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Dynamics and Expression of Solution of a Sixth Order Difference Equation

Abdul Khaliq[1]

[1]*Department of Mathematics, Riphah Institute of Computing and Applied Sciences, Riphah International University, Lahore Campus, Pakistan*

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Abstract

This paper deals with the solution behavior and periodic nature of the solutions of the difference equation

$$s_{n+1} = \alpha s_n + \frac{\beta s_n s_{n-4}}{\gamma s_{n-4} + \delta s_{n-5}}, \quad n = 0, 1, \ldots \tag{0.1}$$

where the initial conditions $s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary positive real numbers and $\alpha,\ \beta,\ \gamma,\ \delta$ are positive constants. Also we obtain the closed form of the solutions of some special cases of this equation.

1. Introduction

This paper deals with the solution behaviour of the difference equation

$$s_{n+1} = \alpha s_n + \frac{\beta s_n s_{n-4}}{\gamma s_{n-4} + \delta s_{n-5}}, \quad n = 0, 1, .. \tag{1.1}$$

where the initial conditions $s_{-5},\ s_{-4},\ s_{-3},\ s_{-2,}\ s_{-1},\ s_0$, are arbitrary positive real numbers and $\alpha,\ \beta,\ \gamma,\ \delta$ are positive constants. Also we obtain the form of solution of some special cases of this equation.
Various biological systems naturally leads to their study by means of a discrete variable. Appropriate examples include population dynamics and medicine. Some fundamental models of biological phenomena, including harvesting of fish, a single species population model, ventilation volume and blood CO2 levels, the production of red blood cells, a simple epidemics model, and a model of waves of disease that can be analyzed by difference equations are shown in [1]. Newly, there has been interest in so-called dynamical diseases, which correspond to physiological disorders for which a generally stable control system becomes unstable. One of the first papers on this subject was that of Mackey and Glass [2]. In which they investigated a first-order difference-delay equation that models the concentration of blood-level CO2. They also discussed models of a second class of diseases associated with the production of red cells, white cells, and platelets in the bone marrow. The dynamical characteristics of population system have been modeled, among others by differential equations in the case of species with overlapping generations and by difference equations in the case of species with non-overlapping generations. In process, one can developed a discrete model directly from observations and experiments. Periodically, for numerical purposes, one wants to propose a finite-difference scheme to numerically solved a given differential equation model, especially when the differential equation cannot be solved explicitly. For a given differential equation, a difference equation approximation would be most acceptable if the solution of the difference equation is the same as the differential equation at the discrete points [3]. But unless we can explicitly solve both equations, it is impossible to satisfy this requirements. Most of the time, it is fascinating that a differential equation, when extracted from a difference equation, marmalade the dynamical features of the corresponding continuous-time model such as equilibria, their local and global stability characteristics, and bifurcation behaviors. If alike discrete models can be derived from continuous time models, and it will preserve the considered realities, such discrete-time models can be called ‘dynamically consistent’ with the continuous-time models.

Email address and ORCID number: **khaliqsyed@gmail.com, 0000-0001-8802-9200**

The study of oscillatory and asymptotic stability properties of solution behavior of difference equations is extremely advantageous in the behavior of various biological system and other applications. This is because difference equations are relevant models for expressing situations where the variable is assumed to take only a discrete set of values and they appear frequently in the formulation and analysis of discrete time systems, in the study of biological systems, the study of deterministic chaos, the numerical integration of differential equations by finite difference schemes and so on. Difference equations are good models for describing situations where population growth is not continuous but seasonal with overlapping generations. "For example, the difference equation

$$\omega_{n+1}=\omega_n e^{\left[r(1-\frac{\phi_n}{k})\right]}$$

has been expressed to model different animal populations.
The generalized Beverton-Holt stock recruitment model has been investigated in [4, 5].

$$z_{n+1}=Az_n+\frac{Bz_{n-1}}{1+Cz_{n-1}+Dz_n}.$$

Several other researchers have studied the behavior of the solution of difference equations for example in [6] E. M. Elsayed investigated the solution of the following non-linear difference equation.

$$w_{n+1}=aw_n+\frac{bw_n^2}{cw_n+dw_{n-1}}.$$

Elabbasy et al. [7] studied the boundedness, global stability, periodicity character and gave the solution of some special cases of the difference equation.

$$y_{n+1}=\frac{Ay_{n-l}+By_{n-k}}{\alpha y_{n-l}+\beta y_{n-k}}.$$

Keratas et al. [8] gave the solution of the following difference equation

$$\ell_{n+1}=\frac{\ell_{n-5}}{1+\ell_{n-2}\ell_{n-5}}.$$

Elabbasy et al. [9] investigated the global stability, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1}=\frac{ax_{n-l}x_{n-k}}{bx_{n-p}+cx_{n-q}}.$$

Yalçınkaya et al. [10] has studied the following difference equation

$$x_{n+1}=\alpha+\frac{x_{n-m}}{x_n^k}.$$

Saleh et. al. [11] study the solution of difference

$$y_{n+1}=A+\frac{y_n}{y_{n-k}}.$$

Elsayed et al. [12] studied the global behavior of rational recursive sequence

$$x_{n+1}=ax_{n-l}+\frac{bx_{n-k}+cx_{n-s}}{d+ex_{n-t}}$$

As a matter of fact, numerous papers negotiate with the problem of solving nonlinear difference equations in any way possible, see, for instance [4]-[6], [13]-[18]. The long-term behavior and solutions of rational difference equations of order greater than one has been extensively studied during the last decade. For example, various results about periodicity, boundedness, stability, and closed form solution of the second-order rational difference equations have been investigated see [12]-[15], [19]-[25]. Other related work on rational difference equations see in refs. [26]-[28].
Here, we recall some basic definitions and some theorems that we need in the sequel.

2. Basic definitions

Let I be some interval of real numbers and let

$$F:I^{k+1}\rightarrow I,$$

be a continuously differentiable function. Then for every set of initial conditions $s_{-k},\ s_{-k+1},...,s_0\in I$, the difference equation

$$s_{n+1}=F(s_n,s_{n-1},...,s_{n-k}),\ \ n=0,1,..., \tag{2.1}$$

has a unique solution $\{s_n\}_{n=-k}^{\infty}$.
A point $\overline{s}\in I$ is called an equilibrium point of 2.1 if

$$\overline{s}=F(\overline{s},\overline{s},...,\overline{s}).$$

That is, $s_n=\overline{s}$ for $n\geq 0$, is a solution of 2.1 or equivalently $\overline{s}$ is a fixed point of F.

Definition 2.1. *(Periodicity) A Sequence* $\{s_n\}_{n=-k}^{\infty}$ *is said to be periodic with period p if* $s_{n+p} = s_n$ *for all* $n \geq -k$.

