Results in Nonlinear Analysis 4 (2021) No. 3, 169–178
https://doi.org/10.53006/rna.916750
Available online at www.nonlinear-analysis.com

Results in Nonlinear Analysis
ISSN 2636-7556

Peer Reviewed Scientific Journal

# Bifurcation and Chaos Control of a System of Rational Difference Equations

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**Abstract**

We study a system of rational difference equations in this article. For equilibrium points, we present the stability conditions. In addition, we show that the system encounters period-doubling bifurcation at the trivial equilibrium point $O$ and Neimark-Sacker bifurcation at the non-trivial equilibrium point $E$. To control the chaotic behavior of the system, we use the hybrid control approach. We also verify our theoretical outcomes at the end with some numerical applications.

*Keywords:* Stability, Bifurcation, Chaos control.
*2020 MSC:* 39A28, 39A30, 92D25.

## 1. Introduction

Owing to their uses in various directions, nonlinear differential equations have considerable significance. It is of great interest to study the existence and qualitative characteristics of the solutions to equilibrium points for nonlinear discrete systems. The nonlinear difference equations system, such as [1, 2], has been extensively studied by several researchers.

In [3], the author studied the following system

$$\begin{cases} x_{n+1} = \frac{a_1 x_{n-1}}{a_2 y_n x_{n-1} + a_3}, \\ y_{n+1} = \frac{b_1 y_{n-1}}{b_2 x_n y_{n-1} + b_3}, \end{cases}$$

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*Received April 16, 2021, Accepted August 10, 2021, Online August 13, 2021.*

where $a_1, a_2, a_3, b_1, b_2, b_3$ and starting values $x_0, y_0, x_{-1}, y_{-1}$ are all real parameters.
The author researched the following $3^{rd}$ order system in [4]

$$\begin{cases} x_{n+1} = \frac{a_1 x_{n-2}}{a_2 y_n z_{n-1} x_{n-2} + a_3}, \\ y_{n+1} = \frac{b_1 y_{n-2}}{b_2 z_n x_{n-1} y_{n-2} + b_3}, \\ z_{n+1} = \frac{c_1 z_{n-2}}{c_2 x_n y_{n-1} z_{n-2} + c_3}, \end{cases}$$

where $a_i, b_i, c_i(\ i \in \{1,2,3\})$, and initial conditions $x_{-j}, y_{-j}, z_{-j}(j \in \{0,1,2\})$ are all real parameters.
Q. Din (2014) researched the following system of difference equations in [5],

$$\begin{cases} x_{n+1} = \frac{a_1 y_n}{a_2 + a_3 y_n}, \\ y_{n+1} = \frac{b_1 y_n}{b_2 + b_3 x_n}. \end{cases} \tag{1}$$

In [5], the author studied local and global stability of equilibrium points of the system (1) when initial conditions and all parameters are positive. It will be interesting to discuss the model (1) when initial conditions and all parameters are real numbers. In this paper we extended the work of [5] by considering initial conditions and all parameters real numbers. Moreover, we discussed Neimark-Sacker bifurcation, period-doubling bifurcation and hybrid control strategy to control bifurcation in the system (1). In [5], the author did not discuss bifurcation and chaos control. For detailed stability analysis and bifurcation theory we refer the readers to [6, 7, 8, 9].

In the first part of the paper, we discuss the stability of equilibrium points of the system (1) for wider domain of parameters and initial values by taking them as all real numbers. In the second part of the paper, we study the period-doubling bifurcation and the Neimark-Sacker bifurcation of the system (1) at those equilibrium points, which were not discussed in [5]. At the end, we utilize the hybrid control technique to control bifurcation and chaotic behavior of the system (1).

## 2. Topological Classification of Equilibrium Points

Two equilibrium points of the system (1) are

$$E_0\,(0,0) \text{ and } E_1\left(\frac{b_1 - b_2}{b_3}, \frac{a_2(b_1 - b_2)}{a_1 b_3 + a_3 b_2 - a_3 b_1}\right).$$

The jacobian matrix of (1) calculated at point $(x, y)$ is

$$J(x,y) = \begin{bmatrix} 0 & \frac{a_1}{a_2 + a_3 y} - \frac{a_1 a_3 y}{(a_2 + a_3 y)^2} \\ -\frac{b_1 b_3 y}{(b_2 + b_3 x)^2} & \frac{b_1}{b_2 + b_3 x} \end{bmatrix}.$$

