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$\mathcal{F}_s-$contractive mappings in controlled metric type spaces

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Abstract

We investigate in this manuscript, we study a new type of mappings so called $\mathcal{F}_s-$contractive, in addition to we establish some fixed point results related to $\mathcal{F}_s-$contractive type mappings in controlled type metric spaces. Also, examples are provided to illustrate our results.

Keywords: Controlled metric type spaces, $\mathcal{F}_s$-contractive mappings, Fixed point
2020 MSC: 34B10; 34B15.

Banach in [1], proved the existence and uniqueness of a fixed point for a contractive self-mapping on a metric space, which was an inspiration to researchers around the world to generalize his result. That is due to the fact that the more general is the result, the more area it can be applied on such as an examples in computer sciences, differential equations, engineering. Some researchers generalize metric spaces by introduced an new extension to metric spaces such as partial metric spaces by assuming that the self-distance is not necessary zero. One of these extensions called $b-$metric spaces, which is basically changing the triangle inequality by multiplying the right hand side by a constant $s \geq 1$. Another approach to extend the result of Banach is to generalize the contraction principle, to get the necessary background on these extensions, we refer the reader to ([2], [3], [5], [3], [6], [7], [18], [19], [20], [21], [22], [23]). One of the these extensions was given by Wardowski in [8], where he presented a new kind of contraction so referred to $\mathcal{F}$-contraction. In this manuscript, we present improvement and generalization of some results on F-contraction in controlled type metric spaces which was introduced in 2018 by Mlaiki et. al. in [4].

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In the first section, we introduce some feasibility requirements. In the second section, We illustrate some goals along with consequences for $\mathcal{F}_s$-contractive mappings. In the third section, we introduce $\mathcal{F}_s$-expanding type mappings in controlled metric type spaces, along with fixed point results in such mappings.

1. Preliminary

First in this preliminary, we remind the reader of the definition of controlled metric type spaces.

Definition 1.1. *[4] Consider the set $X \neq \emptyset$ and $\theta : X \times X \to [1, \infty)$. If for all $x, y, z \in X$, the function $d : X \times X \to [0, \infty)$ satisfies the following:*
(d1) $d(x, y) = 0 \iff x = y$;
(d2) $d(x, y) = d(y, x)$;
(d3) $d(x, y) \leq \theta(x, z)d(x, z) + \theta(z, y)d(z, y)$,
then the pair (X, d) is referred a controlled type metric space.

Next, we give some examples of controlled metric type spaces.

Example 1.2. *[4] Assume that $X = \{1, 2, \cdots\}$. Define $d : X \times X \to [0, \infty)$ by*

$$d(x,y) = \begin{cases} 0, & \iff x = y \\ \frac{1}{x}, & \text{if } x = 2\kappa \text{ and } y = 2n+1 \\ \frac{1}{y}, & \text{if } x = 2n+1 \text{ and } y = 2\kappa \\ 1, & \text{otherwise.} \end{cases}$$

Suppose $\theta : X \times X \to [1, \infty)$ as

$$\theta(x,y) = \begin{cases} x, & \text{if } x = 2\kappa \text{ and } y = 2n+1 \\ y, & \text{if } x = 2n+1 \text{ and } y = 2\kappa \\ 1, & \text{otherwise.} \end{cases}$$

It is simple to see that (d1) and (d2) hold. To prove that (d3) maintains.
Case 1: If $z = x$ or $z = y$, (d3) holds.
Case 2: If $z \neq x$ and $z \neq y$, (d3) maintains when $x = y$. Now, suspect that $x \neq y$. Then we have $x \neq y \neq z$. It is not difficult to see that (d3) maintains for the proceeds subcases:

- $x = 2\kappa, z = 2n$ *and* $y = 2i + 1$;
- $x = 2\kappa$ *and* $y = 2n + 1, z = 2i + 1$;
- $x = 2n + 1, z = 2i + 1$ *and* $y = 2\kappa$;
- $x = 2n, y = 2\kappa, z = 2i$;
- $x = 2\kappa, y = 2n$ *and* $z = 2i + 1$;
- $x = 2n + 1, y = 2i + 1$ *and* $z = 2\kappa$;
- $x = 2n + 1, y = 2i + 1, z = 2\kappa + 1$,

where n, i, κ are natural numbers.

