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THE CAUCHY PROBLEM OF A PERIODIC KAWAHARA EQUATION IN ANALYTIC GEVREY SPACES

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Abstract. The Cauchy problem for the Kawahara equation with data in analytic Gevrey spaces on the circle is considered and its local well-posedness in these spaces is proved. Using Bourgain-Gevrey type analytic spaces and appropriate bilinear estimates, it is shown that local in time wellposedness holds when the initial data belong to an analytic Gevrey spaces of order σ. Moreover, the solution is not necessarily G^{σ} in time. However, it belongs to $G^{5\sigma}$ near zero for every x on the circle.

1. Introduction, related results and position problem

The shallow water equations describes the flow below a pressure surface in a fluid. They are PDEs of hyperbolic type (or parabolic if we consider viscous shear). For $x \in \mathbb{T},\ t \in \mathbb{R}$, we denote by $u = u(x,t)$. When we write (1.1), we mean the equation number i from the problem (1.1) subjected with the initial data $u(x,0) = u_0(x)$. We consider

$$\begin{cases} \partial_t u + \alpha\partial_x^5 u + \beta\partial_x^3 u + \gamma\partial_x u + \mu\partial_x(u^2) = 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.1}$$

here the parameters $\alpha \neq 0$, β and γ are real numbers and μ is a complex number . To outline our contributions, we will extend the results in [2] and [23] , where the solution was obtained in $X_{s,b}$ to the analytic Gevrey-Bourgain spaces $X_{\sigma,\delta,s,b}$ with also regularity in time.

So, from the mathematical point of view, it is important to study the well-posedness and time regularity for the shallow water equations which happens in the water waves with surface tension, in which the Bond number takes on the critical value (See [3], [5], [6], [8]).
Recently, Y. Jia and Z. Huo [2] considered a Cauchy problems (1.1), the authors obtained local well-posedness for data in $H^s(\mathbb{R})$ with $s > -7/4$ for $\partial_x u^2$.

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Motivated by all the above papers, we investigate the well-posedness of (1.1) in Analytic Gevry spaces to extend results in [23]. The second novelty located in the study of Gevrey's temporal regularity for the unique solution, inspired and motivated by [3] and [5] on the temporal regularity of solutions to KdV-type equations with analytical data of Gevrey.

We begin by presenting some ideas to get the well-posedness, we are working mainly on the integral equivalent formulation of (1.1) as

$$u = S(t)u_0 - \int_0^t S(t-t')\partial_x u^2(t')dt', \tag{1.2}$$

where the unit operator related to the corresponding linear equation is

$$S(t) = \mathcal{F}_x^{-1} e^{-it(\alpha\xi^5 - \beta\xi^3 + \gamma\xi)} \mathcal{F}_x. \tag{1.3}$$

Let us define the phase function as follows

$$\phi(\xi) = \alpha\xi^5 - \beta\xi^3 + \gamma\xi, \tag{1.4}$$

We define the needed spaces beginning by the spaces of analytic Gevrey functions that contain our initial data. For $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, let

$$G^{\sigma,\delta,s}(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}); \|f\|^2_{G^{\sigma,\delta,s}(\mathbb{T})} = \sum_{k\in\mathbb{Z}} e^{2\delta|k|^{1/\sigma}} \langle k\rangle^{2s} |\widehat{f}(k)|^2 < \infty \right\}, \tag{1.5}$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$.

For $\delta = 0$, the space $G^{\sigma,\delta,s}(\mathbb{T})$ coincides with the standard Sobolev space $H^s(\mathbb{T})$.

We then define the analytic Gevrey -Bourgain spaces related to Kawahara equation. The completion of the Schwartz class $S(\mathbb{T}\times\mathbb{R})$ is given by $X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})(resp. Y_{\sigma,\delta,s,b})$, for $s, b \in \mathbb{R}$, $\delta > 0$ and $\sigma \geq 1$, subjected to the norm

$$\|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \int e^{2\delta|k|^{1/\sigma}} \langle k\rangle^{2s} \langle \tau + \phi(k)\rangle^{2b} \mid \widehat{u}(k,\tau) \mid^2 d\tau \right)^{\frac{1}{2}}. \tag{1.6}$$

$$\|u\|_{Y_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \left(\sum_{k\in\mathbb{Z}} \left(\int e^{\delta|k|^{1/\sigma}} \langle k\rangle^{s} \langle \tau + \phi(k)\rangle^{b} \mid \widehat{u}(k,\tau) \mid d\tau \right)^2 \right)^{\frac{1}{2}}. \tag{1.7}$$

In addition, let

$$Z_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b} \cap Y_{\sigma,\delta,s,b-1/2}$$

be the Banach space endowed with the norm

$$\|u\|_{Z_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} = \|u\|_{X_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} + \|u\|_{Y_{\sigma,\delta,s,b-1/2}(\mathbb{T}\times\mathbb{R})}. \tag{1.8}$$

For $\delta = 0$, the space $Z_{\sigma,\delta,s,b} = X_{\sigma,\delta,s,b} \cap Y_{\sigma,\delta,s,b-1/2}$ coincides with the standard Bourgain type space $Z_{s,b} = X_{s,b} \cap Y_{s,b-1/2}$.

