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A coupled non-separated system of Hadamard-type fractional differential equations

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Abstract

In this article, we discuss the existence and uniqueness of solutions of a coupled non-separated system for fractional differential equations involving a Hadamard fractional derivative. The existence and uniqueness results obtained in the present study are not only new but also cover some results corresponding to special values of the parameters involved in the Caputo problems. These developed results are obtained by applying Banach's fixed point theorem and Leray-Schauder's nonlinear alternative. An example is presented to illustrate our main results.

Keywords: Hadamard-type fractional differential equations fixed point theorems boundary conditions.
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1. Introduction

Fractional calculus (FC) is one of the sections of mathematics which is a popularization of classical calculus that include integrals and derivatives of non-integer order. FC has a substantial role in numerous fields of science, engineering, and economics. FC tools have been found to support the development of mathematical methods which is more pragmatic to applied problems an expression of fractional differential equations (FDEs). Recently, many versions of the fractional derivatives, such as Reimann-Liouville, Caputo,

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Hadamard, Katugampola, Hilfer, ψ-Caputo, ψ-Hilfer, Atangana-Baleanu were presented, Here we refer to [2, 4, 15, 16, 19, 20, 21, 23, 24, 26, 27, 30].

Initial and boundary value problems for FDEs have won considerable significance because of their many employment in applied sciences and engineering. Many authors have shown great interest in this topic and obtaining a variety of results for FDEs involving different kinds of the conditions and the fractional operators, see [1, 9, 28, 29, 34] and the references therein. Some numerical methods such as the Adomian decomposition method for solving the nonlinear FDEs and system of nonlinear FDEs were studied by Jafari and Daftardar-Gejji in [12, 17, 18].

Several authors studied the existence and uniqueness theory for FDEs involving Hadamard-type operator, see [2, 3, 5, 11, 14, 22, 25, 31] and references therein. Coupled systems of FDEs are of great value and an important field. This type of system arises in business mathematics, management sciences, and other managerial sciences and so forth. To model such problems and some theoretical works on the coupled systems of FDEs, we refer the reader to some studied works [6, 7, 8, 10, 32, 33]. Motivated by the above discussion, in this article, we consider a coupled system of Hadamard-type FDEs:

$$\left\{\begin{array}{ll} \mathbb{D}_1^{\rho}\omega(\vartheta) = \hbar_1(\vartheta,\omega(\vartheta),\varpi(\vartheta)), & \vartheta\in[1,e], \quad 1<\rho\leq 2 \\ \mathbb{D}_1^{\sigma}\varpi(\vartheta) = \hbar_2(\vartheta,\omega(\vartheta),\varpi(\vartheta)), & \vartheta\in[1,e], \quad 1<\sigma\leq 2 \end{array}\right., \tag{1}$$

with the following non-separated coupled boundary conditions:

$$\left\{\begin{array}{l} \omega(1)=\xi_1\varpi(e) \\ \varpi(1)=\ell_1\omega(e) \end{array}\right. \qquad \left\{\begin{array}{l} \omega'(1)=\xi_2\varpi'(e) \\ \varpi'(1)=\ell_2\omega'(e) \end{array}\right., \tag{2}$$

where ξ_1, ξ_2,ℓ_1, ℓ_2 are real constants, $\mathbb{D}_1^{\rho}$, $\mathbb{D}_1^{\sigma}$ are the Hadamard fractional derivatives of order ρ and σ respectively, and $\hbar_1,\hbar_2$:$[1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are appropriate functions.

The existence and uniqueness of solutions for the system (1)-(2) are mainly investigated. To the best of our knowledge, a coupled system of Hadamard-type FDEs (1) with non-separated coupled boundary conditions (2) have were not widely studied. In consequence, the coupled system of FDEs with non-separated coupled boundary conditions will be studied by the Hadamard fractional derivative. Moreover, the main results are obtained by applying Banach fixed point theorem and Leray-Schauder fixed point theorem.

The outline of the paper is the following. In Section 2, we present some basic definitions and known results related to fractional calculus. Section 3 is devoted to proving the existence and uniqueness of the Hadamard coupled system (1)-(2). In the end, we present an illustrative example to justify our main results.

2. Preliminaries:

First of all, we present some definitions and properties from fractional calculus used throughout this article.

Definition 2.1. *[20] For a continuous function $\omega:[1,+\infty)\to\mathbb{R}$, the Hadamard fractional integral of order $\rho>0$ is defined by*

$$\mathbb{I}_1^{\rho}\omega(\vartheta) = \frac{1}{\Gamma(\rho)}\int_1^{\vartheta}(\log\frac{\vartheta}{s})^{\rho-1}\omega(s)\frac{ds}{s},$$

provided the right-hand side is point-wise defined on $[1,+\infty)$.

Definition 2.2. *[16] Let $n-1<\rho<n$, and $\omega(\vartheta)$ has an absolutely continuous derivative up to order $(n-1)$. Then the Caputo-Hadamard fractional derivative of order ρ is defined as*

$$\mathbb{D}_1^{\rho}\omega(\vartheta) = \frac{1}{\Gamma(n-\rho)}\int_1^{\vartheta}(\log\frac{\vartheta}{s})^{n-\rho-1}\delta^n(\omega)(s)\frac{ds}{s},$$

where $\delta^n=(\vartheta\frac{d}{d\vartheta})^n$, and $n=[\rho]+1$.

Lemma 2.3. *[16] Let $\rho > 0$ and $\omega \in C^n[1,+\infty)$ such that $\delta^{(n)}(\omega)$ exists a.e. on any bounded interval of $[1,+\infty)$. Then we have*

$$\mathbb{I}_1^\rho\left[\mathbb{D}_1^\rho\omega(\vartheta)\right] = \omega(\vartheta) - \sum_{k=0}^{n-1}\frac{\delta^{(k)}\omega(1)}{\Gamma(k+1)}(\log\vartheta)^k.$$

In particular, if $0 < \rho < 1$, then we have $\mathbb{I}_1^\rho\left[\mathbb{D}_1^\rho\omega(\vartheta)\right] = \omega(\vartheta) - \omega(1)$.