As a results, (X, d) is a controlled type metric space.

Example 1.3. *[4] Assume that* $X = \{0, 1, 2\}$*. Define* $d : X \times X \to [0, \infty)$ *as*

$$d(0,0) = d(1,1) = d(2,2) = 0,$$

and

$$d(0,1) = d(1,0) = 1, \quad d(0,2) = d(2,0) = \frac{1}{2}, \quad d(1,2) = d(2,1) = \frac{2}{5}.$$

Take $\theta : X \times X \to [1, \infty)$ *to be symmetric (i.e.,* $\theta(x,y) = \theta(y,x)$ *for all* $x, y \in (X)$ *and be defined by*

$$\theta(0,0) = \theta(1,1) = \theta(2,2) = \theta(0,2) = 1, \quad \theta(1,2) = \frac{5}{4}, \quad \theta(0,1) = \frac{11}{10}.$$

It is simple to see that (X, d) *is a controlled metric type space.*

Now, we remind the reader of the definition of Cauchy and convergent sequences in controlled metric type spaces.

Definition 1.4. *[4] let* (X, d) *be a controlled type metric space and a sequence* $\{x_n\}_{n \geq 0}$ *in* X*.*
(1) We say that the sequence $\{x_n\}$ *is convergent to* $x \in X$*, if for every* $\epsilon > 0$*, there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $d(x_n, x) < \epsilon$ *for others* $n \geq N$*. In this case, we write* $\lim_{n\to\infty} x_n = x$*.*
(2) We say that the sequence $\{x_n\}$ *is Cauchy, if for every* $\epsilon > 0$*, there exists* $N = N(\epsilon) \in \mathbb{N}$ *such as* $d(x_m, x_n) < \epsilon$ *for all* $m, n \geq N$*.*
(3) An controlled type metric space (X, d) *is said to be complete if every Cauchy sequence is convergent.*

Definition 1.5. *[4] Let that* (X, d) *be a controlled type metric space. Presumed that* $x \in X$ *and* $\varepsilon > 0$*.*
(i) The open ball $B(x, \varepsilon)$ *is*

$$B(x, \varepsilon) = \{y \in X, d(x, y) < \varepsilon\}.$$

(ii) The mapping $T : X \to X$ *is said to be continuous at* $x \in X$ *if for all* $\varepsilon > 0$*, there exists* $\delta > 0$ *such as* $T(B(x, \delta)) \subseteq B(Tx, \varepsilon)$*.*

Definition 1.6. *Consider the family* $\mathbb{F}$ *of maps* $\mathcal{F} : (0, \infty) \to \mathbb{R}$ *that satisfies the following four instances;*

($\mathcal{F}_1$) $\mathcal{F}(\alpha) < \mathcal{F}(\gamma)$ *if and only if* $\alpha < \gamma$*.*

($\mathcal{F}_2$) *For any sequence* $\{\gamma_n\}_{n\in N}$ *of positive numbers we have* γ_n *converges to* 0 *if and only if* $\lim_{n\to\infty} \mathcal{F}(\gamma_n) = -\infty$*.*

($\mathcal{F}_3$) *There exists* $0 < \kappa < 1$ *where* $\lim_{\gamma\to 0^+} \gamma^\kappa \mathcal{F}(\gamma) = 0$*.*

($\mathcal{F}_4$) *Let* $s \geq 1$ *be a real number. For each sequence* $\{\gamma_n\}_{n\in N}$ *of positive numbers such as*

$$\tau + \mathcal{F}(s\gamma_n) \leq \mathcal{F}(\gamma_{n-1}), \forall n \in N, \tau > 0,$$

then

$$\tau + \mathcal{F}(s^n\gamma_n) \leq \mathcal{F}(s^{n-1}\gamma_{n-1}), \forall n \in N, \tau > 0,$$

Example 1.7. *Consider the mappings from* $(0, \infty)$ *to* $\mathbb{R}$ *defined by:*

1. $\mathcal{F}_1(x) = \log x$,
2. $\mathcal{F}_2(x) = x + \log x$,
3. $\mathcal{F}_3(x) = \log(x^2 + x)$.

Note that, it is not difficult to see that $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \in \mathbb{F}$.

