}, \frac{1}{\sqrt{3}(t^2 + 1)}\right).$$

makes the fixed angle $\theta = \cos^{-1}\left(\frac{\sqrt{2}}{3}\right)$ with axis $e_1$ in $\mathbb{R}^3$. If we choose $\lambda = 1$ in (3.4),

$$\alpha(t) = \Psi(\gamma(t)) = \left(\frac{3\sqrt{2}t \left(t^2 + 3\right)}{12t^2 + 8}, \frac{3\left(t^2 + 2\right)^{3/2}}{12t^2 + 8}, \frac{9t}{12t^2 + 8}\right)$$

is a timelike helix and its tangent vector

$$V_1(t) = \left(\sqrt{2}, \frac{t \left(t^2 - 2\right) \sqrt{t^2 + 2}}{t^4 - t^2 + 2}, \frac{2 - 3t^2}{t^4 - t^2 + 2}\right)$$

makes the fixed angle $\vartheta = \sinh^{-1}\left(\sqrt{2}\right)$ with timelike axis $e_1$ in $\mathbb{R}^3$ (see Figure 1).
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Figure 1. (a) The helix $\gamma$ in $\mathbb{R}^3$, (b) The timelike helix $\alpha = \Psi(\gamma)$ in $\mathbb{R}^3_1$.

Example 3.7. Let us take the helix

$$\gamma(t) = \left( \frac{5e^t}{\sqrt{29}}, e^t \cos 2t, e^t \sin 2t \right)$$

and its tangent vector

$$V_1(t) = \left( \sqrt{\frac{5}{34}}, \sqrt{\frac{29}{170}}, \sqrt{\frac{29}{170}} \right)$$

makes the fixed angle $\theta = \cos^{-1}(\sqrt{\frac{5}{34}})$ with axis $e_1$ in $\mathbb{R}^3$. If we choose $\lambda = 1$ in (3.4),

$$\alpha(t) = \Psi(\gamma(t)) = \left( -\frac{5e^{-t}}{4\sqrt{29}}, -\frac{1}{4} e^{-t} \cos 2t, -\frac{1}{2} e^{-t} \sin t \cos t \right)$$

is a spacelike helix and its tangent vector

$$V_1(t) = \left( \sqrt{\frac{5}{2\sqrt{6}}}, \sqrt{\frac{29}{2\sqrt{30}}}, \sqrt{\frac{29}{2\sqrt{30}}} \right)$$

makes the fixed angle $\vartheta = \cosh^{-1}(\sqrt{\frac{5}{2\sqrt{6}}})$ with timelike axis $e_1$ in $\mathbb{R}^3_1$. Also, the helices $\gamma$ and $\alpha$ lie on the surface $\left\{ (x, y, z) \in \mathbb{R}^3 \middle| y^2 + z^2 = \frac{29}{25} x^2 \right\}$ (see Figure 2).
3.2. Mappings for helices with axis $e_n$

In this subsection, we introduce mappings which transforms a helix with axis $e_n$ in $\mathbb{R}^n$ to a non-null helix with the spacelike axis $e_n$ in $\mathbb{R}_1^n$.

Now, let $\psi_1 : \mathbb{R}^n \rightarrow \mathbb{R}_1^n$ be the mapping defined by

$$\psi_1(x) = \left( \frac{a^2 - 1}{a} x_n, \frac{\sqrt{a^2 - 1}}{a} x_2, \frac{\sqrt{a^2 - 1}}{a} x_3, \ldots, \frac{\sqrt{a^2 - 1}}{a} x_{n-1}, \frac{1}{a} x_1 \right),$$

(3.13)

where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $a > 1$.

Similar to Lemma 3.3 and Lemma 3.4, we give the following two Lemmas.

**Lemma 3.8.** $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$ is a helix whose tangent vector field $V_1$ makes the fixed angle $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ with axis $e_n$ iff $\|\gamma\|^2 = a^2 (\gamma'_n)^2$ or equivalently,

$$(1 - a^2) (\gamma'_n)^2 + \sum_{i=1}^{n-1} (\gamma'_i)^2 = 0$$

where $a > 1$.