Lemma 2.4. *[20] For all $\ell > 0$ and $\nu > -1$, then we have*

$$\frac{1}{\Gamma(\ell)}\int_1^\vartheta(\log\frac{\vartheta}{s})^{\ell-1}(\log s)^\nu\frac{ds}{s} = \frac{\Gamma(\nu)}{\Gamma(\ell+\nu)}(\log\vartheta)^{\nu+\ell-1}.$$

Theorem 2.5. *[20] (Banach fixed point theorem) Let (W,d) be a nonempty complete metric space with $\Pi : W \to W$ is a contraction mapping. Then map Π has a fixed point.*

Theorem 2.6. *[13] (Leray-Schauder Nonlinear Alternative). Let W be a Banach space and $\mathbb{S} \subseteq W$ closed and convex. Assume that $\mathcal{K}$ is a relatively open subset of $\mathbb{S}$ with $0 \in \mathcal{K}$ and $\Pi : \overline{\mathcal{K}} \longrightarrow \mathbb{S}$ is a compact and continuous mapping. Then ethier*

1. *Π has a fixed point in $\overline{\mathcal{K}}$, or*
2. *there exists $\omega \in \partial\mathcal{K}$ such that $\omega = \lambda\Pi\omega$ for some $\lambda \in (0,1)$, where $\partial\mathcal{K}$ is boundary of $\mathcal{K}$.*

Now we present an auxiliary lemma which plays a key role in the sequel.

Lemma 2.7. *Let u, $v \in C([1,e],\mathbb{R})$. Then the solution of the linear fractional differential system*

$$\left\{\begin{array}{ll} \mathbb{D}_1^\rho\omega(\vartheta) = u(\vartheta), \quad \vartheta\in[1,e], & 1<\rho\leq 2 \\ \mathbb{D}_1^\sigma\varpi(\vartheta) = v(\vartheta), \quad \vartheta\in[1,e], & 1<\sigma\leq 2 \\ \omega(1) = \xi_1\varpi(e), & \omega'(1) = \xi_2\varpi'(e) \\ \varpi(1) = \ell_1\omega(e), & \varpi'(1) = \ell_2\omega'(e) \end{array}\right., \tag{3}$$

is equivalent to the system of integral equations

$$\begin{aligned} \omega(\vartheta) &= \frac{\xi_1}{\eta_2}\frac{1}{\Gamma(\rho)}\int_1^e(\log\frac{e}{s})^{\rho-1}u(s)\frac{ds}{s} + \frac{\ell_1\xi_1}{\eta_2}\frac{1}{\Gamma(\sigma)}\int_1^e(\log\frac{e}{s})^{\sigma-1}v(s)\frac{ds}{s} \\ &+ \left[\frac{\ell_1\xi_1\xi_2}{\eta_1\eta_2} + \frac{\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_2}{e\eta_1}\log\vartheta\right]\frac{1}{\Gamma(\rho-1)}\int_1^e(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s} \\ &+ \left[\frac{\ell_1\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_1}{\eta_1\eta_2} + \frac{\xi_2}{\eta_1}\log\vartheta\right]\frac{1}{\Gamma(\sigma-1)}\int_1^e(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\rho)}\int_1^\vartheta(\log\frac{\vartheta}{s})^{\rho-1}u(s)\frac{ds}{s}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \varpi(\vartheta) &= \frac{\ell_1}{\eta_2}\frac{1}{\Gamma(\sigma)}\int_1^e(\log\frac{e}{s})^{\sigma-1}v(s)\frac{ds}{s} + \frac{\ell_1\xi_1}{\eta_2}\frac{1}{\Gamma(\rho)}\int_1^e(\log\frac{e}{s})^{\rho-1}u(s)\frac{ds}{s} \\ &+ \left[\frac{\ell_1\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_1\xi_2}{\eta_1\eta_2} + \frac{\ell_1}{\eta_1}\log\vartheta\right]\frac{1}{\Gamma(\rho-1)}\int_1^e(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s} \\ &+ \left[\frac{\ell_1\ell_2\xi_1}{\eta_1\eta_2} + \frac{\ell_1\ell_2\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_2}{e\eta_1}\log\vartheta\right]\frac{1}{\Gamma(\sigma-1)}\int_1^e(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s} \\ &+ \frac{1}{\Gamma(\sigma)}\int_1^\vartheta(\log\frac{\vartheta}{s})^{\sigma-1}v(s)\frac{ds}{s}, \end{aligned} \tag{5}$$

where $\eta_1 = 1 - \frac{\ell_2\xi_2}{e^2} \neq 0$, $\eta_2 = 1 - \ell_1\xi_1 \neq 0$.

Proof. We know that the general solution of Hadamard coupled system in (3) can be written as

$$\omega(\vartheta) = c_0 + c_1(\log\vartheta) + \frac{1}{\Gamma(\rho)}\int_1^{\vartheta}(\log\frac{\vartheta}{s})^{\rho-1}u(s)\frac{ds}{s}, \tag{6}$$

$$\varpi(\vartheta) = d_0 + d_1(\log\vartheta) + \frac{1}{\Gamma(\sigma)}\int_1^{\vartheta}(\log\frac{\vartheta}{s})^{\sigma-1}v(s)\frac{ds}{s}, \tag{7}$$

where c_i, d_i $(i = 0, 1)$ are arbitrary real constants.