Definition 2.2. *(Fibonacci Sequence). The sequence* $\{F_m\}_{m=1}^{\infty} = \{1,2,3,5,8,13,....\}$ *i.e.* $F_m = F_{m-1} + F_{m-2} \geq 0,\ F_{-2} = 0,\ F_{-1} = 1$ *is called Fibonacci Sequence.*

Definition 2.3. *(Stability) (i) The equilibrium point* $\bar{s}$ *of Eq.(1.2) is locally stable if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for all* $s_{-k}, s_{-k+1}, ..., s_{-1}, s_0 \in I$ *with*

$$|s_{-k} - \bar{s}| + |s_{-k+1} - \bar{s}| + ... + |s_0 - \bar{s}| < \delta,$$

we have

$$|s_n - \bar{s}| < \varepsilon \quad \text{for all} \quad n \geq -k.$$

(ii) The equilibrium point $\bar{s}$ *of 2.1 is locally asymptotically stable if* $\bar{s}$ *is locally stable solution of Eq.(1.2) and there exists* $\gamma > 0$, *such that for all* $s_{-k}, s_{-k+1}, ..., s_{-1},\ s_0 \in I$ *with*

$$|s_{-k} - \bar{s}| + |s_{-k+1} - \bar{s}| + ... + |s_0 - \bar{s}| < \gamma,$$

we have

$$\lim_{n\to\infty} s_n = \bar{s}.$$

(iii) The equilibrium point $\bar{s}$ *of 2.1 is global attractor if for all* $s_{-k}, s_{-k+1}, ..., s_{-1},\ s_0 \in I$, *we have*

$$\lim_{n\to\infty} s_n = \bar{s}.$$

(iv) The equilibrium point $\bar{s}$ *of 2.1 is globally asymptotically stable if* $\bar{s}$ *is locally stable, and* $\bar{s}$ *is also a global attractor of 2.1.*
(v) The equilibrium point $\bar{s}$ *of 2.1 is unstable if* $\bar{s}$ *is not locally stable.*
(vi) The linearized equation of 2.1 about the equilibrium $\bar{s}$ *is the linear difference equation*

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial F(\bar{s}, \bar{s}, ..., \bar{s})}{\partial s_{n-i}} y_{n-i}.$$

Theorem 2.4. *[2] Assume that* $p, q \in R$ *and* $k \in \{0,1,2,...\}$. *Then*

$$|p| + |q| < 1,$$

is a sufficient condition for the asymptotic stability of the difference equation

$$s_{n+1} + ps_n + qs_{n-k} = 0, \quad n = 0,1,... .$$

Remark 2.5. *Theorem 2.4 can be easily extended to a general linear equation of the form*

$$s_{n+k} + p_1 s_{n+k-1} + ... + p_k s_n = 0, \qquad n = 0,1,... \tag{2.2}$$

where $p_1,\ p_2, ..., p_k \in R$ and $k \in \{1,2,...\}$. Then 2.2 is asymptotically stable provided that

$$\sum_{i=1}^{k} |p_i| < 1.$$

Consider the following equation

$$s_{n+1} = g(s_n,\ s_{n-1}, s_{n-2}). \tag{2.3}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem 2.6. *[1] Let* $[\alpha, \beta]$ *be an interval of real numbers and assume that*

$$g : [\alpha, \beta]^3 \to [\alpha, \beta],$$

is a continuous function satisfying the following properties:
(a) $g(x,y,z)$ *is non-decreasing in x and* $y \in [\alpha, \beta]$ *for each fixed* $z \in [\alpha, \beta]$.*and* $g(x,y,z)$ *is non-increasing in* $z \in [\alpha, \beta]$ *for each fixed x and* $y \in [\alpha, \beta]$
(b) If $(\lambda, \mu) \in [\alpha, \beta] \times [\alpha, \beta]$ *is a solution of the system*

$$\mu = g(\mu,\ \mu,\ \lambda) \qquad \text{and} \qquad \lambda = g(\lambda,\ \lambda,\ \mu),$$

then $\mu = \lambda$,
and 2.3 has a unique equilibrium $\bar{s} \in [\alpha, \beta]$ *and every solution of 2.3 converges to* $\bar{s}$."

3. Main results

3.1. Local stability of the equilibrium point of equation 1.1

This section elaborates the equilibrium point is local stable. Following relation shows the equilibrium points of 1.1

$$\bar{s} = \alpha\bar{s} + \frac{\beta\bar{s}^2}{\gamma\bar{s} + \delta\bar{s}}.$$

or

$$\bar{s}^2(1-\alpha)(\gamma+\delta) = \beta\bar{s}^2$$

If $(1-\alpha)(\gamma+\delta) \neq \beta$, then the unique equilibrium point is $\bar{s} = 0$
Let $f : (0,\infty)^3 \longrightarrow (0,\infty)$ be a continuously differentiable function defined by

$$f(\xi,\eta,\omega) = \alpha\xi + \frac{\beta\xi\eta}{\gamma\eta + \delta\omega}.$$

Therefore at $\bar{s} = 0$

$$\left(\frac{\partial f}{\partial \xi}\right)_{\bar{s}} = \alpha + \frac{\beta v}{(\gamma\eta + \delta\omega)}, \quad \left(\frac{\partial f}{\partial \eta}\right)_{\bar{s}} = \frac{\beta\delta\xi w}{(\gamma\eta + \delta\omega)^2}, \quad \left(\frac{\partial f}{\partial \omega}\right)_{\bar{s}} = \frac{-\beta\delta\xi\eta}{(\gamma\eta + \delta\omega)^2}$$

Then the linearized equation of 1.1 about $\bar{s}$ is

$$y_{n+1} - \left(\alpha + \frac{\beta v}{(\gamma\eta + \delta\omega)}\right)y_n + \left(\frac{\beta\delta u\omega}{(\gamma\eta + \delta\omega)^2}\right)y_{n-1} + \left(\frac{-\beta\delta\xi\eta}{(\gamma\eta + \delta\omega)^2}\right)y_{n-2} = 0. \tag{3.2}$$

Theorem 3.1. *The equilibrium point $\bar{s} = 0$ of (1) is locally asymptotically stable if* $\beta(\gamma+3\delta) < (\gamma+\delta)^2(1-\alpha), \quad \alpha < 1.$

Proof. It is follows by Theorem A that 3.2 is asymptotically stable if

$$\left|\alpha + \frac{\beta}{(\gamma+\delta)}\right| + \left|\frac{\beta\delta}{(\gamma+\delta)^2}\right| + \left|\frac{-\beta\delta}{(\gamma+\delta)^2}\right| < 1,$$

or

$$\alpha + \frac{\beta\gamma + 3\beta\delta}{(\gamma+\delta)^2} < 1$$

and so

$$\beta(\gamma+3\delta) < (\gamma+\delta)^2(1-\alpha).$$

Which completes the proof. □

3.2. Global attractivity of the equilibrium point of equation 1.1

This section investigate the global attractivity character of solutions of 1.1.