Definition 1.8. *For a controlled metric type space* (X, d)*, a mapping* $T : X \to X$ *is said to be an* $\mathcal{F}_s$*-contractive type mapping if there exists* $\mathcal{F} \in \mathbb{F}$*,* $\tau > 0$ *and* $s \geq 1$*, where* $d(x, Tx)d(y, Ty) \neq 0$ *infers*

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x, y)) + \mathcal{F}_s(d(x, Tx)) + \mathcal{F}_s(d(y, Ty))\} \tag{1}$$

and $d(x, Tx)d(y, Ty) = 0$ *infers*

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x, y)) + \mathcal{F}_s(d(x, Ty)) + \mathcal{F}_s(d(y, Tx))\} \tag{2}$$

for all $x, y \in X$.

2. Main results

Now, We present our main result.

Theorem 2.1. *Assume that* (X, d) *be a complete controlled type metric space and let* $T : X \to X$ *be an* $\mathcal{F}_s$*-contractive type mapping. Also, assume that there exists* $x_0 \in T$ *define the sequence* $\{x_n\}$ *by* $x_n = Tx_n, n \in N$ *such that for all natural numbers* n*, we have;*

$$\sup_{m \geq 1} \lim_{i \to \infty} \frac{\theta(x_{i+1}, x_{i+2})}{\theta(x_i, x_{i+1})} \theta(x_{i+1}, x_m) < s. \tag{3}$$

Also, assume for every $x \in X$*, we have reached*

$$\lim_{n \to \infty} \theta(x_n, x) \quad \text{and} \quad \lim_{n \to \infty} \theta(x, x_n) \quad \text{exist and are finite.} \tag{4}$$

Then T *has a unique fixed point.*

Proof. Assume that $x_0 \in X$ be the point satisfying the hypothesis of our theorem, and refer the sequence $\{x_n\}$ by $x_n = Tx_n, n \in N$. Denote $d(x_n, x_{n+1})$ by μ_n. We may assume that $\mu_n > 0$ for all $n \in N$. Otherwise, if there exists n such that $\mu_n > 0$, then $x_{n+1} = x_n$ and we are done because x_n is a fixed point of T. Since T is an $\mathcal{F}_s$-contractive type mapping and $Tx_n \neq x_n$ for all $n \in N$, We have reached

$$\mathcal{F}_s(s\mu_n) \leq \frac{1}{3}\{\mathcal{F}_s(d(x_{n-1}, x_n)) + \mathcal{F}_s(d(x_{n-1}, x_n)) + \mathcal{F}_s(d(x_n, x_{n+1}))\} - \tau.$$

Thus,

$$\mathcal{F}_s(s\mu_n) \leq \frac{1}{3}\{\mathcal{F}_s(d(x_{n-1}, x_n)) + \mathcal{F}_s(d(x_{n-1}, x_n)) + \mathcal{F}_s(d(x_n, x_{n+1}))\} - \tau.$$

Hence,

$$\mathcal{F}_s(s\mu_n) \leq \mathcal{F}_s(\mu_{n-1}) - \frac{3}{2}\tau$$

By condition $(\mathcal{F}_4)$, We have reached

$$\mathcal{F}_s(s^n\mu_n) \leq \mathcal{F}_s(s^{n-1}\mu_{n-1}) - \frac{3}{2}\tau.$$

Therefore, we can simply deduce the following;

$$\mathcal{F}_s(s^n\mu_n) \leq \mathcal{F}_s(s^{n-1}\mu_{n-1}) - \frac{3}{2}\tau \leq ... \leq \mathcal{F}_s(\mu_0) - \frac{3}{2}n \leq \mathcal{F}_s(\mu_0), \tag{5}$$