**Lemma 3.9.** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) : J \subset \mathbb{R} \rightarrow \mathbb{R}_1^n$ is a non-null helix whose tangent vector field $V_1$ makes the fixed angle with the spacelike axis $e_n$ iff

$$\langle \alpha', \alpha'_n \rangle = \epsilon b^2 (\alpha'_n)^2,$$

(3.14)

or equivalently,

$$-(\alpha'_1)^2 + \sum_{j=2}^{n-1} (\alpha'_j)^2 + (1 - \epsilon b^2)(\alpha'_n)^2 = 0,$$

(3.15)

where $b$ is a nonzero constant and $\epsilon = \pm 1$.

By the following theorem, we say that the mapping $\psi_1$ transforms a helix with axis $e_n$ in $\mathbb{R}^n$ to a non-null helix with the spacelike axis $e_n$ in $\mathbb{R}_1^n$.

**Theorem 3.10.** Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be a regular curve. Then, the curve $\gamma$ is a helix whose tangent vector field $V_1$ makes the fixed angle $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ with axis $e_n$ where $a \in \left(1, \sqrt{2}\right) \cup \left(\sqrt{2}, \infty\right)$ iff the curve

$$\alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}_1^n, \quad \alpha = \psi_1(\gamma)$$

(3.16)

is a timelike (spacelike) helix whose tangent vector field $V_1$ makes the fixed angle $\bar{\theta} = \sinh^{-1}\left(\frac{e}{\sqrt{2 - a^2}}\right)$ (or $\bar{\theta} = \cosh^{-1}\left(\frac{e}{\sqrt{2 - a^2}}\right)$) with the spacelike axis $e_n$ in $\mathbb{R}_1^n$, where $a > \sqrt{2}$ ($1 < a < \sqrt{2}$).
Proof. Suppose that the curve \( \gamma \) is a helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{a}) \) with \( e_n \) where \( a \in (1, \infty) \backslash \sqrt{2} \). Now, by using (3.16), we have

\[
\alpha_1 = \frac{a^2 - 1}{a} \gamma_1, \quad \alpha_j = \frac{\sqrt{a^2 - 1}}{a} \gamma_j, \quad j = 2, 3, \ldots, n - 1, \quad \alpha_n = \frac{1}{a} \gamma_1.
\]

(3.17) (3.18) (3.19)

So,

\[
\langle \alpha', \alpha' \rangle = (\alpha'_n)^2 \left( 2 - a^2 \right),
\]

(3.20)

and from (3.14), the curve \( \alpha \) is a non-null helix with the spacelike axis \( e_n \) where

\[
e^2 = 2 - a^2.
\]

(3.21)

Thus, by using Lemma 3.9 and (3.21),

Case 1: If \( 1 < a < \sqrt{2} \) then \( \alpha \) is spacelike helix (\( \epsilon = 1 \)) whose tangent vector field \( V_1 \) makes the fixed angle \( \overline{\theta} = \cosh^{-1}(\frac{\epsilon}{\sqrt{2}-a}) \) with spacelike axis \( e_n \).

Case 2: If \( a > \sqrt{2} \) then \( \alpha \) is timelike helix (\( \epsilon = -1 \)) whose tangent vector field \( V_1 \) makes the fixed angle \( \overline{\theta} = \sinh^{-1}(\frac{\epsilon}{\sqrt{2}-a}) \) with spacelike axis \( e_n \).

Conversely, let us take the non-null helix \( \alpha \) with spacelike axis \( e_n \) in \( \mathbb{R}^n_1 \) that satisfies Case 1 or Case 2. Then, it is clear that the curve \( \gamma \) is a helix with axis \( e_n \) in \( \mathbb{R}^n \). □

Let \( \psi_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n_1 \) be the mapping defined by

\[
\psi_2(x) = \left( \sqrt{a^4 - 1} x_n, \sqrt{a^2 + 1} x_2, \sqrt{a^2 + 1} x_3, \ldots, \sqrt{a^2 + 1} x_{n-1}, ax_1 \right),
\]

(3.22)

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( a > 1 \).

By the following theorem, we say that the mapping \( \psi_2 \) transforms a helix with axis \( e_n \) in \( \mathbb{R}^n \) to a timelike helix with the spacelike axis \( e_n \) in \( \mathbb{R}^n_1 \).