Theorem 3.2. *The equilibrium point $\bar{s}$ of 1.1 is global attractor. if*

$$\gamma(1-\alpha) \neq \beta$$

Proof. Let α, β are real numbers and assume that $g : [\alpha,\beta]^3 \to [\alpha,\beta]$, be a function defined by $g(u,v,w) = \alpha u + \frac{\beta uv}{\gamma v + \delta w}$ then we can easily see that the function $g(u,v,w)$ is increasing in u, v and decreasing in w.
Suppose that (λ, μ) is a solution of the system

$$\mu = g(\mu, \mu, \lambda) \quad and \quad \lambda = g(\lambda,\lambda, \mu).$$

Then from 1.1 we see that

$$\mu = \alpha\mu + \frac{\beta\mu^2}{\gamma\mu + \delta\lambda}, \quad \lambda = \alpha\lambda + \frac{\beta\lambda^2}{\gamma\lambda + \delta\mu},$$

Therefore,

$$\mu(1-\alpha) = \frac{\beta\mu^2}{\gamma\mu + \delta\lambda}, \quad \lambda(1-\alpha) = \frac{\beta\lambda^2}{\gamma\lambda + \delta\mu},$$

or

$$\beta\mu^2 = \gamma(1-\alpha)\mu^2 + \delta(1-\alpha)\mu\lambda \qquad and \beta\lambda^2 = \gamma(1-\alpha)\lambda^2 + \delta(1-\alpha)\mu\lambda,$$

subtracting

$$\gamma(1-\alpha)(\mu^2 - \lambda^2) = \beta(\mu^2 - \lambda^2), \qquad \gamma(1-\alpha) \neq \beta.$$

Thus

$$\mu = \lambda.$$

It follows by the Theorem B that $\bar{x}$ is a global attractor of 1.1 and then the proof is complete. □

4. Boundedness of solutions of 1.1

In this section we study the boundedness of solution of 1.1.

Theorem 4.1. *Every solution of 1.1 is bounded if*

$$(\alpha+\frac{\beta}{\gamma})<1.$$

Proof. Let $\{s_n\}_{n=-5}^{\infty}$ be a solution of 1.1. It follows from 1.1 that □

$$s_{n+1}=\alpha s_n+\frac{\beta s_n s_{n-4}}{\gamma s_{n-4}+\delta s_{n-5}}\leq \alpha s_n+\frac{\beta s_n s_{n-4}}{\gamma s_{n-4}}=(\alpha+\frac{\beta}{\gamma})s_n$$

Then

$$s_{n+1}\leq s_n,\quad \text{for all } n\geq 0$$

Then the sub-sequence $\{s_{5n-1}\}_{n=-5}^{\infty}$, $\{s_{5n-2}\}_{n=-5}^{\infty}$, $\{s_{5n-3}\}_{n=-5}^{\infty}$, $\{s_{5n-4}\}_{n=-5}^{\infty}$, and $\{s_{5n-4}\}_{n=-5}^{\infty}$ are decreasing and so are bounded from above by $M=\max\{s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0\}$

Example 4.2. *Let* $\alpha=0.03, \beta=0.6,\ \gamma=0.8,\ \delta=0.01$ *and* $\alpha=0.3,\ \beta=0.06,\ \gamma=0.7\ \delta=0.01$ *Then 1.1 in this case will be*

$$s_{n+1}=0.03s_n+\frac{0.6s_n s_{n-4}}{0.8s_{n-4}+0.01s_{n-5}} \tag{4.1}$$

$$s_{n+1}=0.3s_n+\frac{0.06s_n s_{n-4}}{0.7s_{n-4}+0.1s_{n-5}} \tag{4.2}$$

with initial condition for 4.1 $s_{-5}=9,\ s_{-4}=4,\ s_{-3}=7,\ s_{-2}=1,\ s_{-1}=10,\ s_0=8$. *The plot for solution of* s_n *is shown in (Figure 4.1) and for 4.2* $s_{-5}=5.2,\ s_{-4}=2.3,\ s_{-3}=1.2,\ s_{-2}=0.07,\ s_{-1}=0.2,\ s_0=0.01$. *The plot for solution of* s_n *is shown in (Figure 4.2.)*

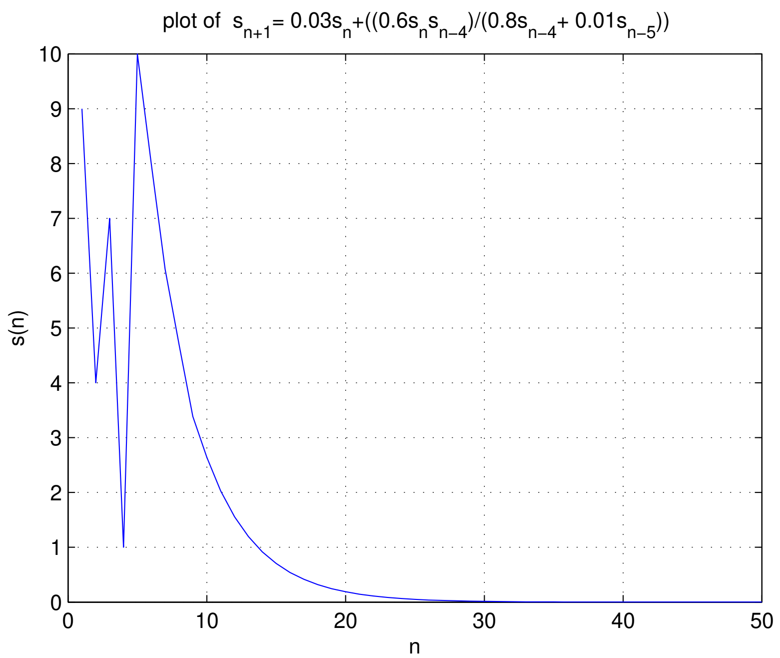

Figure 4.1: Shows Bounded Solution of 4.1

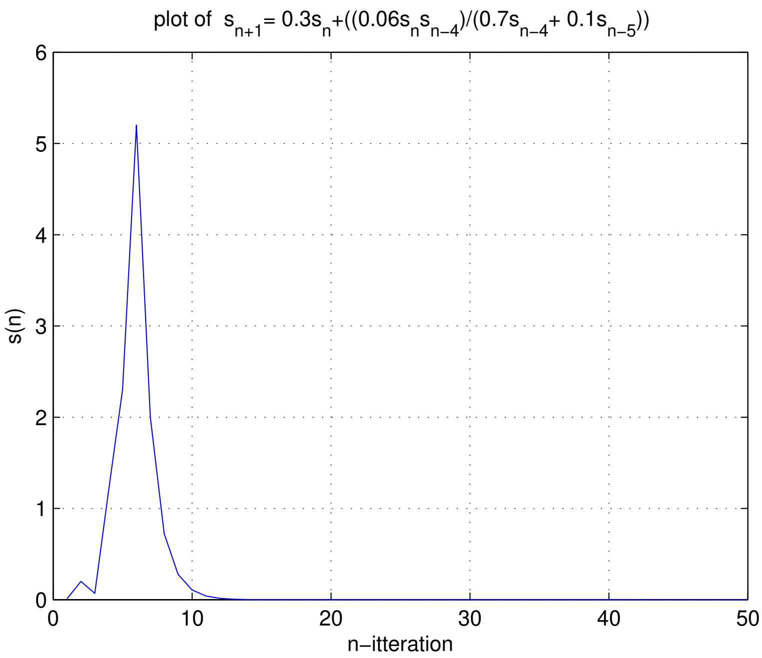

Figure 4.2: Shows Bounded Solution of 4.2

5. Some special cases of 1.1

5.1. First equation

Here we will find the closed form expression of solution of special case of 1.1

$$s_{n+1}=s_n+\frac{s_ns_{n-4}}{s_{n-4}+s_{n-5}},\qquad n=0,1,... \tag{5.1}$$

where the initial conditions $s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary positive real numbers.