which suggests that

$$s^n\mu_n \leq \mu_0, \tag{6}$$

For all natural numbers $n < m$, We have reached

$$
\begin{aligned}
d(x_n, x_m) &\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \theta(x_{n+1}, x_m)d(x_{n+1}, x_m) \\
&\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \theta(x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&+ \theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)d(x_{n+2}, x_m) \\
&\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \theta(x_{n+1}, x_m)\theta(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\
&+ \theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)\theta(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&+ \theta(x_{n+1}, x_m)\theta(x_{n+2}, x_m)\theta(x_{n+3}, x_m)d(x_{n+3}, x_m) \\
&\leq \cdots \\
&\leq \theta(x_n, x_{n+1})d(x_n, x_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\theta(x_j, x_m)\right)\theta(x_i, x_{i+1})d(x_i, x_{i+1}) \\
&+ \prod_{\kappa=n+1}^{m-1}\theta(x_\kappa, x_m)d(x_{m-1}, x_m) \\
&\leq \theta(x_n, x_{n+1})\frac{1}{s^n}d(x_0, x_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\theta(x_j, x_m)\right)\theta(x_i, x_{i+1})\frac{1}{s^i}d(x_0, x_1) \\
&+ \prod_{i=n+1}^{m-1}\theta(x_i, x_m)\frac{1}{s^{m-1}}d(x_0, x_1) \\
&= \theta(x_n, x_{n+1})\frac{1}{s^n}d(x_0, x_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\theta(x_j, x_m)\right)\theta(x_i, x_{i+1})\frac{1}{s^i}d(x_0, x_1).
\end{aligned}
$$

Hence,

$$
d(x_n, x_m) \leq \theta(x_n, x_{n+1})\frac{1}{s^n}d(x_0, x_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\theta(x_j, x_m)\right)\theta(x_i, x_{i+1})\frac{1}{s^i}d(x_0, x_1).
$$

Now, Assume that

$$
S_p = \sum_{i=0}^{p}\left(\prod_{j=0}^{i}\theta(x_j, x_m)\right)\theta(x_i, x_{i+1})\frac{1}{s^i}.
$$

Hence, we have reached

$$
d(x_n, x_m) \leq d(x_0, x_1)\left[\frac{1}{s^n}\theta(x_n, x_{n+1}) + (S_{m-1} - S_n)\right]. \tag{7}
$$

By the ratio test and conditions 3, and 4, it not difficult to see that

$$
\lim_{n,m\to\infty} d(x_n, x_m) = 0.
$$

So, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete controlled metric type spaces, we deduce that converges $\{x_n\}$ to some $z \in X$, that is

$$
\lim_{n\to\infty} x_n = z.
$$

Also, using (1), we deduce that for all $n \in N$

$$
\tau + \mathcal{F}_s(sd(Tz, Tx_n)) \leq \frac{1}{3}\{\mathcal{F}_s(d(z, x_n)) + \mathcal{F}_s(d(z, Tz)) + \mathcal{F}_s(d(x_n, x_{n+1}))\}.
$$

Hence, as $n \to \infty$, and since $d(z, x_n) \to 0$ we deduce that

$$\tau + \lim_{n\to\infty} \mathcal{F}_s(sd(Tz, Tx_n)) \leq -\infty$$

this implies

$$\lim_{n\to\infty} d(Tz, x_{n+1}) = \lim_{n\to\infty} d(Tz, Tx_n) = 0.$$

Thus, $\{x_n\}$ converges to Tz. Therefore, by the uniqueness of the limit we conclude that

$$Tz = z.$$

Now, we may assume that T has more than one fixed point say z^* with $z \neq z^*$. Thus,

$$\tau + \mathcal{F}_s(sd(Tz, Tz^*)) \leq \frac{1}{3}\{\mathcal{F}_s(d(z, z^*)) + \mathcal{F}_s(d(z, Tz^*)) + \mathcal{F}_s(d(Tz, z^*))\}$$

or

$$\mathcal{F}_s(sd(z, z^*)) < \mathcal{F}_s(d(z, z^*)),$$

that is a contradiction. Therefore, The fixed point is unique as desired. □

The following example is an application of Theorem 2.1.

Example 2.2. *Assume that* $X = [0, 1] \cup [2, \infty)$. *Define* $d : X \times X \to [0, \infty)$ *by*

$$d(x, y) = \begin{cases} 0, & \textit{if and only if } x = y, \\ \min\{x + y, 2\}, & \textit{if } x \neq y \ . \end{cases}$$

Consider $\theta : X \times X \to [1, \infty)$ *as*

$$\theta(x, y) = \begin{cases} x, & \textit{if } x = 2\kappa \textit{ and } y = 2n + 1 \\ y, & \textit{if } x = 2n + 1 \textit{ and } y = 2\kappa \\ 1, & \textit{otherwise.} \end{cases}$$

Note that, (X, d) *is complete controlled metric type space. Define the mapping* $T : X \to X$ *as follows;*

$$Tx = \begin{cases} \frac{1}{2}, & \textit{if } 0 \leq x < 1, \\ 0, & \textit{if } x = 1, \\ \frac{1}{2} - \frac{1}{x}, & \textit{if } x \geq 2, \end{cases}$$

It is simple to see that T *is an* $\mathcal{F}_s-$*contractive mapping with* $\mathcal{F}(x) = \log x$, $\tau = \frac{2\ln 3}{3}$ *and* $s = 1$, *which also satisfies all hypothesis of Theorem 2.1. Thus,* T *has a unique fixed point that is* $x = \frac{1}{2}$.