**Proposition 2.1.** *The equilibrium point $E_0(0,0)$ of (1) is*

*(i) locally asymptotically stable if* $|\frac{b_1}{b_2}| < 1,\ b_2 \neq 0,$

*(ii) unstable if* $|\frac{b_1}{b_2}| > 1,\ b_2 \neq 0,$

*(iii) non-hyperbolic point if either* $b_1 = b_2 \neq 0$ *or* $b_1 = -b_2 \neq 0.$

*Proof.* The Jacobian matrix of (1) evaluated at $E_0(0,0)$ is

$$J(E_0) = \begin{bmatrix} 0 & \frac{a_1}{a_2} \\ 0 & \frac{b_1}{b_2} \end{bmatrix}.$$

The matrix $J(E_0)$ have eigenvalues $\lambda_1 = 0$ and $\lambda_2 = \frac{b_1}{b_2}$.

□

**Proposition 2.2.** *The point $E_1$ of* (1) *is locally stable if and only if one of the following holds:*
*(i)* $0.25 + K > 0$ *and* $\sqrt{0.25 + K} < 0.5$,

*(ii)* $0.25 + K < 0$ *and* $|0.5 + \sqrt{0.25 + K}| < 1$.

*Proof.* The Jacobian matrix evaluated at the equilibrium point $E_1$ is

$$J(E_1) = \begin{bmatrix} 0 & \frac{(a_3(-b_1+b_2)+a_1b_3)^2}{a_1a_2{b_3}^2} \\ \frac{a_2b_3(b_1-b_2)}{b_1(a_3(b_1-b_2)-a_1b_3)} & 1 \end{bmatrix}.$$

The characteristic polynomial of $J(E_1)$ is

$$F(\lambda) = \lambda^2 - \lambda + \frac{-a_3{b_1}^2 + 2a_3b_1b_2 - a_3{b_2}^2 + a_1b_1b_3 - a_1b_2b_3}{a_1b_1b_3}. \tag{2}$$

The eigenvalues of $J(E_1)$ are

$$\begin{aligned}\lambda_{1,2} &= \frac{1 \pm \sqrt{1 - 4(\frac{-a_3{b_1}^2+2a_3b_1b_2-a_3{b_2}^2+a_1b_1b_3-a_1b_2b_3}{a_1b_1b_3})}}{2} \\ &= 0.5 \pm \sqrt{0.25 + \frac{a_3{b_1}^2 - 2a_3b_1b_2 + a_3{b_2}^2 - a_1b_1b_3 - a_1b_2b_3}{a_1b_1b_3}}.\end{aligned}$$

We can describe the eigenvalues of $J(E_1)$ as

$$\begin{aligned}\lambda_1 &= 0.5 - \sqrt{0.25 + \frac{a_3{b_1}^2 - 2a_3b_1b_2 + a_3{b_2}^2 - a_1b_1b_3 - a_1b_2b_3}{a_1b_1b_3}} \\ &= 0.5 - \sqrt{0.25 + K}\end{aligned}$$

and

$$\begin{aligned}\lambda_2 &= 0.5 + \sqrt{0.25 + \frac{a_3{b_1}^2 - 2a_3b_1b_2 + a_3{b_2}^2 - a_1b_1b_3 - a_1b_2b_3}{a_1b_1b_3}} \\ &= 0.5 + \sqrt{0.25 + K}\end{aligned}$$

where $K = \frac{a_3{b_1}^2-2a_3b_1b_2+a_3{b_2}^2-a_1b_1b_3-a_1b_2b_3}{a_1b_1b_3}$.

We know that the point $E$ is locally stable if and only if $|\lambda_{1,2}| < 1$. In our case $|\lambda_1| < |\lambda_2|$, so it is sufficient to show that $|\lambda_2| < 1$.

For $0.25 + K > 0$, we have

$$|\lambda_2| < 1 \Leftrightarrow \sqrt{0.25 + K} < 0.5,$$

and, for $0.25 + K < 0$, we have

$$|\lambda_2| < 1 \Leftrightarrow |0.5 + \sqrt{0.25 + K}| < 1.$$

□

## 3. Bifurcation and Chaos Control

The chaotic behavior and the bifurcation are unpredictable situations which may risk densities to extinction. The period doubling bifurcation at $O$ and the Neimark-Sacker bifurcation at $E$ in the (1) method are discussed in this section. The bifurcation theory can be studied from [10, 11, 12].