Theorem 3.11. Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n \) be a regular curve. Then, the curve \( \gamma \) is a helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{a}) \) with axis \( e_n \) where \( a > 1 \) iff the curve,

\[
\alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}^n_1, \quad \alpha = \psi_2(\gamma)
\]

(3.23)

is a timelike helix whose tangent vector field \( V_1 \) makes the fixed angle \( \overline{\theta} = \sinh^{-1}(\epsilon a) \) with the spacelike axis \( e_n \) in \( \mathbb{R}^n_1 \).

Proof. We omit the proof since it is analogous to the proof of Theorem 3.10. □

Let \( \psi_3 : \mathbb{R}^n \rightarrow \mathbb{R}^n_1 \) be the mapping defined by

\[
\psi_3(x) = \left( a\sqrt{a^2 - 1} x_n, ax_2, ax_3, \ldots, ax_{n-1}, \sqrt{a^2 - 1} x_1 \right),
\]

(3.24)

where \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) and \( a > 1 \).

By the following theorem, we say that the mapping \( \psi_3 \) transforms a helix with axis \( e_n \) in \( \mathbb{R}^n \) to a timelike helix with the spacelike axis \( e_n \) in \( \mathbb{R}^n_1 \).

Theorem 3.12. Let \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n \) be a regular curve. Then, the curve \( \gamma \) is a helix whose tangent vector field \( V_1 \) makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{a}) \) with axis \( e_n \) where \( a > 1 \) iff the curve,

\[
\alpha : J \subset \mathbb{R} \rightarrow \mathbb{R}^n_1, \quad \alpha = \psi_3(\gamma)
\]

(3.25)

is a timelike helix whose tangent vector field \( V_1 \) makes the fixed angle \( \overline{\theta} = \sinh^{-1}(\epsilon \sqrt{a^2 - 1}) \) with the spacelike axis \( e_n \) in \( \mathbb{R}^n_1 \).
**Proof.** We omit the proof since it is analogous to the proof of Theorem 3.10. □

**Example 3.13.** Let us take the helix

$$\gamma(t) = \left(\sqrt{11} \cos \frac{t}{6}, \sqrt{11} \sin \frac{t}{6}, \frac{5t}{6}\right)$$

and its tangent vector

$$V_1(t) = \left(-\frac{\sqrt{11}}{6} \sin \frac{t}{6}, \frac{\sqrt{11}}{6} \cos \frac{t}{6}, \frac{5}{6}\right)$$

makes the fixed angle $\theta = \cos^{-1}\left(\frac{5}{6}\right)$ with axis $e_3$ in $\mathbb{R}^3$. Also, the curve

$$\alpha(t) = \psi_1(\gamma(t)) = \left(\frac{11t}{36}, \frac{11}{6} \sin \frac{t}{6}, \frac{5\sqrt{11}}{6} \cos \frac{t}{6}\right)$$

is a spacelike helix and its tangent vector

$$V_1(t) = \left(\sqrt{\frac{11}{14}} \csc \frac{t}{6}, \sqrt{\frac{11}{14}} \cot \frac{t}{6}, -\frac{5}{\sqrt{14}}\right)$$

makes the fixed angle $\bar{\theta} = \cosh^{-1}\left(\frac{5}{\sqrt{14}}\right)$ with the spacelike axis $e_3$ in $\mathbb{R}^3$. Moreover, the curve $\gamma$ lies on the helicoid $\{(x,y,z) \in \mathbb{R}^3 \mid y = \tan \frac{t}{x}\}$ in $\mathbb{R}^3$ and the curve $\alpha$ lies on the surface $\{(x,y,z) \in \mathbb{R}^3 \mid y = \sqrt{\frac{11}{5}} \tan \frac{6x}{11}\}$ in $\mathbb{R}^3_1$ (see Figure 3).