Corollary 2.3. *Consider* (X, d) *to be a complete controlled type metric space and* $T : X \to X$ *be a mapping such that, for some* $\tau > 0, d(x, Tx)d(y, Ty) \neq 0$ *implies*

$$\tau + \mathcal{F}_s(sd(T^n x, T^n y)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x, y)) + \mathcal{F}_s(d(x, T^n x)) + \mathcal{F}_s(d(y, T^n y))\}$$

and $d(x, Tx)d(y, Ty) = 0$ *infers*

$$\tau + \mathcal{F}_s(sd(T^n x, T^n y)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x, y)) + \mathcal{F}_s(d(x, T^n y)) + \mathcal{F}_s(d(y, T^n x))\}$$

for some natural number n. *Then* T *has a unique fixed point.*

Proof. Consider the map $S = T^n$, it not difficult to see that by Theorem 2.1, S has a unique fixed point, say w, that is $T^n w = Sw = w$. Since $T^{n+1}w = Tw$,

$$STw = T^n(Tw) = T^{n+1}w = Tw,$$

Now, since the fixed point of S is unique, we deduce that $Tw = w$. □

Theorem 2.4. *For a controlled type metric space (X, d), assume that for any closed subset Y of X any $\mathcal{F}_s$-contractive type mapping T on Y has a fixed point, then X is complete.*

Proof. Assume that $\{x_n\}$ be a Cauchy sequence in X. Assume that $\{x_n\}$ does not have any convergent subsequence. Thus,

$$\beta(x_n) := \inf\{d(x_n, x_m) : m > n\} > 0, \forall n \in N.$$

Note that $\beta(x_n) \leq \beta(x_m)$ for $m \geq n$. For a given γ with $0 < \gamma < 1$, we construct inductively a subsequence $\{x_{n_\kappa}\}$ such that

$$sd(x_i, x_j) < \gamma\beta(x_{n_{\kappa-1}}), \forall i, j \geq n_\kappa.$$

Then $Y = \{x_{n_\kappa} : \kappa \in N\}$ is a closed subset of X. Define $T : Y \to Y$ by

$$Tx_{n_\kappa} = x_{n_{\kappa+1}} \forall \kappa \in N$$

Then it is clear that T is fixed point free. Now,

$$sd(Tx_{n_\kappa}, Tx_{n_{\kappa+i}}) = d(x_{n_{\kappa+1}}, x_{n_{\kappa+i+1}}) < \gamma\beta(x_{n_\kappa})$$

By definition,

$$\begin{aligned} \beta(x_{n_\kappa}) &\leq d(x_{n_\kappa}, x_{n_{\kappa+i}}) = d(x, y) \\ &\leq d(x_{n_\kappa}, x_{n_{\kappa+1}}) = d(x, Tx) \\ &\leq \beta(x_{n_{\kappa+i}}) = d(y, Ty). \end{aligned}$$

Thus, we can easily conclude that

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x, y)) + \mathcal{F}_s(d(x, Tx)) + \mathcal{F}_s(d(y, Ty))\}.$$

where $\tau > 0$, which leads us to a contradiction. □

Now, we define Kannan $\mathcal{F}_s$-contractive type mappings and prove some fixed point results for the same in a controlled type metric space.

Definition 2.5. *Assume that (X, d) be a controlled type metric space. A mapping $T : X \to X$ is said to be a Kannan $\mathcal{F}_s$-contractive type mapping if there exists $\tau > 0$ and $s \geq 1$ such that $d(x, Tx)d(y, Ty) \neq 0$ infers*

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \frac{1}{2}\{\mathcal{F}_s(d(x, Tx)) + \mathcal{F}_s(d(y, Ty))\} \tag{8}$$

and $d(x, Tx)d(y, Ty) = 0$ infers

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \frac{1}{2}\{\mathcal{F}_s(d(x, Ty)) + \mathcal{F}_s(d(y, Tx))\} \tag{9}$$

for all $x, y \in X$.