If we allow $x_n, y_n$ and all parameters to run through the set of all real numbers, then the system (1) experiences the period doubling bifurcation at point $E_0(0,0)$ for $b_1 = -b_2$. We define the set

$$\Gamma = \Big\{(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{R}^6 \mid b_1 = -b_3\Big\}.$$

The system (1) experiences the period doubling bifurcation at point $E_0(0,0)$ when $b_1$ varies in a small neighbourhood of $d_1 = -b_2$.

If we consider the following set

$$\Omega = \Big\{(a_1, a_2, a_3, b_1, b_2, b_3) \in \mathbb{R}^6 \mid a_1 = -\frac{a_3(b_1-b_2)^2}{b_2b_3},\ b_2b_3 \neq 0,\ a_3{b_1}^2 - a_3b_1b_2 \neq 0\Big\},$$

then the system (1) experiences the Neimark-Sacker bifurcation at the point $E_1\left(\frac{b_1-b_2}{b_3}, \frac{a_2(b_1-b_2)}{a_1b_3+a_3b_2-a_3b_1}\right)$ when $a_1$ varies in a small neighbourhood of $d_2 = -\frac{a_3(b_1-b_2)^2}{b_2b_3}$.

There are different control techniques to eliminate or delay the bifurcation. We use the hybrid control strategy.

Consider the following controlled system associated with system (1)

$$\begin{cases} x_{n+1} = \frac{\beta a_1 y_n}{a_2+a_3y_n} + (1-\beta)x_n, \\ y_{n+1} = \frac{\beta b_1 y_n}{b_2+b_3x_n} + (1-\beta)y_n, \end{cases} \tag{3}$$

where $0 < \beta \leq 1$.

Notice that the equilibrium points of the original system (1) and the controlled system (3) are exactly the same.

The controlled system (3) have Jacobian matrix at $(x, y)$

$$J(x,y) = \begin{bmatrix} 1-\beta & \frac{a_1\beta}{a_2+a_3y} - \frac{a_1\beta a_3 y}{(a_2+a_3y)^2} \\ -\frac{\beta b_1 b_3 y}{(b_2+b_3x)^2} & 1-\beta+\frac{\beta b_1}{b_2+b_3x} \end{bmatrix}.$$

**Proposition 3.1.** *The equilibrium point $O(0,0)$ of the controlled system* (3) *is locally stable if and only if*

$$\left|\frac{\beta b_1 + b_3 - \beta b_2}{b_2}\right| < 1.$$

*Proof.* The controlled system (3) have Jacobian matrix at $O(0,0)$

$$J(O) = \begin{bmatrix} 1-\beta & \frac{a_1\beta}{a_2} \\ 0 & 1+\beta(\frac{b_1}{b_2}-1) \end{bmatrix}.$$

Eigenvalues of the matrix $J(O)$ are $\lambda_1 = 1-\beta$ and $\lambda_2 = \frac{\beta d+e-\beta e}{e}$. □

**Proposition 3.2.** *The point $E$ of* (3) *is locally stable if and only if*

$$0 < \frac{(b_1-b_2)(b_3(-b_1+b_3)+a_1b_3)}{a_1b_1b_3} < \frac{1}{\beta}.$$

*Proof.* The controlled system (3) have Jacobian matrix at $E$

$$J(E) = \begin{bmatrix} 1-\beta & \frac{\beta(a_3(-b_1+b_2)+a_1b_3)^2}{a_1a_2{b_3}^2} \\ \frac{b\beta f(b_1-b_2)}{b_1(a_3(b_1-b_2)-a_1b_3)} & 1 \end{bmatrix}.$$

The characteristic polynomial of $J(E)$ is

$$F(\lambda) = \lambda^2 - p\lambda + q, \tag{4}$$

where

$$p = 2-\beta,$$
$$q = 1-\beta + \frac{\beta^2(b_1-b_2)(a_3(-b_1+b_2)+a_1b_3)}{a_1b_1b_3}.$$

The roots of $F(\lambda)=0$ satisfy the following

$$|\lambda| < 1 \Leftrightarrow |p| < 1+q < 2.$$

This gives our desired result

$$0 < \frac{(b_1-b_2)(a_3(-b_1+b_2)+a_1b_3)}{a_1b_1b_3} < \frac{1}{\beta}.$$

□

## 4. NUMERICAL EXAMPLES

We provide some interesting numerical examples in this section to validate our theoretical discussions about different qualitative and chaotic aspects of the model.