Using the non-separated coupled boundary conditions in (6) and (7), we have

$$\omega(1) = \xi_1\varpi(e) \quad \Rightarrow c_0 = \xi_1\left[d_0 + d_1 + \frac{1}{\Gamma(\sigma)}\int_1^{e}(\log\frac{e}{s})^{\sigma-1}v(s)\frac{ds}{s}\right], \tag{8}$$

$$\varpi(1) = \ell_1\omega(e) \quad \Rightarrow h_o = \ell_1\left[c_0 + c_1 + \frac{1}{\Gamma(\rho)}\int_1^{e}(\log\frac{e}{s})^{\rho-1}u(s)\frac{ds}{s}\right]. \tag{9}$$

Similarly, by using the non-separated coupled boundary conditions in (6) and (7), we have

$$\omega'(1) = \xi_2\varpi'(e) \quad \Rightarrow c_1 = \xi_2\left[\frac{d_1}{e} + \frac{1}{\Gamma(\sigma-1)}\int_1^{e}(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s}\right], \tag{10}$$

$$\varpi'(1) = \ell_2\omega'(e) \quad \Rightarrow d_1 = \ell_2[\frac{c_1}{e} + \frac{1}{\Gamma(\rho-1)}\int_1^{e}(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s}]. \tag{11}$$

From the relations (10) and (11), we get

$$\begin{aligned} c_1 &= \frac{\ell_2\xi_2}{e\eta_1}\frac{1}{\Gamma(\rho-1)}\int_1^{e}(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s} \\ &\quad + \frac{\xi_2}{\eta_1}\frac{1}{\Gamma(\sigma-1)}\int_1^{e}(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s}, \end{aligned} \tag{12}$$

$$\begin{aligned} d_1 &= \frac{\ell_2\xi_2}{e\eta_1}\frac{1}{\Gamma(\sigma-1)}\int_1^{e}(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s} \\ &\quad + \frac{\ell_2}{\eta_1}\frac{1}{\Gamma(\rho-1)}\int_1^{e}(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s}. \end{aligned} \tag{13}$$

Substituting c_1 and d_1 into (8) and (9), we get

$$\begin{aligned} c_0 &= \frac{\xi_1}{\eta_2}\frac{1}{\Gamma(\rho)}\int_1^{e}(\log\frac{e}{s})^{\rho-1}u(s)\frac{ds}{s} + \frac{\ell_1\xi_1}{\eta_2}\frac{1}{\Gamma(\sigma)}\int_1^{e}(\log\frac{e}{s})^{\sigma-1}v(s)\frac{ds}{s} \\ &\quad + \left[\frac{\ell_1\xi_1\xi_2}{\eta_1\eta_2} + \frac{\ell_2\xi_1\xi_2}{e\eta_1\eta_2}\right]\frac{1}{\Gamma(\rho-1)}\int_1^{e}(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s} \\ &\quad + \left[\frac{\ell_1\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_1}{\eta_1\eta_2}\right]\frac{1}{\Gamma(\sigma-1)}\int_1^{e}(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s}, \end{aligned}$$

and

$$\begin{aligned} d_0 &= \frac{\ell_1}{\eta_2}\frac{1}{\Gamma(\sigma)}\int_1^{e}(\log\frac{e}{s})^{\sigma-1}v(s)\frac{ds}{s} + \frac{\ell_1\xi_1}{\eta_2}\frac{1}{\Gamma(\rho)}\int_1^{e}(\log\frac{e}{s})^{\rho-1}u(s)\frac{ds}{s} \\ &\quad + \left[\frac{\ell_1\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_1\xi_2}{\eta_1\eta_2}\right]\frac{1}{\Gamma(\rho-1)}\int_1^{e}(\log\frac{e}{s})^{\rho-2}u(s)\frac{ds}{s} \\ &\quad + \left[\frac{\ell_1\ell_2\xi_1}{\eta_1\eta_2} + \frac{\ell_1\ell_2\xi_2}{e\eta_1\eta_2}\right]\frac{1}{\Gamma(\sigma-1)}\int_1^{e}(\log\frac{e}{s})^{\sigma-2}v(s)\frac{ds}{s}. \end{aligned}$$

Substituting the values of $c_i, d_i (i = 0, 1)$ in (6) and (7), we get solutions (4) and (5). The converse follows by direct computation. This completes the proof. □

3. Main results:

Let us introduce the space

$$W = \{\omega(\vartheta)|\omega(\vartheta) \in C([1,e],\mathbb{R})\},$$

endowed with the norm

$$\|\omega\| = \sup\{|\omega(\vartheta)|, \vartheta \in [1,e]\}.$$

Clearly, $(W, \|\cdot\|)$ is a Banach space. Then the product space

$$W \times G = \{\omega \times \varpi : \|\omega, \varpi\| = \sup_{\vartheta\in[1,e]} |(\omega,\varpi)(\vartheta)|\},$$

is also a Banach space equipped with the norm

$$\|(\omega,\varpi)\| = \|\omega\| + \|\varpi\|.$$

In view of Lemma (2.7), we define the operator $\Pi : W \times G \to W \times G$ by

$$\Pi(\omega,\varpi)(\vartheta) = \begin{pmatrix} \Pi_1(\omega,\varpi)(\vartheta) \\ \Pi_2(\omega,\varpi)(\vartheta) \end{pmatrix},$$

where

$$\begin{aligned}
\Pi_1(\omega,\varpi)(\vartheta) &= \frac{\xi_1}{\eta_2}\frac{1}{\Gamma(\rho)}\int_1^e (\log\frac{e}{s})^{\rho-1}\hbar_1(s,\omega(s),\varpi(s))\frac{ds}{s} \\
&+\frac{\ell_1\xi_1}{\eta_2}\frac{1}{\Gamma(\sigma)}\int_1^e (\log\frac{e}{s})^{\sigma-1}\hbar_2(s,\omega(s),\varpi(s))\frac{ds}{s} \\
&+\left[\frac{\ell_1\xi_1\xi_2}{\eta_1\eta_2} + \frac{\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_2}{e\eta_1}\log\vartheta\right] \\
&\times\left[\frac{1}{\Gamma(\rho-1)}\int_1^e (\log\frac{e}{s})^{\rho-2}\hbar_1(s,\omega(s),\varpi(s))\frac{ds}{s}\right] \\
&+\left[\frac{\ell_1\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_1}{\eta_1\eta_2} + \frac{\xi_2}{\eta_1}\log\vartheta\right] \\
&\times\left[\frac{1}{\Gamma(\sigma-1)}\int_1^e (\log\frac{e}{s})^{\sigma-2}\hbar_2(s,\omega(s),\varpi(s))\frac{ds}{s}\right] \\
&+\frac{1}{\Gamma(\rho)}\int_1^\vartheta (\log\frac{\vartheta}{s})^{\rho-1}\hbar_1(s,\omega(s),\varpi(s))\frac{ds}{s},
\end{aligned}$$