Theorem 5.1. *Let $\{s_n\}_{n=-5}^{\infty}$ be a solution of 5.1. Then for $n=0,1,2,....$*

$$s_{5n}=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right),$$

$$s_{5n+1}=\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\rho+f_{2i}\varepsilon}{f_{2i}\rho+f_{2i-1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}l+f_{2i-1}\sigma}\right),$$

$$s_{5n+2}=\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}l+f_{2i-1}k}\right),$$

$$s_{5n+3}=\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

$$s_{5n+4}=\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

where $s_{-5}=\varepsilon,\ s_{-4}=\varepsilon,\ s_{-3}=\rho,\ s_{-2}=\iota,\ s_{-1}=\sigma,\ s_0=\varkappa,\quad \{f_m\}_{m=1}^{\infty}=\{1,1,2,3,5,8,13,.......\},\ \ f_{-1}=f_0=1$

Proof. For $n=0$ result holds. Now suppose that $n>0$ and that our assumption holds for $n-1$, $n-2$. That is,

$$s_{5n-6}=\varkappa\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

$$s_{5n-5}=\varkappa\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right),$$

$$s_{5n-4}=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\rho+f_{2i}\varepsilon}{f_{2i}\rho+f_{2i-1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

$$s_{5n-3}=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

$$s_{5n-2}=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

$$s_{5n-1}=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right),$$

Now, we see from 5.1 that

$$s_{5n+1}=s_{5n}+\frac{s_{5n}s_{5n-4}}{s_{5n-4}+s_{5n-5}}$$

$$=\left(\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)\right)$$

$$+\frac{\left(\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}q}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)\right)\times\left(\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\rho+f_{2i}\varepsilon}{f_{2i}\rho+f_{2i-1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}p}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\right)}{\left[\begin{array}{c}\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}k}\right)\\+\varkappa\prod_{i=0}^{n-2}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)\end{array}\right]}$$

$$=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)$$

$$+\frac{\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)\left(\frac{f_{2n+1}\varepsilon+f_{2n}\varepsilon}{f_{2n}\varepsilon+f_{2n-1}\varepsilon}\right)}{\left(\frac{f_{2n+1}\varepsilon+f_{2n}\varepsilon}{f_{2n}\varepsilon+f_{2n-1}\varepsilon}\right)+1}$$

$$=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\rho}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)$$

$$+\frac{\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)(f_{2n}\varepsilon+f_{2n-1}\varepsilon)}{f_{2n+1}\varepsilon+f_{2n}\varepsilon+f_{2n}\varepsilon+f_{2n-1}\varepsilon}$$

$$=\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+3}\sigma+f_{2i+2}\iota}{f_{2i+2}\sigma+f_{2i+1}\iota}\right)\left(\frac{f_{2i+3}\varkappa+f_{2i+2}\sigma}{f_{2i+2}\varkappa+f_{2i+1}\sigma}\right)\left(1+\frac{f_{2n+1}\varepsilon+f_{2n}\varepsilon}{f_{2n+2}\varepsilon+f_{2n+1}\varepsilon}\right)$$

Thus,

$$s_{5n+1}=\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\rho+f_{2i}\varepsilon}{f_{2i}\rho+f_{2i-1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)$$

Again, it follows from equation 5.1 that,

$$s_{5n+3}=s_{5n+2}+\frac{s_{5n+2}s_{5n-2}}{s_{5n-2}+s_{5n-3}}$$

$$= \varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)$$

$$+\frac{\left(\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\right)\times\left(\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\right)}{\left[\left(\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\right)+\left(\varkappa\prod_{i=0}^{n-1}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\right)\right]}$$

$$=\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)$$

$$+\left[\frac{\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\left(\frac{f_{2n+1}\iota+f_{2n}\rho}{f_{2n}\iota+f_{2n-1}\rho}\right)}{\left(\frac{f_{2n+1}\iota+f_{2n}\rho}{f_{2n}\iota+f_{2n-1}\rho}\right)+1}\right]$$

$$=\varkappa\prod_{i=0}^{n}\left[\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\left(1+\frac{f_{2n+1}\iota+f_{2n}\rho}{f_{2n+1}\iota+f_{2n}\rho+f_{2n}\iota+f_{2n-1}\rho}\right)\right]$$

$$=\varkappa\prod_{i=0}^{n}\left[\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+1}\iota+f_{2i}\rho}{f_{2i}\iota+f_{2i-1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right)\left(1+\frac{f_{2n+1}\iota+f_{2n}\rho}{f_{2n+2}\iota+f_{2n+1}\rho}\right)\right]$$

Therefore

$$s_{5n+3}=\varkappa\prod_{i=0}^{n}\left(\frac{f_{2i+3}\varepsilon+f_{2i+2}\varepsilon}{f_{2i+2}\varepsilon+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\rho+f_{2i+2}\varepsilon}{f_{2i+2}\rho+f_{2i+1}\varepsilon}\right)\left(\frac{f_{2i+3}\iota+f_{2i+2}\rho}{f_{2i+2}\iota+f_{2i+1}\rho}\right)\left(\frac{f_{2i+1}\sigma+f_{2i}\iota}{f_{2i}\sigma+f_{2i-1}\iota}\right)\left(\frac{f_{2i+1}\varkappa+f_{2i}\sigma}{f_{2i}\varkappa+f_{2i-1}\sigma}\right).$$

Other relations can be done similarly. So, the proof is completed.

□

Example 5.2. *To confirm the result in this case we consider numerical example. Let α =1, β =1, γ =1, $\delta=1$. Then 1.1 in this case will be*

$$s_{n+1}=s_n+\frac{s_n s_{n-4}}{s_{n-4}+s_{n-5}} \tag{5.2}$$

with initial condition $s_{-5}=2,\ s_{-4}=8,\ s_{-3}=5,\ s_{-2}=3,\ s_{-1}=1,\ s_0=6$. The plot for solution of s_n is shown in Figure 5.1.)