Theorem 2.6. *Assume that (X, d) be a complete controlled type metric space and let $T : X \to X$ be a Kannan $\mathcal{F}_s$-contractive type mapping. Then T has a unique fixed point.*

Proof. The goal follows, following the proof of Theorem 2.1. □

Definition 2.7. *We say that a self mapping on a controlled matric type space T is an asymptotically regular mapping, if*

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0 \text{ forall } x \in X.$$

Theorem 2.8. *Assume that (X, d) be a complete controlled type metric space and $T : X \to X$ be an asymptotically regular mapping such that, for some $\tau > 0$, $d(x, Tx)d(y, Ty) \neq 0$ implies*

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \mathcal{F}_s(d(x, Tx)) + \mathcal{F}_s(d(y, Ty) \tag{10}$$

and $d(x, Tx)d(y, Ty) = 0$ infers

$$\tau + \mathcal{F}_s(sd(Tx, Ty)) \leq \mathcal{F}_s(d(x, Ty)) + \mathcal{F}_s(d(y, Tx) \tag{11}$$

assume that there exists $x_0 \in T$ such that for all natural numbers n, we have reached;

$$\sup_{m\geq 1} \lim_{i\to\infty} \frac{\theta(x_{i+1}, x_{i+2})}{\theta(x_i, x_{i+1})}\theta(x_{i+1}, x_m) < s. \tag{12}$$

Also, assume for every $x \in X$, we have reached

$$\lim_{n\to\infty} \theta(x_n, x) \text{ and } \lim_{n\to\infty} \theta(x, x_n) \text{ exist and are finite.} \tag{13}$$

for all $x, y \in X$. Then T has a fixed point $z \in X$.

Proof. Assume that $x_0 \in X$ be an arbitrary point (but fixed) and consider the sequence x_n, where $x_n = T^n x_0, n \in N$. Assume that$d(x_n, x_{n+1}) = \mu_n$ and suppose that $\mu_n > 0$ for all $n \in N$. Since T is asymptotically regular, we have reached

$$\lim_{n\to\infty} \mu_n = 0. \tag{14}$$

Now, since $Tx_n \neq x_n$ for all $n \in N$, we have for $n < m \in N$,

$$\begin{aligned}\tau + \mathcal{F}_s(sd(x_n, x_m)) &\leq \mathcal{F}_s(d(T^{n-1}x_0, T^n x_0)) + \mathcal{F}_s(d(T^{m-1}x_0, T^m x_0)) \\ &= \mathcal{F}_s(\mu_{n-1}) + \mathcal{F}_s(\mu_{m-1}).\end{aligned}$$

Now, by (14) we can simply deduce that;

$$\lim_{n\to\infty} \mathcal{F}_s(sd(x_n, x_m)) = -\infty.$$

Hence, by condition $(\mathcal{F}_2)$ we have reached;

$$\lim_{n\to\infty} d(x_n, x_m) = 0,$$

Therefore, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete controlled metric type spaces, we deduce that converges $\{x_n\}$ to some $z \in X$, that is

$$\lim_{n\to\infty} x_n = z,$$

that is $\lim_{n\to\infty} d(x_n, z) = 0$. Also, we have for all $n \in N$

$$\tau + \mathcal{F}_s(sd(Tz, Tx_n)) \leq \mathcal{F}_s((d(z, Tz)) + \mathcal{F}_s(d(x_n, Tx_n)).$$

Hence,

$$\tau + \lim_{n\to\infty} \mathcal{F}_s(sd(Tz, Tx_n)) \leq -\infty.$$

that is, $\lim_{n\to\infty} d(Tz, x_{n+1}) = 0$.
Since the convergent sequence $\{x_n\}$ converges to both z and Tz. Therefore, by the uniqueness of the limit we have $Tz = z$. □

3. $\mathcal{F}$-expanding type mappings

In this section, we define new kinds of $\mathcal{F}_s$-expanding mapping and we prove a fixed point goals in controlled type metric spaces.