and

$$\begin{aligned}
\Pi_2(\omega,\varpi)(\vartheta) &= \frac{\ell_1}{\eta_2}\frac{1}{\Gamma(\sigma)}\int_1^e (\log\frac{e}{s})^{\sigma-1}\hbar_2(s,\omega(s),\varpi(s))\frac{ds}{s} \\
&+\frac{\ell_1\xi_1}{\eta_2}\frac{1}{\Gamma(\rho)}\int_1^e (\log\frac{e}{s})^{\rho-1}\hbar_1(s,\omega(s),\varpi(s))\frac{ds}{s} \\
&+\left[\frac{\ell_1\ell_2\xi_1\xi_2}{e\eta_1\eta_2} + \frac{\ell_1\xi_2}{\eta_1\eta_2} + \frac{\ell_1}{\eta_1}\log\vartheta\right] \\
&\times\left[\frac{1}{\Gamma(\rho-1)}\int_1^e (\log\frac{e}{s})^{\rho-2}\hbar_1(s,\omega(s),\varpi(s))\frac{ds}{s}\right] \\
&+\left[\frac{\ell_1\ell_2\xi_1}{\eta_1\eta_2} + \frac{\ell_1\ell_2\xi_2}{e\eta_1\eta_2} + \frac{\ell_2\xi_2}{e\eta_1}\log\vartheta\right] \\
&\times\left[\frac{1}{\Gamma(\sigma-1)}\int_1^e (\log\frac{e}{s})^{\sigma-2}\hbar_2(s,\omega(s),\varpi(s))\frac{ds}{s}\right] \\
&+\frac{1}{\Gamma(\sigma)}\int_1^\vartheta (\log\frac{\vartheta}{s})^{\sigma-1}\hbar_2(s,\omega(s),\varpi(s))\frac{ds}{s}.
\end{aligned}$$

To simplify, we put

$$N_1 = \frac{1}{\Gamma(\rho)}\left[\frac{|\ell_1||\xi_1||\xi_2|}{|\eta_1||\eta_2|} + \frac{|\ell_2||\xi_1||\xi_2|}{e|\eta_1||\eta_2|} + \frac{|\ell_2||\xi_2|}{e|\eta_1|}\right] + \frac{1}{\Gamma(\rho+1)}\left[\frac{|\xi_1|}{|\eta_2|}+1\right], \tag{14}$$

$$N_2 = \frac{1}{\Gamma(\sigma)}\left[\frac{|\ell_1||\ell_2||\xi_1||\xi_2|}{e|\eta_1||\eta_2|} + \frac{|\ell_2||\xi_1|}{|\eta_1||\eta_2|} + \frac{|\xi_2|}{|\eta_1|}\right] + \frac{1}{\Gamma(\sigma+1)}\left[\frac{|\ell_1||\xi_1|}{|\eta_2|}\right], \tag{15}$$

$$N_3 = \frac{1}{\Gamma(\rho)}\left[\frac{|\ell_1||\ell_2||\xi_1||\xi_2|}{e|\eta_1||\eta_2|} + \frac{|\ell_1||\xi_2|}{|\eta_1||\eta_2|} + \frac{|\ell_1|}{|\eta_1|}\right] + \frac{1}{\Gamma(\rho+1)}\left[\frac{|\ell_1||\xi_1|}{|\eta_2|}\right], \tag{16}$$

$$N_4 = \frac{1}{\Gamma(\sigma)}\left[\frac{|\ell_1||\ell_2||\xi_1|}{|\eta_1||\eta_2|} + \frac{|\ell_1||\ell_2||\xi_2|}{e|\eta_1||\eta_2|} + \frac{|\ell_2||\xi_2|}{e|\eta_1|}\right] + \frac{1}{\Gamma(\sigma+1)}\left[\frac{|\ell_1|}{|\eta_2|}+1\right]. \tag{17}$$

The first result is based on Banach's fixed point theorem. To this end, we need the following hypotheses:

(H$_1$) $\hbar_1, \hbar_2 : [1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are continuous and there exist two positive constants κ_1 and κ_2 such that

$$\begin{aligned} |\hbar_1(\vartheta,\omega_1,\omega_2) - \hbar_1(\vartheta,\varpi_1,\varpi_2)| &\le \kappa_1\left(|\omega_1-\varpi_1| + |\omega_2-\varpi_2|\right), \\ |\hbar_2(\vartheta,\omega_1,\omega_2) - \hbar_2(\vartheta,\varpi_1,\varpi_2)| &\le \kappa_2\left(|\omega_1-\varpi_1| + |\omega_2-\varpi_2|\right), \end{aligned}$$

for all $\vartheta\in[1,e]$ and $\omega_i,\varpi_i\in\mathbb{R}$ $i=1,2$.

(H$_2$) There exist real constants δ_i, $\beta_i\ge 0$, $i=0,1,2$, such that $\forall\omega_j\in\mathbb{R}(j=1,2)$, we have

$$\begin{aligned} |\hbar_1(\vartheta,\omega_1,\omega_2)| &\le \delta_0 + \delta_1|\omega_1| + \delta_2|\omega_2|, \\ |\hbar_2(\vartheta,\omega_1,\omega_2)| &\le \beta_0 + \beta_1|\omega_1| + \beta_2|\omega_2|. \end{aligned}$$

Theorem 3.1. *Assume that* (H_1) *holds. If*

$$\phi_1 := \left[(N_1+N_3)\kappa_1 + (N_2+N_4)\kappa_2\right] < 1, \tag{18}$$

then system (1)-(2) has a unique solution, where N_i*,* $i=1,2,3,4$ *are given by (14)-(17).*