**Example 4.1.** *Consider the following values of the parameters*

$$a_1 = 55, a_2 = 70, a_3 = 80, b_2 = -50, b_3 = 35$$

*and the initial conditions*

$$x(0) = 0.1, y(0) = 0.1.$$

*We take $b_1$ as bifurcation parameter. For above set of values,*
♦ *The system undergoes the period doubling bifurcation at the equilibrium point $O$ as it passes through $b_1 = 50$.*
♦ *The eigenvalues of $J(O)$ are $\lambda_1 = -1, \lambda_2 = 0$ which confirm that the system undergoes the period doubling bifurcation at the point $O$.*

*We plot the bifurcation graphs of $x_n$ and $y_n$ against $b_1$ which show that both $x_n$ and $y_n$ undergoes the period doubling bifurcation. The bifurcation diagrams tell us that the point $O$ is stable for $b_1 < 50$ and loses its stability at $b_1 = 50$. This happens due to the occurrence of the period doubling bifurcation which leads to chaos for large values of $b_1$. It can easily be observed that the periodic prbits $2, 4, 8$ and $16$ occur for $b_1 \in [50, 68]$ and chaotic set occurs for $b_1 > 68$. In the window of the chaotic region, there exist some more periodic orbits. (See figure (1))*

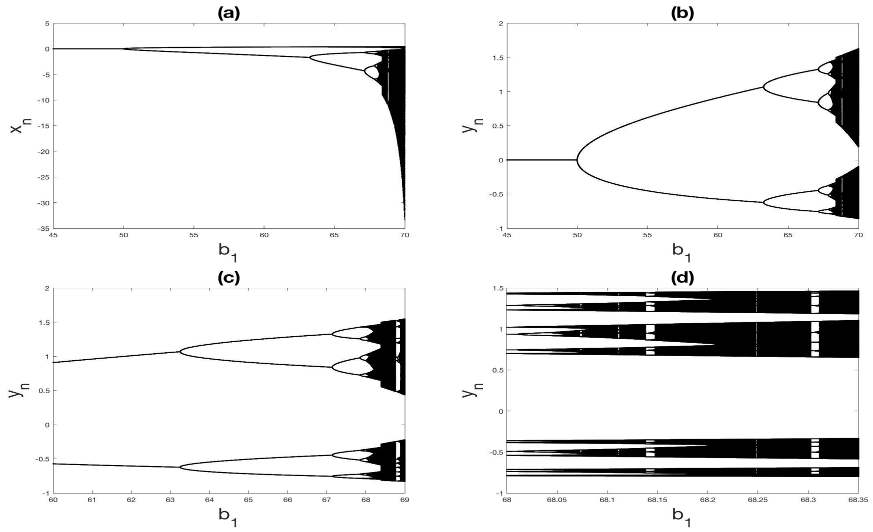


Figure 1: Plot of bifurcation diagrams for $x_n$ and $y_n$ for $a_1 = 55, a_2 = 70, a_3 = 80, b_2 = -50, b_3 = 35$ and initial conditions $x(0) = 0.1, y(0) = 0.1$ for $b_1 \in [45, 70]$, local amlification of bifurcation diagram for $y_n$ in subintervals $[60, 69]$ and $[68, 68.35]$.

**Example 4.2.** *Consider the parameters with following values*

$$a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$$

*and initial conditions*

$$x(0) = 0.15, y(0) = -0.05.$$

*We take $a_1$ as bifurcation parameter. For above set of values,*

♦ *The system undergoes the Neimark-Sacker bifurcation at point $E$ as it passes through $a_1 = -1.54113$.*

♦ *The eigenvalues of $J(E)$ are $\lambda_1 = 0.5 + 0.866025i$ and $\lambda_2 = 0.5 - 0.866025i$ with the property $|\lambda_{1,2}| = 1$ which confirms that the system undergoes the Neimark-Sacker bifurcation at the equilibrium point $E$.*

*We plot the bifurcation graphs and their amplification of $x_n$ and $y_n$ against parameter $a_1$ which show that both $x_n$ and $y_n$ undergoes the Neimark-Sacker bifurcation (see figure 2).*