We organize this paper as follows. In Section 2, our main results regarding the well-posedness (Theorem 2.1) and regularity (Theorem 2.2) in the analytic Gevrey-Bourgain spaces for (1.1) are stated. In Section 3, all Theorems by deriving the bilinear estimates are proved in details.

2. Main results

Theorem 2.1. *Let $s > 0, \sigma \geqslant 1, \delta > 0$ and $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$. Then for some real number $b > \frac{1}{2}$, which is near enough to $\frac{1}{2}$, and a constant $T > 0$, such that (1.1) admits a unique local solution $u \in C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right) \cap Z_{\sigma,\delta,s,\frac{1}{2}}$. Moreover, given $t \in (0,T)$, the map $u_0 \to u(t)$ is Lipschitz continuous from $G^{\sigma,\delta,s}(\mathbb{T})$ to $C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right)$.*

Our next goal is to study Gevrey's temporal regularity of the unique solution obtained in Theorem 2.1. A periodic function $f(x)$ is the Gevrey class of order σ, if there exists a constant $C > 0$ such that

$$\sup_{x\in\mathbb{T}} | \partial_x^l f(x) | \leq C^{l+1}(l!)^\sigma \quad l = 0, 1, 2, \ldots .$$

Theorem 2.2. *Let $s > 0$, $\sigma \geqslant 1, \delta > 0$ and $\beta = \gamma = \mu = \alpha = 1$. If $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, then the solution $u \in C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right)$ given by Theorem 2.1 belongs to the Gevrey class $G^{5\sigma}$ in time variable.*

3. Proof of main Theorems

We are going to prepare the prove of our main theorems, let us beginning by the embedded result in the next lemma, which is useful for Theorem 2.1.

Lemma 3.1. *Let $s \in \mathbb{R}$, $\sigma \geqslant 1$ and $\delta > 0$, we have*

$$Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R}) \hookrightarrow C\left([0,T], G^{\sigma,\delta,s}(\mathbb{T})\right).$$

3.1. **Existence of solution.** Taking the Fourier transform with respect to x of the Cauchy problems (1.1), after an ordinary calculation, we get

$$u = S(t)u_0 - \int_0^t S(t-t')\partial_x u^2(t')dt',$$

we localize it t by using a cut-off function, satisfying $\psi \in C_0^\infty(\mathbb{R})$, with $\psi = 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and supp$\psi \subset [-1, 1]$. We consider the operator Φu given by

$$\Phi(u) = \psi(t)S(t)u_0 - \psi(t)\int_0^t S(t-t')\partial_x u^2(t')dt', \tag{3.1}$$

We now estimate the fist part in the right hand side of (3.1).

Lemma 3.2. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geqslant 1$, for some constant $C > 0$, we have*

$$\|\psi(t)S(t)u_0\|_{Z_{\sigma,\delta,s,b}(\mathbb{T}\times\mathbb{R})} \leq C\ \|u_0\|_{G^{\sigma,\delta,s}(\mathbb{T})}, \tag{3.2}$$

for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$.

Proof. Define the operator A defined by

$$\widehat{Au}^x(k,t) = e^{\delta|k|^{1/\sigma}}\widehat{u}^x(k,t), \tag{3.3}$$

for $\delta = 0$ can be found in Lemma 2.1 of [23] . These inequalities clearly remain valid for $\delta > 0$, as one merely has to replace u_0 by Au_0 in these results. □

We estimate the second part in right hand side of (3.1).

Lemma 3.3. *Let $s \in \mathbb{R}$, $\delta > 0$ and $\sigma \geqslant 1$, then for some constant $C > 0$, we have*

$$\|\psi(t) \int_0^t S(t-t')F(x,t')\mathrm{d}t'\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C\|F\|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}. \tag{3.4}$$

Proof. Define $U = \psi_T(t)\int_0^t S(t-t')F(x,t')\mathrm{d}t'$ and using the operator A.

$$\begin{aligned} \widehat{AU}^x(k,t) &= \psi(t)\int_0^t \left(e^{-i(t-t')\phi(k)}\right) e^{\delta|k|^{1/\sigma}} \widehat{F}^x(k,t')\mathrm{d}t', \\ &= \psi(t)\int_0^t \ \widehat{[S(t-t')(AF)]}^x(k,t')\mathrm{d}t'. \end{aligned}$$

Thus,

$$\| U \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} = \| AU \|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} = \|\psi(t)\int_0^t S(t-t')AF(x,t')dt'\|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}.$$

Using Lemma 2.1. in [23], we have

$$\|\psi(t)\int_0^t S(t-t')AF(x,t')dt'\|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C\|AF\|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} = C\|F\|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}.$$

□

In order to treat the different nonlinear terms, we will see several lemmas. Here the bilinear estimate is given in the next lemma.