Definition 3.1. *A mapping $T : X \to X$ is said to be an $\mathcal{F}_s$-expanding type mapping if there exists $t > 0$ such that $d(x,Tx)d(y,Ty) \neq 0$ infers*

$$t + \mathcal{F}_s(sd(x,y)) \leq \frac{1}{3}\{\mathcal{F}_s(d(Tx,Ty)) + \mathcal{F}_s(x,Tx) + \mathcal{F}_s(d(y,Ty))\} \tag{15}$$

and $d(x,Tx)d(y,Ty) = 0$ infers

$$t + \mathcal{F}_s(sd(x,y)) \leq \frac{1}{3}\{\mathcal{F}_s(d(Tx,Ty)) + \mathcal{F}_s(x,Ty) + F_s(d(y,Tx))\} \tag{16}$$

for all $x, y \in X$.

Next, we remind the reader of the following well knowning lemma.

Lemma 3.2. *[16] Assume that T be a surjective, self-mapping on a controlled type metric space (X,d). Then there exists a mapping $T^* : X \to X$ such that $T \circ T^*$ is the identity map on X.*

In the next theorem We prove the existence and uniqueness of a fixed point for $\mathcal{F}_s$-expanding type mappings in controlled metric type spaces.

Theorem 3.3. *Assume that T be a surjective, self-mapping on a controlled type metric space (X,d) as a result T is additionally an $\mathcal{F}_s$-expanding type mapping. Then T has a unique fixed point $z \in X$.*

Proof. Lemma (3.2) implies that there exists a self-mapping mapping T^* on X such that $T \circ T^*$ is the identity map on X. Take any arbitrary points$x, y \in X$ such that $x \neq y$, and define $u = T^*x$ and $v = T^*y$. It is obvious that $u \neq v$. Applying (15) on u and v, we have, for $d(u,Tu)d(v,Tv) \neq 0$,

$$\tau + \mathcal{F}_s(sd(u,v)) \leq \frac{1}{3}\{\mathcal{F}_s(d(Tu,Tv)) + \mathcal{F}_s(u,Tu) + \mathcal{F}_s(d(v,Tv))\}.$$

and, for $d(x,Tx)d(y,Ty) = 0$,

$$\tau + \mathcal{F}_s(sd(u,v)) \leq \frac{1}{3}\{\mathcal{F}_s(d(Tu,Tv)) + \mathcal{F}_s(u,Tv) + \mathcal{F}_s(d(v,Tu))\}.$$

Since $Tu = T(T^*(x)) = x$ and $Tv = T(T^*y) = y$, we get

$$\tau + \mathcal{F}_s(sd(T^*x,T^*y)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x,y)) + \mathcal{F}_s(x,T^*x) + \mathcal{F}_s(d(y,T^*y))\}.$$

for $d(x,Tx)d(y,Ty) \neq 0$ and

$$\tau + \mathcal{F}_s(sd(T^*x,T^*y)) \leq \frac{1}{3}\{\mathcal{F}_s(d(x,y)) + \mathcal{F}_s(x,T^*y) + \mathcal{F}_s(d(y,T^*x))\}.$$

for $d(x,Tx)d(y,Ty) = 0$, showing that T^* is an $\mathcal{F}_s$-contractive type mapping. By Theorem (2.1), T^* has a unique fixed point $z \in X$ and for every $x_0 \in X$ the sequence $\{T^{*n}x_0\}$ converges to z. In particular, z is also a fixed point of T since $T^*z = z$ reveals that
$Tz = T(T^*z) = z$.
At long last, if $w = Tw$ is another fixed point, then from (16).

$$\tau + \mathcal{F}_s(sd(z,w)) \leq \frac{1}{3}\{\mathcal{F}_s((d(Tz,Tw)) + \mathcal{F}_s(d(z,Tw)) + \mathcal{F}_s(d(w,Tz))\}.$$

or

$$\tau + \frac{2}{3}\mathcal{F}_s(sd(z,w)) \leq \frac{2}{3}\mathcal{F}_s((d(z,w)).$$

which is impossible. Additionally, the fixed point of T is unique. □

4. conclusion

In closing, we would like to present the following questions;

Question Under what conditions an $\mathcal{F}_s-$contractive mapping in *double controlled metric type space* has a unique fixed point?

Question Under what conditions an $\mathcal{F}_s-$expanding mapping in *double controlled metric type space* has a unique fixed point?