*The bifurcation diagrams tell us that the equilibrium point $E$ is stable for $a_1 < -1.54113$ and loses its stability at $a_1 = -1.54113$. In Chaotic set $a_1 \in [-1.54113, -1.162]$ an attracting invariant curve appears. The invariant curve suddenly disappears for $a_1 = -1.162$ and a period 7 orbit occurs in the window $a_1 \in [-1.162, -1.136]$ which disappears at $a_1 = -1.135$ where again invariant curve appears. (see figure 3)*

**Example 4.3.** *Consider the parameters with the following values*

$$a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$$

*and initial conditions*

$$x(0) = 0.15, y(0) = -0.05.$$

*We take $a$ as arbitrary real number. For above set of values,*

♦ *The equilibrium point of (1) is $E(0.175, \frac{10.5}{40a_1-91})$.*

♦ *The Jacobian matrix of (1) at the point $E$ is*

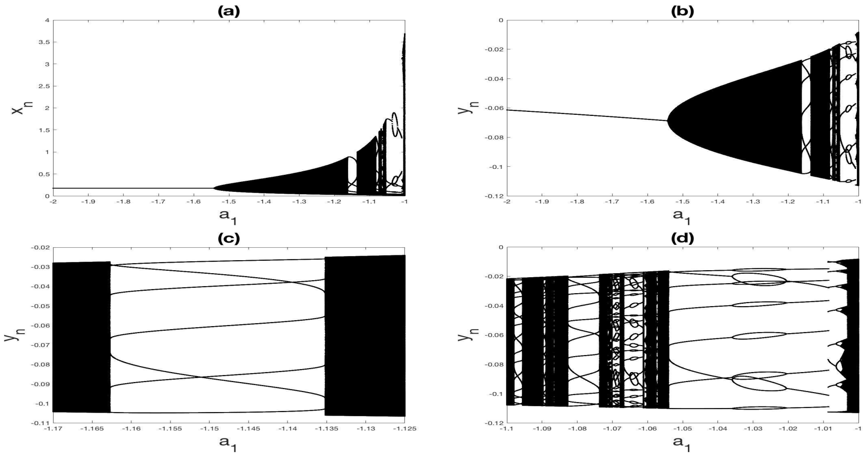


Figure 2: Plot of bifurcation diagrams for $x_n$ and $y_n$ for $a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$ and initial conditions $x(0) = 0.15, y(0) = -0.05$ for $a_1 \in [-2, -1]$.

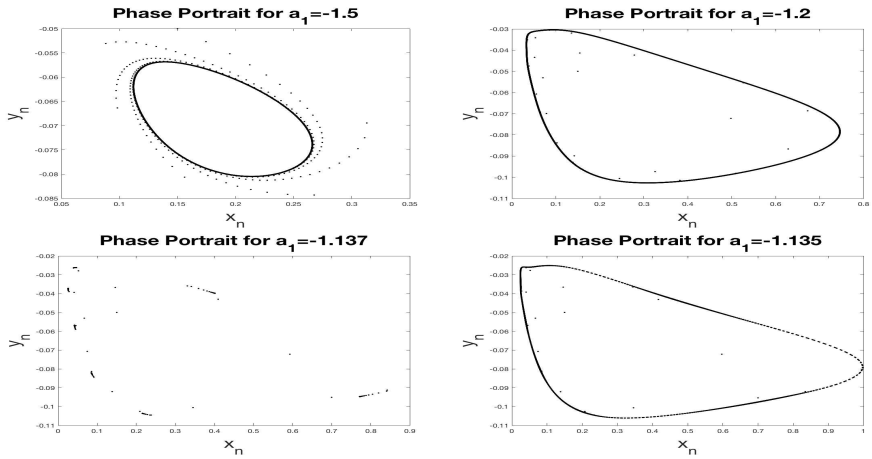


Figure 3: Plot of phase portrait for $x_n$ and $y_n$ for $a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$ and initial conditions $x(0) = 0.15, y(0) = -0.05$ for different values of $a_1$.

$$J(E) = \begin{bmatrix} 0 & \frac{a_1}{1.5+13y} - \frac{13a_1 y}{(1.5+13y)^2} \\ -\frac{62.4y}{(3.1+12x)^2} & \frac{5.2}{3.1+12x} \end{bmatrix}.$$

♦ *The characteristic polynomial of $J(E)$ is*

$$F(\lambda) = \lambda^2 - \lambda + \frac{2.09016 - 1.8375a_1 + 0.403846{a_1}^2}{a_1(a_1 - 2.275)}.$$