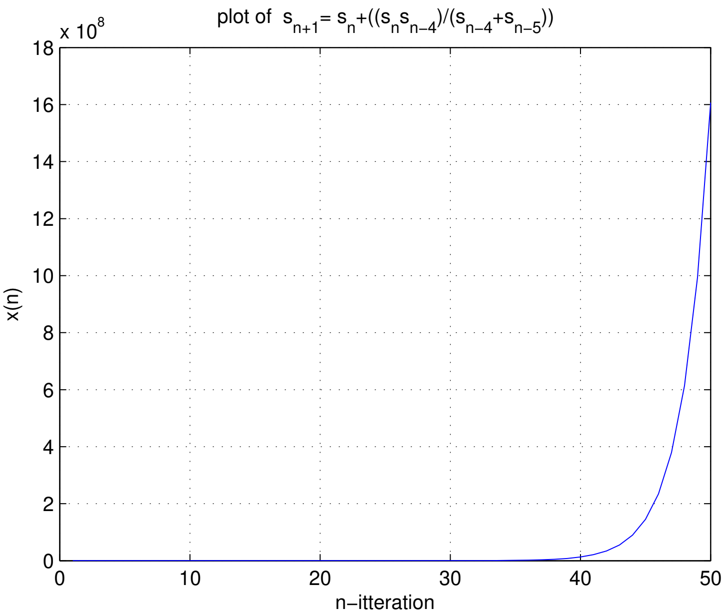

Figure 5.1: Shows Unbounded Solution of 5.2

5.2. Second equation

In this section we solve the specific form of the 1.1

$$s_{n+1}=s_n+\frac{s_ns_{n-4}}{s_{n-4}-s_{n-5}},\qquad n=0,1,\dots \tag{5.3}$$

where the initial conditions $s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary positive real numbers.

Theorem 5.3. *Let $\{s_n\}_{n=-5}^{\infty}$ be a solution of 5.3. Then for $n=0,1,2,\dots$.*

$$s_{5n}=\vartheta\prod_{i=0}^{n-1}\left(\frac{f_{i+3}\sigma-f_{i+1}\omega}{f_{i+1}\sigma-f_{i-1}\omega}\right)\left(\frac{f_{i+3}\rho-f_{i+1}\sigma}{f_{i+1}\rho-f_{i-1}\sigma}\right)\left(\frac{f_{i+3}r-f_{i+1}\rho}{f_{i+1}r-f_{i-1}\rho}\right)\left(\frac{f_{i+3}k-f_{i+1}r}{f_{i+1}k-f_{i-1}r}\right)\left(\frac{f_{i+3}\vartheta-f_{i+1}k}{f_{i+1}\vartheta-f_{i-1}k}\right),$$

$$s_{5n+1}=\vartheta\left(\tfrac{2\sigma-\omega}{\sigma-\omega}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+4}\sigma-f_{i+2}\omega}{f_{i+2}\sigma-f_{i}\omega}\right)\left(\frac{f_{i+3}\rho-f_{i+1}\sigma}{f_{i+1}\rho-f_{i-1}\sigma}\right)\left(\frac{f_{i+3}r-f_{i+1}\rho}{f_{i+1}r-f_{i-1}\rho}\right)\left(\frac{f_{i+3}k-f_{i+1}r}{f_{i+1}k-f_{i-1}r}\right)\left(\frac{f_{i+3}\vartheta-f_{i+1}k}{f_{i+1}\vartheta-f_{i-1}k}\right),$$

$$s_{5n+2}=\vartheta\left(\tfrac{2\sigma-\omega}{\sigma-\omega}\right)\left(\tfrac{2\rho-\sigma}{\rho-\sigma}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+4}\sigma-f_{i+2}\omega}{f_{i+2}\sigma-f_{i}\omega}\right)\left(\frac{f_{i+4}\rho-f_{i+2}\sigma}{f_{i+2}\rho-f_{i}\sigma}\right)\left(\frac{f_{i+3}r-f_{i+1}\rho}{f_{i+1}r-f_{i-1}\rho}\right)\left(\frac{f_{i+3}k-f_{i+1}r}{f_{i+1}k-f_{i-1}r}\right)\left(\frac{f_{i+3}\vartheta-f_{i+1}k}{f_{i+1}\vartheta-f_{i-1}k}\right),$$

$$s_{5n+3}=\vartheta\left(\tfrac{2\sigma-\omega}{\sigma-\omega}\right)\left(\tfrac{2\rho-\sigma}{\rho-\sigma}\right)\left(\tfrac{2r-\rho}{r-\rho}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+4}\sigma-f_{i+2}\omega}{f_{i+2}\sigma-f_{i}\omega}\right)\left(\frac{f_{i+4}\rho-f_{i+2}\sigma}{f_{i+2}\rho-f_{i}\sigma}\right)\left(\frac{f_{i+4}r-f_{i+2}\rho}{f_{i+2}r-f_{i}\rho}\right)\left(\frac{f_{i+3}k-f_{i+1}r}{f_{i+1}k-f_{i-1}r}\right)\left(\frac{f_{i+3}\vartheta-f_{i+1}k}{f_{i+1}\vartheta-f_{i-1}k}\right),$$

$$s_{5n+4}=\vartheta\left(\tfrac{2\sigma-\omega}{\sigma-\omega}\right)\left(\tfrac{2\rho-\sigma}{\rho-\sigma}\right)\left(\tfrac{2r-\rho}{r-\rho}\right)\left(\tfrac{2k-r}{k-r}\right)\prod_{i=0}^{n-1}\left(\frac{f_{i+4}\sigma-f_{i+2}\omega}{f_{i+2}\sigma-f_{i}\omega}\right)\left(\frac{f_{i+4}\rho-f_{i+2}\sigma}{f_{i+2}\rho-f_{i}\sigma}\right)\left(\frac{f_{i+4}r-f_{i+2}\rho}{f_{i+2}r-f_{i}\rho}\right)\left(\frac{f_{i+4}k-f_{i+2}r}{f_{i+2}k-f_{i}r}\right)\left(\frac{f_{i+3}\vartheta-f_{i+1}k}{f_{i+1}\vartheta-f_{i-1}k}\right),$$

where $s_{-5}=\omega,\ s_{-4}=\sigma,\ s_{-3}=\rho,\ s_{-2}=r,\ s_{-1}=k,\ s_0=\vartheta,\quad \{f_m\}_{m=1}^{\infty}=\{1,1,2,3,5,8,13,\dots\dots\},\ f_{-1}=f_0=1$

Proof. Same as the Theorem 5.1 and is omitted. □

Example 5.4. *We will confirm our result by considering some numerical examples. Assume $s_{-5}=1,\ s_{-4}=3,\ s_{-3}=2,\ s_{-2}=9,\ s_{-1}=6,\ s_0=7$ (see Figure 5.2) and $s_{-5}=13,\ s_{-4}=12,\ s_{-3}=18,\ s_{-2}=16,\ s_{-1}=15,\ s_0=10$ (see behavior of solution of 5.3 Figure 5.3).*