**Example 3.14.** Let us take the spherical helix

$$\gamma(t) = \left(-\frac{3}{5} \cos 4t - \frac{2}{5} \cos 6t, -\frac{3}{5} \sin 4t - \frac{2}{5} \sin 6t, \frac{2\sqrt{6}}{5} \sin t\right)$$

and its tangent vector

$$V_1(t) = \left(\frac{2\sqrt{6}}{5} \sin 5t, -\frac{2\sqrt{6}}{5} \cos 5t, \frac{1}{5}\right)$$

makes the fixed angle $\theta = \cos^{-1}\left(\frac{1}{5}\right)$ with axis $e_3$ in $\mathbb{R}^3$. Also, the curve

$$\alpha(t) = \psi_2(\gamma(t)) = \left(\frac{24\sqrt{26}}{5} \sin t, -\frac{\sqrt{26}}{5} (3 \sin 4t + 2 \sin 6t), -3 \cos 4t - 2 \cos 6t\right)$$

![Figure 3. (a) The helix $\gamma$ in $\mathbb{R}^3$. (b) The spacelike helix $\alpha = \psi_1(\gamma)$ in $\mathbb{R}^3_1$.](image)
Mappings that transform helices from Euclidean space to Minkowski space

is a timelike helix and its tangent vector

\[ V_1(t) = \left( \frac{2\sqrt{26}\cos t}{\sin 4t + \sin 6t}, -\frac{\sqrt{26}(\cos 4t + \cos 6t)}{\sin 4t + \sin 6t}, 5 \right) \]

makes the fixed angle \( \bar{\theta} = \sinh^{-1}(5) \) with spacelike axis \( e_3 \) in \( \mathbb{R}^3_1 \). Moreover, the curve \( \gamma \) lies on the unit sphere in \( \mathbb{R}^3 \) and the curve \( \alpha \) lies on the surface \( \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{6t^2} + \frac{y^2}{26} + \frac{z^2}{25} = 1 \right\} \) in \( \mathbb{R}^3_1 \) (see Figure 4).

**Figure 4.** (a) The spherical helix \( \gamma \) in \( \mathbb{R}^3 \), (b) The timelike helix \( \alpha = \psi_2(\gamma) \) in \( \mathbb{R}^3_1 \).

**Example 3.15.** Let us take the helix

\[ \gamma(t) = \left( \frac{2t \sin t + 2\cos t}{\sqrt{3}}, \frac{2\sin t - 2t \cos t}{\sqrt{3}}, \frac{t^2 + 1}{3} \right) \]

and its tangent vector

\[ V_1(t) = \left( \frac{\sqrt{3}}{2} \cos t, \frac{\sqrt{3}}{2} \sin t, \frac{1}{2} \right) \]

makes the fixed angle \( \theta = \cos^{-1}(\frac{1}{2}) \) with axis \( e_3 \) in \( \mathbb{R}^3 \). Also, the curve

\[ \alpha(t) = \psi_3(\gamma(t)) = \left( \frac{2t^2 + 2}{\sqrt{3}}, 4 \sin t - 4t \cos t, \frac{2t \sin t + 2\cos t}{\sqrt{3}} \right) \]

is a timelike helix and its tangent vector

\[ \overline{V}_1(t) = \left( 2 \sec t, 2 \tan t, \sqrt{3} \right) \]

makes the fixed angle \( \bar{\theta} = \sinh^{-1}(\sqrt{3}) \) with spacelike axis \( e_3 \) in \( \mathbb{R}^3_1 \). Moreover, the curve \( \gamma \) lies on the paraboloid \( \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{y^2}{4} + z^2 = 2\sqrt{3}x \right\} \) in \( \mathbb{R}^3_1 \) and the curve \( \alpha \) lies on the surface \( \left\{ (x, y, z) \in \mathbb{R}^3_1 \mid z^2 = 2\sqrt{3}x \right\} \) in \( \mathbb{R}^3_1 \) (see Figure 5).
4. Mappings that transform helices from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}_1$

Now, we introduce a mapping that transforms a helix with axis $e_1$ in $\mathbb{R}^n$ to a non-null helix with the timelike axis $(e_1,0)$ in $\mathbb{R}^{n+1}_1$.

Let $\Phi : \mathbb{R}^n \setminus \Omega \to \mathbb{R}^{n+1}_1$ be the mapping defined by

$$\Phi(x_1,x_2,\ldots,x_n) = \frac{\mu}{d^2 - a^2 x_1^2 + \|x\|^2} (x_1,x_2,\ldots,x_n,d),$$

where $\Omega = \{x \in \mathbb{R}^n \mid \|x\|^2 - a^2 x_1^2 + d^2 \neq 0\}$, $\mu \neq 0$, $d \neq 0$ and $a > 1$.