*By simple computations, we get*

$$F(0) = F(1) = \frac{2.09016 - 1.8375a_1 + 0.403846{a_1}^2}{a_1(a_1 - 2.275)},$$
$$F(-1) = \frac{2.09016 - 6.3875a_1 + 2.40385{a_1}^2}{a_1(a_1 - 2.275)}.$$

*By using lemma 2.2 of [13], the point $E$ is locally asymptotically stable iff either $a_1 < -1.54113$ or $a_1 > 2.275$.*
*We plot phase portraits for $a_1 = -3, -1.55, -1.54, -1.52, -1.4, 2.5$ and observe the following:*
♦ *For $a_1 = -3$ and $a_1 = -1.55$, the equilibrium point $E(0.175, \frac{10.5}{40a_1-91})$ is locally asymptotically stable.*
♦ *For $a_1 = -1.54, a_1 = -1.52$ and $a_1 = -1.4$, the equilibrium point $E(0.175, \frac{10.5}{40a_1-91})$ is unstable.*
♦ *For $a_1 = 2.5$, the point $E(0.175, \frac{10.5}{40a_1-91})$ is locally asymptotically stable.*

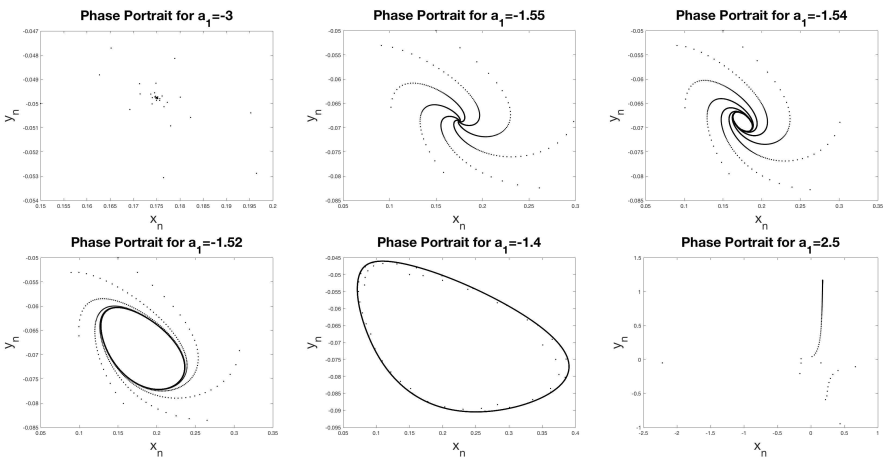


Figure 4: Plot of phase portraits of system (1) for $a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$ and initial conditions $x(0) = 0.15, y(0) = -0.05$ for different values of $a_1$.

**Example 4.4.** *Consider the parameters with following values*

$$a_1 = 55, a_2 = 70, a_3 = 80, b_2 = -50, b_3 = 35$$

*and initial conditions*

$$x(0) = 0.1, y(0) = 0.1.$$

*We present the bifurcation diagrams of $x_n$ and $y_n$ of the controlled system (3) for different values of $b_1$ by taking $\beta$ as bifurcation parameter.*
♦ *In figure 5(a,b), we take $b_1 = 50$.*
♦ *In figure 5(c,d), we take $b_1 = 55$.*
♦ *In figure 5(e,f), we take $b_1 = 65$.*
♦ *In figure 5(g,h), we take $b_1 = 68.5$.*
*From bifurcation diagrams, figure (5), we observe that the period doubling bifurcation at equilibrium point $O$ is delayed. The point $O$ of (3) shows the stability for wide domain of control parameter $\beta$.*