Proof. Let $\sup_{\vartheta\in[1,e]}\hbar_1(\vartheta,0,0)=\epsilon_1<\infty$ and $\sup_{\vartheta\in[1,e]}\hbar_2(\vartheta,0,0)=\epsilon_2<\infty$ and $r>0$, we show that $\Pi\mathbb{B}_r\subset\mathbb{B}_r$, where

$$\mathbb{B}_r = \{(\omega,\varpi)\in W\times G : \|(\omega,\varpi)\| < r\},$$

with $r\ge\frac{\phi_2}{1-\phi_1}$, where $\phi_1<1$ and

$$\phi_2 := (N_1+N_3)\epsilon_1 + (N_2+N_4)\epsilon_2.$$

By hypotheses (H_1) and for $(\omega,\varpi)\in\mathbb{B}_r$, $\vartheta\in[1,e]$, we have

$$\begin{aligned} |\hbar_1(\vartheta,\omega(\vartheta),\varpi(\vartheta))| &\le |\hbar_1(\vartheta,\omega(\vartheta),\varpi(\vartheta)) - \hbar_1(\vartheta,0,0)| + |\hbar_1(\vartheta,0,0)| \\ &\le \kappa_1(|\omega(\vartheta)| + |\varpi(\vartheta)|) + \epsilon_1 \\ &\le \kappa_1 r + \epsilon_1. \end{aligned}$$

As same way, we have

$$|\hbar_2(\vartheta,\omega(\vartheta),\varpi(\vartheta))| \le \kappa_2 r + \epsilon_2,$$

which give to

$$
\begin{aligned}
|\Pi_1(\omega,\varpi)(\vartheta)| &\leq \left[\frac{|\xi_1|}{|\eta_2|}\right]\frac{1}{\Gamma(\rho)}\int_1^e (\log\frac{e}{s})^{\rho-1}\left|\hbar_1(s,\omega(s),\varpi(s))\right|\frac{ds}{s} \\
&+\left[\frac{|\ell_1|\,|\xi_1|}{|\eta_2|}\right]\frac{1}{\Gamma(\sigma)}\int_1^e (\log\frac{e}{s})^{\sigma-1}\left|\hbar_2(s,\omega(s),\varpi(s))\right|\frac{ds}{s} \\
&+\left[\frac{|\ell_1|\,|\xi_1|\,|\xi_2|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_2|}{e\,|\eta_1|}\left|\log\vartheta\right|\right] \\
&\times\left[\frac{1}{\Gamma(\rho-1)}\int_1^e (\log\frac{e}{s})^{\rho-2}\left|\hbar_1(s,\omega(s),\varpi(s))\right|\frac{ds}{s}\right] \\
&+\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_1|}{|\eta_1|\,|\eta_2|}+\frac{|\xi_2|}{|\eta_1|}\left|\log\vartheta\right|\right] \\
&\times\left[\frac{1}{\Gamma(\sigma-1)}\int_1^e (\log\frac{e}{s})^{\sigma-2}\left|\hbar_2(s,\omega(s),\varpi(s))\right|\frac{ds}{s}\right] \\
&\leq \left[\frac{|\xi_1|}{|\eta_2|}+1\right]\frac{(\kappa_1 r+\epsilon_1)}{\Gamma(\rho+1)}+\left[\frac{|\ell_1|\,|\xi_1|}{|\eta_2|}\right]\frac{(\kappa_2 r+\epsilon_2)}{\Gamma(\sigma+1)} \\
&+\left[\frac{|\ell_1|\,|\xi_1|\,|\xi_2|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_2|}{e\,|\eta_1|}\right]\frac{(\kappa_1 r+\epsilon_1)}{\Gamma(\rho)} \\
&+\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_1|}{|\eta_1|\,|\eta_2|}+\frac{|\xi_2|}{|\eta_1|}\right]\frac{(\kappa_2 r+\epsilon_2)}{\Gamma(\sigma)} \\
&\leq (\kappa_1 N_1+\kappa_2 N_2)r+(\epsilon_1 N_1+\epsilon_2 N_2).
\end{aligned}
$$

Hence

$$\|\Pi_1(\omega,\varpi)(\vartheta)\| \leq (\kappa_1 N_1+\kappa_2 N_2)r+(\epsilon_1 N_1+\epsilon_2 N_2).$$

Similarly, we obtain

$$
\begin{aligned}
|\Pi_2(\omega,\varpi)(\vartheta)| &\leq \left[\frac{|\ell_1|}{|\eta_2|}\right]\frac{(\kappa_2 r+\epsilon_2)}{\Gamma(\sigma+1)}+\left[\frac{|\ell_1|\,|\xi_1|}{|\eta_2|}\right]\frac{(\kappa_1 r+\epsilon_1)}{\Gamma(\rho+1)} \\
&+\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_1|\,|\xi_2|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_1|}{|\eta_1|}\right]\frac{(\kappa_1 r+\epsilon_1)}{\Gamma(\rho)} \\
&+\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_1|\,|\ell_2|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_2|}{e\,|\eta_1|}\right]\frac{(\kappa_2 r+\epsilon_2)}{\Gamma(\sigma)}+\frac{(\kappa_2 r+\epsilon_2)}{\Gamma(\sigma+1)} \\
&= \left(\frac{1}{\Gamma(\rho)}\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_1|\,|\xi_2|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_1|}{|\eta_1|}\right]\right. \\
&\left.+\frac{1}{\Gamma(\rho+1)}\left[\frac{|\ell_1|\,|\xi_1|}{|\eta_2|}\right]\right)(\kappa_1 r+\epsilon_1) \\
&+\left(\frac{1}{\Gamma(\sigma)}\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_1|\,|\ell_2|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_2|}{e\,|\eta_1|}\right]\right. \\
&\left.+\frac{1}{\Gamma(\sigma+1)}\left[\frac{|\ell_1|}{|\eta_2|}+1\right]\right)(\kappa_2 r+\epsilon_2) \\
&= (\kappa_2 N_3+\kappa_1 N_4)r+(\epsilon_2 N_3+\epsilon_1 N_4).
\end{aligned}
$$

Hence

$$\|\Pi_2(\omega,\varpi)(\vartheta)\| \leq (\kappa_1 N_3+\kappa_2 N_4)r+(\epsilon_1 N_3+\epsilon_2 N_4).$$