Analogous to the proof of Theorem 3.5, we can prove that the following theorem.

**Theorem 4.1.** Let $\gamma = (\gamma_1,\gamma_2,\ldots,\gamma_n) : I \subset \mathbb{R} \to \mathbb{R}^n$ be a regular curve. Then, the curve $\gamma$ is a helix whose tangent vector field $V_1$ makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis $e_1 \in \mathbb{R}^n$ where $a \in (1,\infty) \setminus \sqrt{2}$ if the curve $\beta : J \subset \mathbb{R} \to \mathbb{R}^{n+1}_1$,

$$\beta = \Phi(\gamma) = \frac{\mu}{d^2 - a^2 \gamma_1^2 + \|\gamma\|^2} (\gamma_1,\gamma_2,\ldots,\gamma_n,d)$$

is a timelike (spacelike) helix whose tangent vector field $V_1$ makes the fixed angle $\theta = \cosh^{-1} \left( \frac{\mu}{\sqrt{2-a^2}} \right)$ with the timelike axis $(e_1,0) \in \mathbb{R}^{n+1}_1$, where $\mu \neq 0$, $d \neq 0$ and $1 < a < \sqrt{2}$ ($a > \sqrt{2}$).

**Example 4.2.** Let us take the helix

$$\gamma(t) = \left( \frac{t^3}{3} + t, \frac{2}{3} \left( t^2 + 2 \right)^{3/2}, 2t \right)$$

and its tangent vector

$$V_1(t) = \left( \frac{1}{\sqrt{5}}, \frac{2t\sqrt{t^2 + 2}}{\sqrt{5}(t^2 + 1)}, \frac{2}{\sqrt{5}(t^2 + 1)} \right)$$

makes the fixed angle $\theta = \cos^{-1}(\frac{1}{a})$ with axis $e_1$ in $\mathbb{R}^3$. If we choose $\mu = 2$ and $d = 2$ in (4.2),

$$\beta(t) = \Phi(\gamma(t)) = \left( \frac{3t}{24t^2 + 34 + 9t^2 + 17}, \frac{9t}{12t^2 + 17}, \frac{9}{12t^2 + 17} \right)$$

is a spacelike helix and its tangent vector

$$V_1(t) = \left( \frac{1}{\sqrt{3}}, \frac{2t(4t^2 + 1)}{\sqrt{3}(4t^4 + 5t^2 + 17)}, \frac{34 - 24t^2}{\sqrt{3}(4t^4 + 5t^2 + 17)}, \frac{-16\sqrt{3}t}{4t^4 + 5t^2 + 17} \right)$$
makes the fixed angle $\bar{\theta} = \cosh^{-1}\left(\frac{1}{\sqrt{2}}\right)$ with timelike axis $(e_1, 0) = (1, 0, 0, 0)$ in $\mathbb{R}_1^4$.

**Example 4.3.** Let us take the cylindrical helix

$$\gamma(t) = \left(\frac{\sqrt{3}}{\sqrt{2}}t, \cos \frac{t}{\sqrt{2}}, \sin \frac{t}{\sqrt{2}}\right)$$

and its tangent vector

$$V_1(t) = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2} \sin \frac{t}{\sqrt{2}}, \frac{1}{2} \cos \frac{t}{\sqrt{2}}\right)$$

makes the fixed angle $\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ with axis $e_1$ in $\mathbb{R}^3$. If we choose $\mu = 1$ and $d = \frac{1}{2}$ in (4.2),

$$\beta(t) = \Phi(\gamma(t)) = \left(\frac{2\sqrt{6}t}{5-2t^2}, \frac{4}{5-2t^2}\cos \frac{t}{\sqrt{2}}, \frac{4}{5-2t^2}\sin \frac{t}{\sqrt{2}}, \frac{2}{5-2t^2}\right)$$

is a timelike helix and its tangent vector

$$\overline{V}_1(t) = \left(\sqrt{\frac{3}{2}} \frac{\sqrt{2}(2t^2-5)}{4t^2+10} \sin \frac{t}{\sqrt{2}} + 8t \cos \frac{t}{\sqrt{2}}, \frac{\sqrt{2}(2t^2)}{4t^2+10} \cos \frac{t}{\sqrt{2}} + 8t \sin \frac{t}{\sqrt{2}}, 2t\right)$$

makes the fixed angle $\bar{\theta} = \sinh^{-1}\left(\sqrt{\frac{2}{5}}\right)$ with timelike axis $(e_1, 0) = (1, 0, 0, 0)$ in $\mathbb{R}_1^4$.