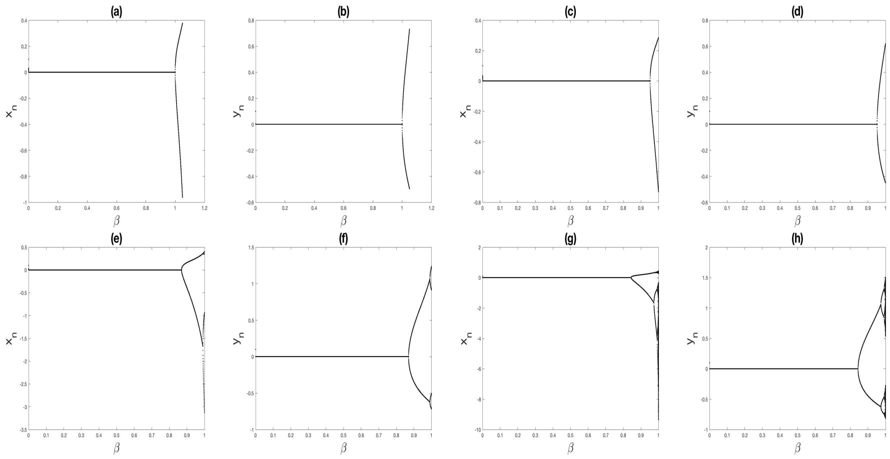

Figure 5: Plots of bifurcation diagrams for $x_n$ and $y_n$ of the controlled system (3) for $a_1 = 55, a_2 = 70, a_3 = 80, b_2 = -50, b_3 = 35$ and initial conditions $x(0) = 0.1, y(0) = 0.1$ for $b_1 = 50, 55, 65$ and $b_1 = 68.5$ with $0 < \beta \leq 1$.

**Example 4.5.** *Consider the following values of the parameters*

$$a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$$

*and initial conditions*

$$x(0) = 0.15, y(0) = -0.05.$$

*We present the bifurcation diagrams of $x_n$ and $y_n$ of the controlled system* (3) *for different values of $a_1$ by taking $\beta$ as bifurcation parameter.* ♦ *In figure 6(a,b), we take $a_1 = -1.54113$.*
♦ *In figure 6(c,d), we take $a_1 = -1.18$.*
♦ *In figure 6(e,f), we take $a_1 = -1.162$.*
♦ *In figure 6(g,h), we take $a_1 = -1.135$.*
*From bifurcation diagrams, figure* (6)*, we observe that the Neimark-Sacker bifurcation at equilibrium point $E$ is delayed. The point $E$ of* (3) *shows the stability for wide domain of control parameter $\beta$.*

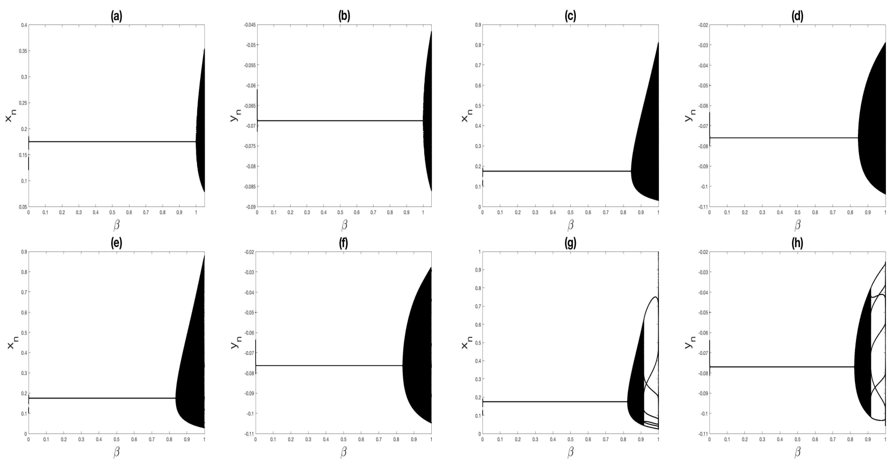

Figure 6: Plots of bifurcation diagrams for $x_n$ and $y_n$ of the controlled system (3) for $a_2 = 1.5, a_3 = 13, b_1 = 5.2, b_2 = 3.1, b_3 = 12$ and initial conditions $x(0) = 0.15, y(0) = -0.05$ for $a_1 = -1.54113, -1.18, -1.162$ and $a_1 = -1.132$ with $0 < \beta \leq 1$.

## 5. CONCLUSION

We explored the model of rational difference equations for broader parameter domains in this paper. By evaluating the Jacobian matrix (1) at the equilibrium points, we provided the stability conditions for equilibrium points. Moreover we discussed the period doubling and the Neimark-Sacker bifurcation at non-hyperbolic equilibrium points. We have shown numerically that the system undergoes the period doubling bifurcation at point $O$ and the Neimark-Sacker bifurcation at point $E$ under certain conditions on parameters. We used the technique of hybrid control to control the system's chaotic behavior. To confirm our theoretical results, we presented some interesting numerical examples.

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