Consequently,

$$\begin{aligned}
\|\Pi(\omega,\varpi)(\vartheta)\| &\leq \|\Pi_1(\omega,\varpi)(\vartheta)\| + \|\Pi_2(\omega,\varpi)(\vartheta)\| \\
&\leq [(N_1+N_3)\kappa_1 + (N_2+N_4)\kappa_2]\, r \\
&\quad [N_1+N_3]\,\epsilon_1 + [N_2+N_4]\,\epsilon_2 \\
&\leq \phi_1 r + \phi_2 = r.
\end{aligned}$$

Now, for $(\omega_1,\varpi_1),(\omega_2,\varpi_2) \in W \times G$ and for any $\vartheta \in [1,e]$, we get

$$\begin{aligned}
& |\Pi_1(\omega_2,\varpi_2)(\vartheta) - \Pi_1(\omega_1,\varpi_1)(\vartheta)| \\
\leq\ & \left(\frac{1}{\Gamma(\rho)}\left[\frac{|\ell_1|\,|\xi_1|\,|\xi_2|}{|\eta_1|\,|\eta_2|} + \frac{|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|} + \frac{|\ell_2|\,|\xi_2|}{e\,|\eta_1|}\right]\right. \\
& \left. + \frac{1}{\Gamma(\rho+1)}\left[\frac{|\xi_1|}{|\eta_2|} + 1\right]\right) \times \kappa_1 \left(\|\omega_2-\omega_1\| + \|\varpi_2-\varpi_1\|\right) \\
& + \left(\frac{1}{\Gamma(\sigma)}\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|} + \frac{|\ell_2|\,|\xi_1|}{|\eta_1|\,|\eta_2|} + \frac{|\xi_2|}{|\eta_1|}\right]\right. \\
& \left. + \frac{1}{\Gamma(\sigma+1)}\left[\frac{|\ell_1|\,|\xi_1|}{|\eta_2|}\right]\right) \times \kappa_2 \left(\|\omega_2-\omega_1\| + \|\varpi_2-\varpi_1\|\right) \\
\leq\ & (N_1\kappa_1 + N_2\kappa_2)\left(\|\omega_2-\omega_1\| + \|\varpi_2-\varpi_1\|\right).
\end{aligned}$$

Hence

$$\|\Pi_1(\omega_2,\varpi_2)(\vartheta) - \Pi_1(\omega_1,\varpi_1)(\vartheta)\| \leq (N_1\kappa_1 + N_2\kappa_2)\left(\|\omega_2-\omega_1\| + \|\varpi_2-\varpi_1\|\right). \tag{19}$$

In the same way,

$$\|\Pi_2(\omega_2,\varpi_2)(\vartheta) - \Pi_2(\omega_1,\varpi_1)(\vartheta)\| \leq (N_3\kappa_1 + N_4\kappa_2)\left(\|\omega_2-\omega_1\| + \|\varpi_2-\varpi_1\|\right). \tag{20}$$

It follows from (19) and (20) that

$$\begin{aligned}
& \|\Pi(\omega_2,\varpi_2)(\vartheta) - \Pi(\omega_1,\varpi_1)(\vartheta)\| \\
\leq\ & [(N_1+N_3)\,\kappa_1 + (N_2+N_4)\,\kappa_2]\left(\|\omega_2-\omega_1\| + \|\varpi_2-\varpi_1\|\right).
\end{aligned}$$

Since $\phi_1 < 1$, the operator Π is a contraction mapping. So, we conclude that the Hadamard coupled system (1)-(2) has a unique solution due to Banach fixed point theorem. The proof is completed. □

The next result is based on Leray-Schauder's fixed point theorem.

Theorem 3.2. *Assume that (H_1) and (H_2) holds. If*

$$[(N_1+N_3)\,\delta_1 + (N_2+N_4)\,\beta_1] < 1,$$

$$[(N_1+N_3)\,\delta_2 + (N_2+N_4)\,\beta_2] < 1,$$

and

$$[(N_1+N_3)\,\mu_1 + (N_2+N_4)\,\mu_2] < 1,$$

where $\mu_1 > 0$, $\mu_2 > 0$ and N_i, $i = 1,2,3,4$, are given by (14)-(17), then the Hadamard coupled system (1)-(2) has at least one solution.

Proof. Firstly, we will prove that the operator $\Pi : W \times G \to W \times G$ is completely continuous. Since $\hbar_1$ and $\hbar_2$ are continuous functions on $[1,e]$, it is obvious that Π is continuous too. Let $\mathbb{S} \subset W \times G$ be bounded. Then there exist two positive constants $\mu_1 > 0$ and $\mu_2 > 0$ such that

$$|\hbar_1(\vartheta,\omega(\vartheta),\varpi(\vartheta))| < \mu_1, \text{ for } (\vartheta,\omega,\varpi) \in [1,e] \times \mathbb{S},$$

and

$$|\hbar_2(\vartheta,\omega(\vartheta),\varpi(\vartheta))| < \mu_2, \text{ for } (\vartheta,\omega,\varpi) \in [1,e]\times\mathbb{S}.$$

Then, for any $(\omega,\varpi)\in\mathbb{S}$, we have

$$\begin{aligned}
|\Pi_1(\omega,\varpi)(\vartheta)| &\le \left(\frac{1}{\Gamma(\rho)}\left[\frac{|\ell_1|\,|\xi_1|\,|\xi_2|}{|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_2|}{e\,|\eta_1|}\right]\right.\\
&\quad\left.+\frac{1}{\Gamma(\rho+1)}\left[\frac{|\xi_1|}{|\eta_2|}+1\right]\right)\mu_1\\
&\quad+\left(\frac{1}{\Gamma(\sigma)}\left[\frac{|\ell_1|\,|\ell_2|\,|\xi_1|\,|\xi_2|}{e\,|\eta_1|\,|\eta_2|}+\frac{|\ell_2|\,|\xi_1|}{|\eta_1|\,|\eta_2|}+\frac{|\xi_2|}{|\eta_1|}\right]\right.\\
&\quad\left.+\frac{1}{\Gamma(\sigma+1)}\left[\frac{|\ell_1|\,|\xi_1|}{|\eta_2|}\right]\right)\mu_2\\
&\le N_1\mu_1+N_2\mu_2,
\end{aligned}$$

which gives $\|\Pi_1(\omega,\varpi)\|\le N_1\mu_1+N_2\mu_2$. Analogously, we have $\|\Pi_2(\omega,\varpi)\|\le N_3\mu_1+N_4\mu_2$.