Now, we define mappings that transforms a helix with $e_n$ in $\mathbb{R}^n$ to a non-null helix with the spacelike axis $(0, e_n)$ in $\mathbb{R}_1^{n+1}$.

Let $\varphi_i = \psi_i \circ \mathcal{S} : \mathbb{R}^n \rightarrow \mathbb{R}_1^{n+1}$ be a mapping for $i = 1, 2, 3$. Then, the mapping $\psi_i$ transforms a helix from with axis $e_n$ in $\mathbb{R}^n$ to another helix with the spacelike axis $(0, e_n)$ in $\mathbb{R}_1^{n+1}$.

**Corollary 4.4.** Let $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) : I \subset \mathbb{R} \rightarrow \mathbb{R}^n$ be a regular curve. Then, the curve $\gamma$ is a helix whose tangent vector field $V_1$ makes the fixed angle $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ with axis $e_n \in \mathbb{R}^n$ where $a > 1$. Then,

1) Let us take,

$$\beta_1 = \varphi_1(\gamma) = \frac{c}{d^2 - a^2 \gamma_n^2 + \|\gamma\|^2} \left(\frac{a^2-1}{a} \gamma_n, \frac{\sqrt{a^2-1}}{a} \gamma_1, \frac{\sqrt{a^2-1}}{a} \gamma_2, \ldots, \frac{\sqrt{a^2-1}}{a} \gamma_{n-1}, \frac{1}{a}\right)$$

i) $\beta_1 : J \subset \mathbb{R} \rightarrow \mathbb{R}_1^{n+1}$ is a timelike helix whose tangent vector field $V_1$ makes the fixed angle $\bar{\theta} = \sinh^{-1}\left(\frac{c}{\sqrt{a^2-1}}\right)$ with the spacelike axis $(0, e_n)$, where $a > \sqrt{2},$

ii) $\beta_1 : J \subset \mathbb{R} \rightarrow \mathbb{R}_1^{n+1}$ is a spacelike helix whose tangent vector field $V_1$ makes the fixed angle $\bar{\theta} = \cosh^{-1}\left(\frac{c}{\sqrt{a^2-1}}\right)$ with the spacelike axis $(0, e_n)$, where $1 < a < \sqrt{2}$.

2) The curve $\beta_2 : J \subset \mathbb{R} \rightarrow \mathbb{R}_1^{n+1},$

$$\beta_2 = \varphi_2(\gamma) = \frac{c}{d^2 - a^2 \gamma_n^2 + \|\gamma\|^2} \left(\sqrt{a^2-1} \gamma_n, \sqrt{a^2+1} \gamma_1, \sqrt{a^2+1} \gamma_2, \ldots, \sqrt{a^2+1} \gamma_{n-1}, ad\right)$$

is a timelike helix whose tangent vector field $V_1$ makes the fixed angle $\bar{\theta} = \sinh^{-1}(\varepsilon a)$ with the spacelike axis $(0, e_n)$.

3) The curve $\beta_3 : J \subset \mathbb{R} \rightarrow \mathbb{R}_1^{n+1},$

$$\beta_3 = \varphi_3(\gamma) = \frac{c}{d^2 - a^2 \gamma_n^2 + \|\gamma\|^2} \left(\sqrt{a^2-1} \gamma_n, a \gamma_1, a \gamma_2, \ldots, a \gamma_{n-1}, d \sqrt{a^2-1}\right)$$