Note that, *double controlled metric type space* was introduced in 2018 by Abdeljawad et. al in [17].

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fund Math. 3, 133-181 (1922).

[2] J. Jachymski, I. Jóźwik, On Kirk's asymptotic contractions. J Math Anal Appl. 300, 147-159 (2004). doi:10.1016/j.jmaa.2004.06.037.

[3] T. Suzuki, Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces, Non-linear Anal. 64, 971-978 (2006).

[4] N. Mlaiki, H. Aydi, N. Souayah and T. Abdeljawad, Controlled metric type spaces and the related contraction principle, Mathematics, 6, 194, 2018.

[5] A. Meir, E. Keeler, A theorem on contraction mappings. J Math Anal Appl. 28, 326-329 (1969). doi:10.1016/0022-247X(69)90031-6.

[6] T. Abdeljawad, Fixed points for generalized weakly contractive mappings in partial metric spaces. Math Comput Modelling. 54, 2923-2927 (2011). doi:10.1016/j.mcm.2011.07.013.

[7] Choudhury, Binayak, S, Konar, P, Rhoades, BE, Metiya, N: Fixed point theorems for generalized weakly contractive mappings. Nonlinear Anal. 74, 2116-2126 (2011). doi:10.1016/j.na.2010.11.017.

[8] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. 2012, 94 (2012) https://doi.org/10.1186/1687-1812-2012-94.

[9] A. Lukács, S. Kajántó, Fixed point theorems for various types of F-contractions in complete b-metric spaces. Fixed Point Theory 19(1), 321–334 (2018). https://doi.org/10.24193/fpt-ro.2018.1.25.

[10] S. Cobzas, Fixed points and completeness in metric and in generalized metric spaces (2016). arXiv:1508.05173v4 [math.FA]

[11] T.K. Hu, On a fixed-point theorem for metric spaces. Am. Math. Mon. 74, 436-437 (1967).

[12] H. Garai, T. Senapati, L.K. Dey, A study on Kannan type contractive mappings (2017). arXiv:1707.06383v1 [math.FA].

[13] F.E. Browder, W.V. Petryshyn, The solution by iteration of non-linear functional equations in Banach spaces. Bull. Am. Math. Soc. 72, 571-575 (1966).

[14] J.B. Baillon, R.E. Bruck, S. Reich, On the asymptotic behaviour of non-expansive mappings and semi-groups in Banach spaces. Houst. J. Math. 4, 1-9 (1978).

[15] R.E. Bruck, S. Reich, Non-expansive projections and resolvents of accretive operators in Banach spaces. Houst. J. Math. 3, 459-470 (1977).

[16] J. Górnicki, Fixed point theorems for F-expanding mappings. Fixed Point Theory Appl. 2017, 9 (2017). https://doi.org/10.1186/s13663-017-0602-3.

[17] T. Abdeljawad, N. Mlaiki, H. Aydi, and N. Souayah, Double Controlled Metric Type Spaces and Some Fixed Point Results, Mathematics 2018, **6**, 320; doi:10.3390/math6120320

[18] E. Karapınar, S. Czerwik, H. Aydi, (α, ψ)-Meir-Keeler contraction mappings in generalized b-metric spaces, Journal of Function spaces, Volume 2018 (2018), Article ID 3264620, 4 pages.

[19] H. Afshari, H. Aydi, E. Karapınar, On generalized $\alpha - \psi$-Geraghty contractions on b-metric spaces, Georgian Math. J. 27 (2020), 9-21

[20] E. Karapınar, A. Petruşel, and G.Petruşel, On admissible hybrid Geraghty contractions, Carpathian J. Math. 36 (2020), No. 3, 433 - 442.

[21] H. Aydi, M. F. Bota, E. Karapınar, S. Mitrovic, A fixed point theorem for set-valued quasi-contractions in b-metric spaces, Fixed Point Theory Appl. 2012, 2012 :88.

[22] H. Aydi, M.F. Bota, E. Karapınar, S. Moradi, A common fixed point for weak phi-contractions on b-metric spaces, Fixed Point Theory, 13 (2) (2012), 337-346.

[23] M.A. Alghamdi, S. Gulyaz-Ozyurt and E. Karapınar, A Note on Extended $Z-$Contraction, Mathematics, Volume 8 Issue 2 Article Number 195 (2020).