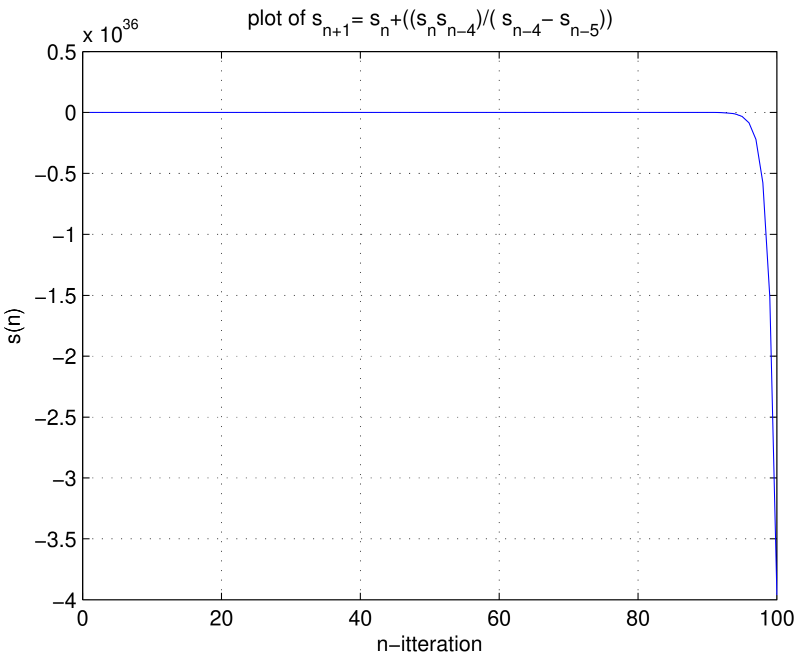

Figure 5.2

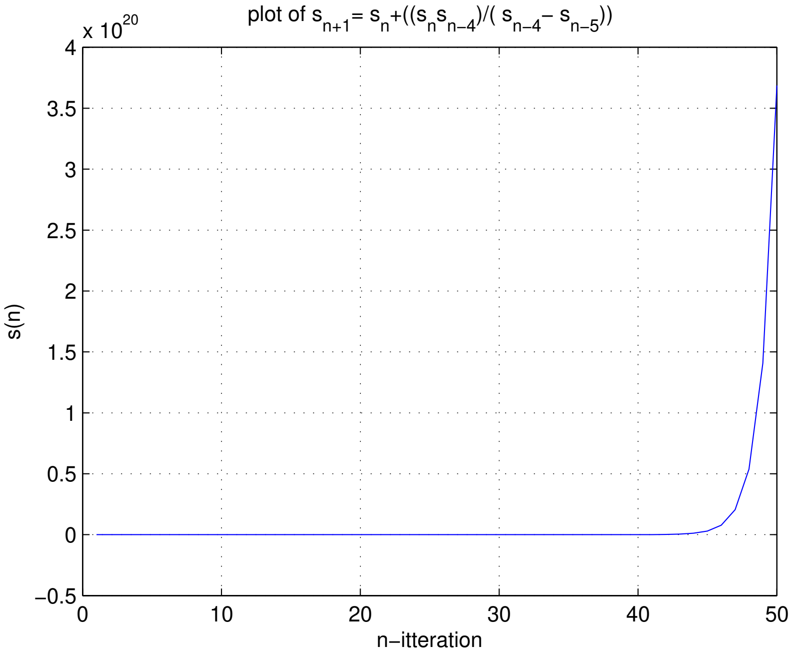

Figure 5.3

5.3. Third equation

In this section we deal with the specific form of the 1.1

$$s_{n+1} = s_n - \frac{s_n s_{n-4}}{s_{n-4} + s_{n-5}}, \qquad n = 0, 1, \ldots \tag{5.4}$$

where the initial conditions $s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary positive real numbers.

Theorem 5.5. *Let*$\{s_n\}_{n=-5}^{\infty}$*be a solution of 5.4. Then for* $n = 0, 1, 2, \ldots.$

$$s_{5n} = \frac{lkrpqt}{(f_n q + f_{n+1} t)(f_n p + f_{n+1} q)(f_n r + f_{n+1} p)(f_n k + f_{n+1} r)(f_n l + f_{n+1} k)},$$

$$s_{5n+1} = \frac{lkrpqt}{(f_{n+1} q + f_{n+2} t)(f_n p + f_{n+1} q)(f_n r + f_{n+1} p)(f_n k + f_{n+1} r)(f_n l + f_{n+1} k)},$$

$$s_{5n+2} = \frac{lkrpqt}{(f_{n+1} q + f_{n+2} t)(f_{n+1} p + f_{n+2} q)(f_n r + f_{n+1} p)(f_n k + f_{n+1} r)(f_n l + f_{n+1} k)},$$

$$s_{5n+3} = \frac{lkrpqt}{(f_{n+1} q + f_{n+2} t)(f_{n+1} p + f_{n+2} q)(f_{n+1} r + f_{n+2} p)(f_n k + f_{n+1} r)(f_n l + f_{n+1} k)},$$

$$s_{5n+4} = \frac{lkrpqt}{(f_{n+1} q + f_{n+2} t)(f_{n+1} p + f_{n+2} q)(f_{n+1} r + f_{n+2} p)(f_{n+1} k + f_{n+2} r)(f_n l + f_{n+1} k)}.$$

Where $s_{-5} = t,\ s_{-4} = q,\ s_{-3} = p,\ s_{-2} = r,\ s_{-1} = k,\ s_0 = l,\quad \{f_m\}_{m=1}^{\infty} = \{1, 1, 2, 3, 5, 8, 13, \ldots\ldots\}\quad f_0 = 1.$

Proof. For $n = 0$, the result holds. Now suppose that $n > 0$ and that our supposition holds for $n-1,\ n-2$. That is

$$s_{5n-6} = \frac{lkrpqt}{(f_{n-1} q + f_n t)(f_{n-1} p + f_n q)(f_{n-1} r + f_n p)(f_{n-1} k + f_n r)(f_{n-2} l + f_{n-1} k)},$$

$$s_{5n-5} = \frac{lkrpqt}{(f_{n-1} q + f_n t)(f_{n-1} p + f_n q)(f_{n-1} r + f_n p)(f_{n-1} k + f_n r)(f_{n-1} l + f_n k)},$$

$$s_{5n-4} = \frac{lkrpqt}{(f_n q + f_{n-1} t)(f_n p + f_{n+1} q)(f_n r + f_{n+1} p)(f_n k + f_{n+1} r)(f_n l + f_{n+1} k)},$$

$$s_{5n-3}=\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-1}l+f_nk)},$$

$$s_{5n-2}=\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_{n-1}k+f_nr)(f_{n-1}l+f_nk)},$$

$$s_{5n-1}=\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)},$$

Now, from equation 5.1, we see that,

$$s_{5n}=s_{5n-1}-\frac{s_{5n-1}s_{5n-5}}{s_{5n-5}+s_{5n-6}}$$

$$\begin{aligned}
&= \frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)} \\
&\quad - \frac{\left(\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}\right)\left(\frac{lkrpqt}{(f_{n-1}q+f_nt)(f_{n-1}p+f_nq)(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-1}l+f_nk)}\right)}{\left[\left(\frac{lkrpqt}{(f_{n-1}q+f_nt)(f_{n-1}p+f_nq)(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-1}l+f_nk)}\right)+\left(\frac{lkrpqt}{(f_{n-1}q+f_nt)(f_{n-1}p+f_nq)(f_{n-1}r+f_np)(f_{n-1}k+f_nr)(f_{n-2}l+f_{n-1}k)}\right)\right]} \\
&= \frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)} \\
&\quad - \frac{\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}\left(\frac{1}{f_{n-1}l+f_nk}\right)}{\left[\frac{1}{(f_{n-1}l+f_nk)}+\frac{1}{(f_{n-2}l+f_{n-1}k)}\right]}
\end{aligned}$$

$$\begin{aligned}
&= \frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)} \\
&\quad - \left[\frac{\left(\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}\right)(f_{n-2}l+f_{n-1}k)}{f_{n-1}l+f_nk+f_{n-2}l+f_{n-1}k}\right] \\
&= \frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}\left(1-\frac{f_{n-2}l+f_{n-1}k}{f_nl+f_{n+1}k}\right) \\
&= \frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}\left(\frac{f_{n-1}l+f_nk}{f_nl+f_{n+1}k}\right)
\end{aligned}$$