Thus, it follows from the above inequalities that the operator Π is uniformly bounded, since

$$\|\Pi(\omega,\varpi)\|\le[(N_1+N_3)\,\mu_1+(N_2+N_4)\,\mu_2]<1.$$

Next, we show that Π is equicontinuous, i.e. we prove that a bounded set $\mathbb{S}$ is mapped into an equicontinuous set of $W\times G$ by Π. Let $\vartheta_1,\vartheta_2\in[1,e]$ with $\vartheta_1<\vartheta_2$. Then we have

$$\begin{aligned}
&|\Pi_1(\omega(\vartheta_2),\varpi(\vartheta_2))-\Pi_1(\omega(\vartheta_1),\varpi(\vartheta_1))|\\
\le\ & \frac{1}{\Gamma(\rho)}\int_1^{\vartheta_2}(\log\frac{\vartheta_2}{s})^{\rho-1}\,|\hbar_1(s,\omega(s),\varpi(s))|\,\frac{ds}{s}\\
&+\frac{1}{\Gamma(\rho)}\int_1^{\vartheta_1}(\log\frac{\vartheta_1}{s})^{\rho-1}\,|\hbar_1(s,\omega(s),\varpi(s))|\,\frac{ds}{s}\\
\le\ & \mu_1\left\{\frac{1}{\Gamma(\rho)}\int_1^{\vartheta_1}\left[(\log\frac{\vartheta_2}{s})^{\rho-1}-(\log\frac{\vartheta_1}{s})^{\rho-1}\right]\frac{ds}{s}\right.\\
&\left.+\frac{1}{\Gamma(\rho)}\int_{\vartheta_1}^{\vartheta_2}(\log\frac{\vartheta_2}{s})^{\rho-1}\frac{ds}{s}\right\}\\
\le\ & \frac{\mu_1}{\Gamma(\rho+1)}\left\{2\,(\log\vartheta_2-\log\vartheta_1)^{\rho}+(\log\vartheta_2)^{\rho}-(\log\vartheta_1)^{\rho}\right\}.
\end{aligned}\tag{21}$$

In a similar way, we can easily get

$$\begin{aligned}
&|\Pi_2(\omega(\vartheta_2),\varpi(\vartheta_2))-\Pi_2(\omega(\vartheta_1),\varpi(\vartheta_1))|\\
\le\ & \frac{\mu_2}{\Gamma(\sigma+1)}\left\{2\,(\log\vartheta_2-\log\vartheta_1)^{\sigma}+(\log\vartheta_2)^{\sigma}-(\log\vartheta_1)^{\sigma}\right\}.
\end{aligned}\tag{22}$$

Since $\log(\vartheta)$ is uniformly continuous on $[1,e]$, the right-hand sides of the inequalities (21) and (22) tend to zero as $\vartheta_2\to\vartheta_1$. Therefore, the operator $\Pi(\omega,\varpi)$ is equicontinuous. The Arzela-Ascoli theorem along with the above steps shows that Π is completely continuous mapping.

Finally, we shall verify that the set

$$\mathbb{S}=\{(\omega,\varpi)\in W\times\ G:(\omega,\varpi)=\lambda\Pi(\omega,\varpi),0\le\lambda\le1\},$$

is bounded. Indeed, Let $(\omega,\varpi)\in\mathbb{S}$ with $(\omega,\varpi)=\lambda\Pi(\omega,\varpi)$. For any $\vartheta\in[1,e]$, we have

$$\omega(\vartheta)=\lambda\Pi_1(\omega,\varpi)(\vartheta),\qquad \varpi(\vartheta)=\lambda\Pi_2(\omega,\varpi)(\vartheta).$$

Then

$$\begin{aligned}
|\omega(\vartheta)| &< |\Pi_1(\omega,\varpi)(\vartheta)| \\
&\leq \left[\frac{|\xi_1|}{|\eta_2|}\right]\frac{1}{\Gamma(\rho)}\int_1^e (\log\frac{e}{s})^{\rho-1}|\hbar_1(s,\omega(s),\varpi(s))|\frac{ds}{s} \\
&+\left[\frac{|\ell_1||\xi_1|}{|\eta_2|}\right]\frac{1}{\Gamma(\sigma)}\int_1^e (\log\frac{e}{s})^{\sigma-1}|\hbar_2(s,\omega(s),\varpi(s))|\frac{ds}{s} \\
&+\left[\frac{|\ell_1||\xi_1||\xi_2|}{|\eta_1||\eta_2|}+\frac{|\ell_2||\xi_1||\xi_2|}{e|\eta_1||\eta_2|}+\frac{|\ell_2||\xi_2|}{e|\eta_1|}|\log\vartheta|\right] \\
&\times\left[\frac{1}{\Gamma(\rho-1)}\int_1^e (\log\frac{e}{s})^{\rho-2}|\hbar_1(s,\omega(s),\varpi(s))|\frac{ds}{s}\right] \\
&+\left[\frac{|\ell_1||\ell_2||\xi_1||\xi_2|}{e|\eta_1||\eta_2|}+\frac{|\ell_2||\xi_1|}{|\eta_1||\eta_2|}+\frac{|\xi_2|}{|\eta_1|}|\log\vartheta|\right] \\
&\times\left[\frac{1}{\Gamma(\sigma-1)}\int_1^e (\log\frac{e}{s})^{\sigma-2}|\hbar_2(s,\omega(s),\varpi(s))|\frac{ds}{s}\right] \\
&+\frac{1}{\Gamma(\rho)}\int_1^\vartheta (\log\frac{\vartheta}{s})^{\rho-1}|\hbar_1(s,\omega(s),\varpi(s))|\frac{ds}{s}.
\end{aligned}$$