is a timelike helix whose tangent vector field $V_1$ makes the fixed angle $\bar{\theta} = \sinh^{-1}\left(\varepsilon \sqrt{a^2-1}\right)$ with the spacelike axis $(0, e_n)$. 
Example 4.5. Let us take the helix
\[ \gamma(t) = \left( \sqrt{11} \cos \frac{t}{6}, \sqrt{11} \sin \frac{t}{6}, \frac{5t}{6} \right) \]
and its tangent vector
\[ V_1(t) = \left( -\frac{\sqrt{11}}{6} \sin \frac{t}{6}, \frac{\sqrt{11}}{6} \cos \frac{t}{6}, \frac{5}{6} \right) \]
makes the fixed angle \( \theta = \cos^{-1}\left(\frac{5}{6}\right) \) with axis \( e_3 \) in \( \mathbb{R}^3 \). If we choose \( c = 1 \) and \( d = 1 \) in Corollary 4.4, then

1) The curve
\[ \varphi_1(\gamma(t)) = \left( \frac{11t}{322 - 11t^2}, \frac{66}{432 - 11t^2} \cos \frac{t}{6}, \frac{66}{432 - 11t^2} \sin \frac{t}{6}, \frac{30}{432 - 11t^2} \right) \]
is a spacelike helix in \( \mathbb{R}^4 \). Then, its tangent vector field
\[ V_1(t) = \left( \frac{11t^2 + 322}{12\sqrt{14}t}, \frac{(11t^2 - 322) \sin \frac{t}{6} + 132t \cos \frac{t}{6}}{12\sqrt{14}t}, \frac{(432 - 11t^2) \cos \frac{t}{6} + 132t \sin \frac{t}{6}}{12\sqrt{14}t}, \frac{5}{\sqrt{14}} \right) \]
makes the fixed angle \( \bar{\theta} = \cosh^{-1}\left(\frac{5}{\sqrt{14}}\right) \) with the spacelike axis \( (0, e_3) = (0, 0, 0, 1) \) in \( \mathbb{R}^4 \).

2) The curve
\[ \varphi_2(\gamma(t)) = \left( \frac{6\sqrt{67}t}{420 - 55t^2}, \frac{36\sqrt{67}t}{420 - 55t^2} \cos \frac{t}{6}, \frac{36\sqrt{67}t}{420 - 55t^2} \sin \frac{t}{6}, \frac{216}{420 - 55t^2} \right) \]
is a timelike helix in \( \mathbb{R}^4 \). Then, its tangent vector field
\[ V_1(t) = \left( \sqrt{\frac{61}{11}} \frac{11t^2 + 322}{60t}, \sqrt{\frac{61}{11}} \frac{(11t^2 - 322) \sin \frac{t}{6} + 132t \cos \frac{t}{6}}{60t}, \sqrt{\frac{61}{11}} \frac{(432 - 11t^2) \cos \frac{t}{6} + 132t \sin \frac{t}{6}}{60t}, \frac{6}{\sqrt{14}} \right) \]
makes the fixed angle \( \bar{\theta} = \sinh^{-1}\left(\frac{6}{\sqrt{14}}\right) \) with the spacelike axis \( (0, e_3) = (0, 0, 0, 1) \) in \( \mathbb{R}^4 \).

3) The curve
\[ \varphi_3(\gamma(t)) = \left( \frac{36\sqrt{11}t}{2100 - 55t^2}, \frac{216\sqrt{11}t}{2100 - 55t^2} \cos \frac{t}{6}, \frac{216\sqrt{11}t}{2100 - 55t^2} \sin \frac{t}{6}, \frac{36\sqrt{11}t}{2100 - 55t^2} \right) \]
is a timelike helix in \( \mathbb{R}^4 \). Then, its tangent vector field
\[ V_1(t) = \left( \frac{11t^2 + 322}{10\sqrt{11}t}, \frac{(11t^2 - 322) \sin \frac{t}{6} + 132t \cos \frac{t}{6}}{10\sqrt{11}t}, \frac{(432 - 11t^2) \cos \frac{t}{6} + 132t \sin \frac{t}{6}}{10\sqrt{11}t}, \frac{\sqrt{11}}{5} \right) \]
makes the fixed angle \( \bar{\theta} = \sinh^{-1}\left(\frac{\sqrt{11}}{5}\right) \) with the spacelike axis \( (0, e_3) = (0, 0, 0, 1) \) in \( \mathbb{R}^4 \).

References