Therefore,

$$s_{5n}=\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_nl+f_{n+1}k)}$$

Now, from equation 5.4

$$\begin{aligned}
s_{5n+4} &= s_{5n+3}-\frac{s_{5n+3}s_{5n-1}}{s_{5n-1}+s_{5n-2}} \\
&= \frac{lkrpqt}{(f_{n+1}q+f_{n+2}t)(f_{n+1}p+f_{n+2}q)(f_{n+1}r+f_{n+2}p)(f_nk+f_{n+1}r)(f_nl+f_{n+1}k)} \\
&\quad - \frac{\left[\left(\frac{lkrpqt}{(f_{n+1}q+f_{n+2}t)(f_{n+1}p+f_{n+2}q)(f_{n+1}r+f_{n+2}p)(f_nk+f_{n+1}r)(f_nl+f_{n+1}k)}\right)\times\left(\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}\right)\right]}{\left[\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_nk+f_{n+1}r)(f_{n-1}l+f_nk)}+\frac{lkrpqt}{(f_nq+f_{n+1}t)(f_np+f_{n+1}q)(f_nr+f_{n+1}p)(f_{n-1}k+f_nr)(f_{n-1}l+f_nk)}\right]}
\end{aligned}$$

$$= \frac{\left(\frac{lkrpqt}{(f_{n+1}q+f_{n+2}t)(f_{n+1}p+f_{n+2}q)(f_{n+1}r+f_{n+2}p)(f_nk+f_{n+1}r)(f_nl+f_{n+1}k)}\right)\left(\frac{1}{f_nk+f_{n+1}r}\right)}{\left[\frac{1}{f_nk+f_{n+1}r}+\frac{1}{f_{n-1}k+f_nr}\right]}$$

$$= \frac{lkrpqt}{(f_{n+1}q+f_{n+2}t)(f_{n+1}p+f_{n+2}q)(f_{n+1}r+f_{n+2}p)(f_nk+f_{n+1}r)(f_nl+f_{n+1}k)}\left[1-\frac{f_{n-1}k+f_nr}{(f_{n-1}k+f_nr)+(f_nk+f_{n+1}r)}\right]$$

$$= \frac{lkrpqt}{(f_{n+1}q+f_{n+2}t)(f_{n+1}p+f_{n+2}q)(f_{n+1}r+f_{n+2}p)(f_nk+f_{n+1}r)(f_nl+f_{n+1}k)}\left[1-\frac{f_{n-1}k+f_nr}{f_{n+1}k+f_{n+2}r}\right]$$

Therefore,

$$s_{5n+4} = \frac{lkrpqt}{(f_{n+1}q+f_{n+2}t)(f_{n+1}p+f_{n+2}q)(f_{n+1}r+f_{n+2}p)(f_{n+1}k+f_{n+2}r)(f_nl+f_{n+1}k)}.$$

Remaining relations can be found similarly. Hence, the proof is completed. □

Example 5.6. *Assume* $s_{-5}=1,\ s_{-4}=3,\ s_{-3}=6,\ s_{-2}=5,\ s_{-1}=2,\ s_0=7$. *(Figure 5.4, shows behavior of solution of 5.4)*

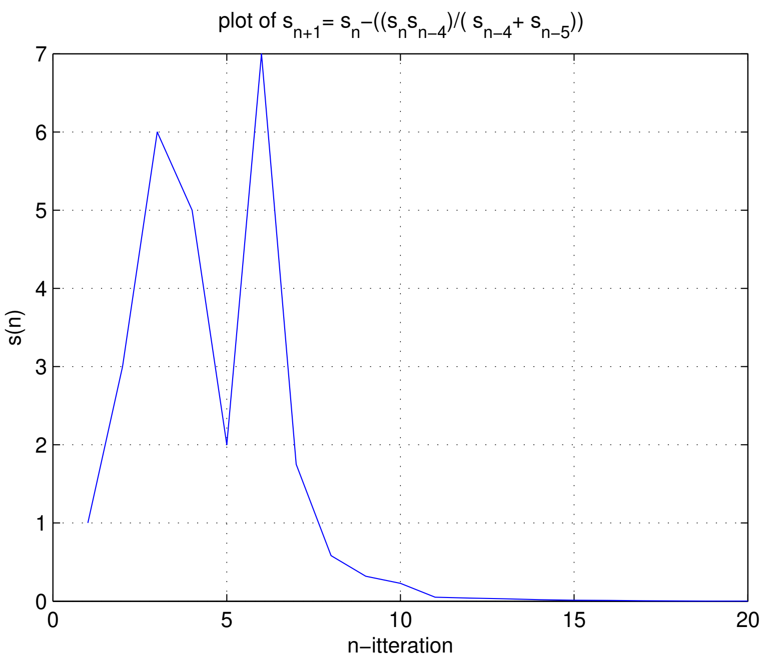

Figure 5.4: Shows behavior of Solution of 5.4

5.4. Fourth equation

In this section we deal with the specific form of the 1.1

$$s_{n+1} = s_n - \frac{s_ns_{n-4}}{s_{n-4}-s_{n-5}}, \qquad n=0,1,... \tag{5.5}$$

where the initial conditions $s_{-5},\ s_{-4},\ s_{-3},\ s_{-2},\ s_{-1},\ s_0$ are arbitrary non-zero real numbers with $s_{-5}\neq s_{-4}\neq s_{-3}\neq s_{-2}\neq s_{-1}\neq s_0$.