Assumption (H_2) gives

$$\begin{aligned}
|\omega(\vartheta)| &\leq N_1(\delta_0+\delta_1|\omega|+\delta_2|\varpi|)+N_2(\beta_0+\beta_1|\omega|+\beta_2|\varpi|) \\
&= N_1\delta_0+N_2\beta_0+(N_1\delta_1+N_2\beta_1)|\omega|+(N_1\delta_2+N_2\beta_2)|\varpi|,
\end{aligned}$$

and

$$\begin{aligned}
|\varpi(\vartheta)| &\leq N_3(\delta_0+\delta_1|\omega|+\delta_2|\varpi|)+N_4(\beta_0+\beta_1|\omega|+\beta_2|\varpi|) \\
&\leq N_3\delta_0+N_4\beta_0+(N_3\delta_1+N_4\beta_1)|\omega|+(N_3\delta_2+N_4\beta_2)|\varpi|.
\end{aligned}$$

Hence

$$\|\omega\|\leq N_1\delta_0+N_2\beta_0+(N_1\delta_1+N_2\beta_1)\|\omega\|+(N_1\delta_2+N_2\beta_2)\|\varpi\|,$$

and

$$\|\varpi\|\leq N_3\delta_0+N_4\beta_0+(N_3\delta_1+N_4\beta_1)\|\omega\|+(N_3\delta_2+N_4\beta_2)\|\varpi\|,$$

which imply that

$$\begin{aligned}
\|\omega\|+\|\varpi\| \leq\ & (N_1+N_3)\delta_0+(N_2+N_4)\beta_0+ \\
& [(N_1+N_3)\delta_1+(N_2+N_4)\beta_1]\|\omega\| \\
& +[(N_1+N_3)\delta_2+(N_2+N_4)\beta_2]\|\varpi\|.
\end{aligned}$$

Consequently,

$$\|\omega+\varpi\|\leq\frac{(N_1+N_3)\delta_0+(N_2+N_4)\beta_0}{N_0},$$

where $N_0=\min\{1-(N_1+N_3)\delta_1+(N_2+N_4)\beta_1, 1-(N_1+N_3)\delta_2+(N_2+N_4)\beta_2\}$, which shows that the set $\mathbb{S}$ is bounded. Thus, as a consequence Theorem 2.6, the Hadamard coupled system (1)-(2) has at least one solution. The proof is complete. □

Example 3.3. *Let* $\rho=\sigma=\frac{4}{3}$, $\xi_1=\frac{3}{4}$, $\xi_2=\frac{1}{2}$, $\ell_1=\frac{3}{2}$, $\ell_2=\frac{1}{2}$,

$$\hbar_1(\vartheta,\omega,\varpi)=\frac{1}{16(\vartheta+1)^2}\frac{|\omega(\vartheta)|}{1+|\omega(\vartheta)|}+1+\frac{1}{128}\sin^2(\varpi(\vartheta))+\frac{1}{\sqrt{\vartheta^2+1}},$$

and

$$\hbar_2(\vartheta,\omega,\varpi)=\frac{1}{128\pi}\sin(2\pi\omega(\vartheta))+\frac{|\varpi(\vartheta)|}{64(1+|\varpi(\vartheta)|)}+\frac{1}{2}.$$

Consider the Hadamard-type coupled system

$$\begin{cases} \mathbb{D}_1^{\frac{4}{3}}\omega(\vartheta) = \hbar_1(\vartheta,\omega,\varpi), & \vartheta \in [1,e], \\ \mathbb{D}_1^{\frac{4}{3}}\varpi(\vartheta) = \hbar_2(\vartheta,\omega,\varpi), & \vartheta \in [1,e], \end{cases} \tag{23}$$

with the non-separated coupled boundary conditions

$$\begin{cases} \omega(1) = \frac{3}{4}\varpi(e) \\ \varpi(1) = \frac{3}{2}\omega(e) \end{cases}, \qquad \begin{cases} \omega'(1) = \frac{1}{2}\varpi'(e) \\ \varpi'(1) = \frac{1}{2}\omega'(e) \end{cases}. \tag{24}$$

For $\vartheta \in [1,e]$ and $\omega,\omega^*,\varpi,\varpi^* \in \mathbb{R}^+$, we have

$$|\hbar_1(\vartheta,\omega,\omega^*) - \hbar_1(\vartheta,\varpi,\varpi^*)| \leq \frac{1}{64}|\omega-\varpi| + \frac{1}{64}|\omega^*-\varpi^*|$$

and

$$|\hbar_2(\vartheta,\omega,\omega^*) - \hbar_2(\vartheta,\varpi,\varpi^*)| \leq \frac{1}{64}|\omega-\varpi| + \frac{1}{64}|\omega^*-\varpi^*|.$$

Hence the condition (H_1) holds with $\kappa_1 = \kappa_2 = \frac{1}{64}$, $\eta_1 = 0.966\,17 \neq 0$, $\eta_2 = -0.125 \neq 0$, $\sup_{\vartheta\in[1,e]}\hbar_1(\vartheta,0,0) = 1 + \frac{1}{\sqrt{2}} = \epsilon_1 < \infty$ and $\sup_{\vartheta\in[1,e]}\hbar_2(\vartheta,0,0) = \frac{1}{2} = \epsilon_2 < \infty$. We shall check that condition (18) holds. Indeed, by some simple calculations we find that $N_1 = 11.841$, $N_2 = 12.575$, $N_3 = 17.211$ and $N_4 = 17.52$ With the given data, we see that $\phi_1 = 0.924\,17 < 1$.

Therefore, by Theorem 3.1, we conclude that problem (23)-(24) has a unique solution.

4. conclusion:

In this article, we have established the existence and uniqueness results of a new type of Hadamard-type fractional differential equations with coupled non-separated boundary conditions. Our analysis is based on the reduction of convert differential equations to integral equations and using some fixed point theorems which are completely general and effective. We are confident the reported results here will have a favorable impact on the expansion of further applications in applied sciences and engineering.

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