Theorem 5.7. *Let* $\{s_n\}_{n=-5}^{\infty}$*be a solution of 5.2. Then every solution of it is periodic with period 24. Moreover,* $\{s_n\}_{n=-5}^{\infty}$ *takes the form*

$$t,\ q,\ p,\ r,\ k,\ l,\ \frac{tl}{l-k},\ \frac{tql}{(t-q)(q-p)},\ \frac{tqpl}{(t-q)(q-p)(p-r)},\ \frac{tqprl}{(t-q)(q-p)(p-r)(r-k)},\ \frac{tqprkl}{(t-q)(q-p)(p-r)(r-k)(k-l)},\ \frac{-tprkl}{(q-p)(p-r)(r-k)(k-l)},$$

or,

$$\begin{aligned}
s_{24n-5} &= t, & s_{24n-4} &= q,\\
s_{24n-3} &= p, & s_{24n-2} &= r,\\
s_{24n-1} &= k, & s_{24n} &= l,\\
s_{24n+1} &= \frac{tl}{l-k}, & s_{24n+2} &= \frac{tql}{(t-q)(q-p)},\\
s_{24n+3} &= \frac{tqpl}{(t-q)(q-p)(p-r)}, & s_{24n+4} &= \frac{tqprl}{(t-q)(q-p)(p-r)(r-k)},\\
s_{24n+5} &= \frac{tqprkl}{(t-q)(q-p)(p-r)(r-k)(k-l)}, & s_{24n+6} &= \frac{-tprkl}{(q-p)(p-r)(r-k)(k-l)},\\
s_{24n+7} &= \frac{trkl}{(p-r)(r-k)(k-l)}, & s_{24n+8} &= \frac{-tkl}{(r-k)(k-l)},\\
s_{24n+9} &= \frac{tl}{(k-l)}, & s_{24n+10} &= -t,\\
s_{24n+11} &= -q, & s_{24n+12} &= -p,\\
s_{24n+13} &= -r, & s_{24n+14} &= -k,\\
s_{24n+15} &= -l, & s_{24n+16} &= \frac{-tl}{(t-q)},\\
s_{24n+17} &= \frac{-tql}{(t-q)(q-p)}, & s_{24n+18} &= \frac{-tqpl}{(t-q)(q-p)(p-r)},\\
s_{24n+19} &= \frac{-tqprl}{(t-q)(q-p)(p-r)(r-k)}, & s_{24n+20} &= \frac{-tqprkl}{(t-q)(q-p)(p-r)(r-k)(k-l)},\\
s_{24n+21} &= \frac{tprkl}{(q-p)(p-r)(r-k)(k-l)}, & s_{24n+22} &= \frac{-trkl}{(p-r)(r-k)(k-l)},\\
s_{24n+23} &= \frac{tkl}{(r-k)(k-l)}, & s_{24n+24} &= \frac{-tl}{(k-l)}.
\end{aligned}$$

where $s_{-5} = t,\ s_{-4} = q,\ s_{-3} = p,\ s_{-2} = r,\ s_{-1} = k,\ s_0 = l$

Proof. The proof is left to the reader. □

Example 5.8. *Assume* $s_{-5} = 2,\ s_{-4} = 17,\ s_{-3} = 15,\ s_{-2} = 14,\ s_{-1} = 19,\ s_0 = 11$. *(See Figure 5.5 for the periodic behavior of 5.5) and*$s_{-5} = -2,\ s_{-4} = 17,\ s_{-3} = 15,\ s_{-2} = -8,\ s_{-1} = 19,\ s_0 = 1$. *(See Figure 5.6)*

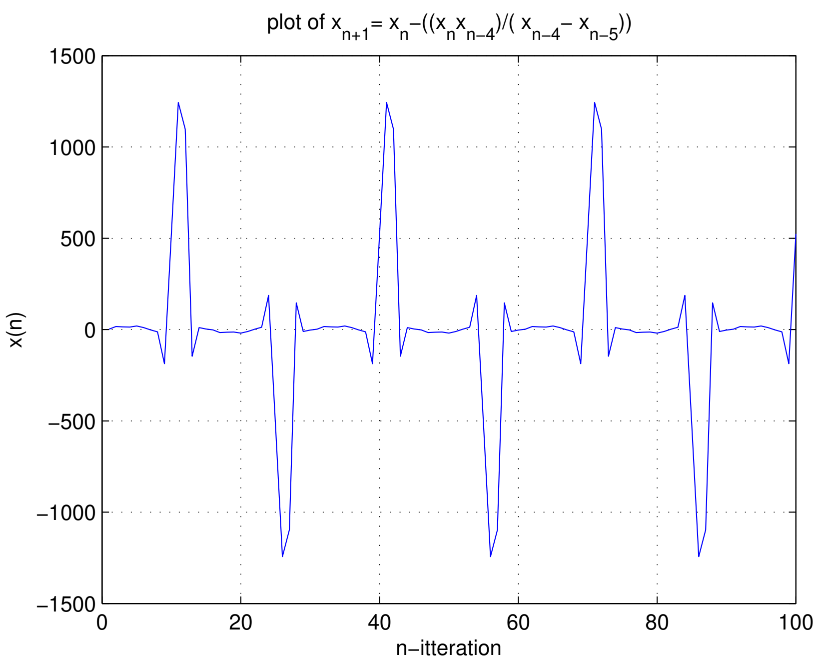

Figure 5.5: Shows periodic solution of 5.5

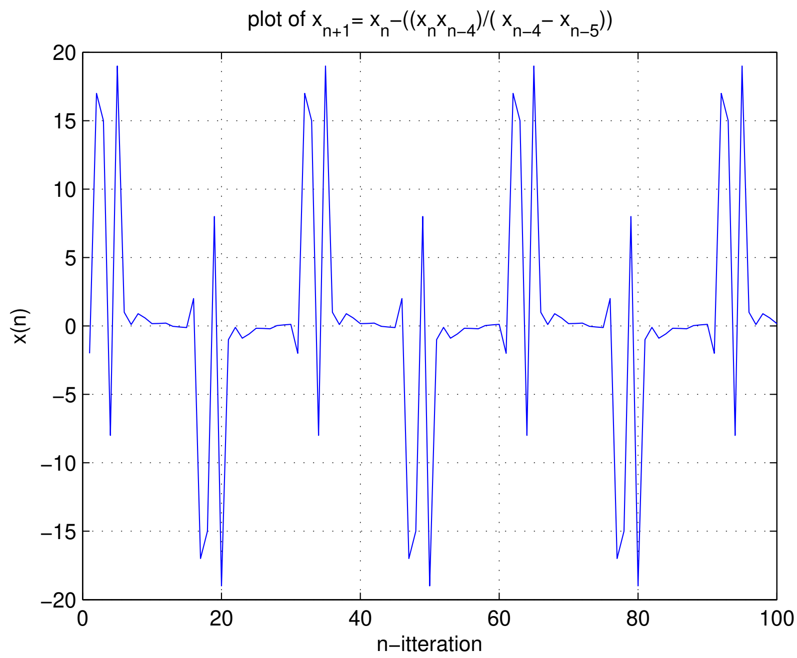

Figure 5.6: Shows periodic solution of 5.5

6. Conclusion

In This paper we studied global stability, boundedness and the solutions of some special cases of equation 1.1. In Section 3 we proved when $\beta(\gamma+3\delta)<(\gamma+\delta)^2(1-\alpha)$, 1.1 has local stability. We proved in the same section that the unique equilibrium of equation 1.1 is globally asymptotically stable if $\gamma(1-\alpha)\neq\beta$. In Section 4 we showed that the solution of equation 1.1 is bounded if $(\alpha+\frac{\beta}{\gamma})<1$. In Section 5, we obtained the expression and closed form solution of four special cases of equation 1.1 and gave numerical examples of each of the case, with different initial values.

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