Lemma 3.4. *If $s > 0$, let $\sigma \geqslant 1$, $\delta > 0$. Then*

$$\| \partial_x(u_1u_2) \|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leqslant C \| u_1 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| u_2 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}. \tag{3.5}$$

Proof. We observe, by considering the operator A in (3.3), that

$$\begin{aligned} e^{\delta|k|^{1/\sigma}}\widehat{u_1u_2} &= (2\pi)^{-2}e^{\delta|k|^{1/\sigma}}\widehat{u_1} * \widehat{u_2} \\ &\leqslant (2\pi)^{-2}\int_{\mathbb{R}^2} e^{\delta|k-\eta|^{1/\sigma}}\widehat{u_1}(k-\eta,\tau-\rho)e^{\delta|\eta|^{1/\sigma}}\widehat{u_2}(\eta,\rho)d\eta d\rho \qquad (3.6) \\ &= \widehat{Au_1Au_2}, \end{aligned}$$

since $\delta \mid k \mid^{1/\sigma} \leqslant \delta \mid k-\eta \mid^{1/\sigma} + \delta \mid \eta \mid^{1/\sigma}, \quad \forall \sigma \geqslant 1$. Then

$$\begin{aligned} \| \partial_x(u_1u_2) \|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} &= \| \partial_x(A(u_1u_2)) \|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\ &\leq \| \partial_x(Au_1Au_2) \|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}. \end{aligned}$$

Now, by using Lemma 2.3. of [23], there exists $C > 0$ such that

$$\begin{aligned} \| \partial_x(Au_1Au_2) \|_{Z_{s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} &\leqslant C \| Au_1 \|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| Au_2 \|_{Z_{s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\ &= C \| u_1 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| u_2 \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}. \end{aligned}$$

□

We are now ready to estimate all the terms in (3.1) by using the bilinear estimates in the above lemmas.

Lemma 3.5. *Let $s > 0$, and $\sigma \geq 1$, $\delta > 0$. Then, for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, with some constant $C > 0$, we have*

$$\| \Phi(u) \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C \left(\| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{T})} + \| u \|^2_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \right), \tag{3.7}$$

and

$$\| \Phi(u) - \Phi(v) \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq C \| u - v \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \| u + v \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})}, \tag{3.8}$$

for all $u, v \in Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$

Proof. To prove estimate (3.7), we follow

$$\begin{aligned}
\| \Phi(u) \|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} &\leq \|\psi_T(t) S(t) u_0\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\
&+ \|\psi_T(t) \int_0^t S(t-t') \partial_x u^2(t') dt'\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\
&\leq C \| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{T})} + C \| \partial_x u^2 \|_{Z_{\sigma,\delta,s,-\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \\
&\leq C \| u_0 \|_{G^{\sigma,\delta,s}(\mathbb{T})} + C \| u \|^2_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} .
\end{aligned}$$

For the estimate (3.8), we observe that

$$\Phi(u) - \Phi(v) = \psi_T(t) \int_0^t S(t-t') \left(\partial_x u^2 - \partial_x v^2 \right)(x, t') dt',$$

where $\omega = \partial_x u^2 - \partial_x v^2$ is now given by

$$\omega = \partial_x (u^2 - v^2) = \partial_x [(u+v)(u-v)],$$

Thus, from the previous results, we obtain (3.8). □

We will show that the map Φ is a contraction on the ball $\mathbb{B}(0, r)$ to $\mathbb{B}(0, r)$. where u_0 satisfies the smallness condition $\|u_0\|_{G^{\sigma,\delta,s}(\mathbb{T})} \leq \frac{1}{18C^2}$ and $r = \frac{1}{6C}$

Lemma 3.6. *Let $s > 0$ and $\sigma \geq 1$. Then, for all $u_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, such that the map $\Phi : \mathbb{B}(0, r) \to \mathbb{B}(0, r)$ is a contraction, where $\mathbb{B}(0, r)$ is given by*

$$\mathbb{B}(0, r) = \{u \in Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T} \times \mathbb{R}); \|u\|_{Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T}\times\mathbb{R})} \leq r\}.$$

Proof. To prove Lemma3.6 we need to use Lemma3.5.

□

This completes the prove of existence Theorem2.1.

3.2. **Continuous dependence of the initial data.** To prove continuous dependence of the initial data in $Z_{\sigma,\delta,s,\frac{1}{2}}(\mathbb{T} \times \mathbb{R})$ we will prove the following.

Lemma 3.7. *Let $s > 0$ and $\sigma \geq 1$, $\delta > 0$. Then for all $u_0, v_0 \in G^{\sigma,\delta,s}(\mathbb{T})$, if u and v are two solutions to (1.1) corresponding to initial data u_0 and v_0. We have*

$$\|u - v\|_{C([0,T], G^{\sigma,\delta,s}(\mathbb{T}))} \leq 2C_0 C \|u_0 - v_0\|_{G^{\sigma,\delta,s}(\mathbb{T})}. \tag{3.9}$$

Proof. To prove Lemma3.7 we need to use Lemma3.1.

□

This completes the prove of Theorem 2.1.

3.3. **Time regularity.**

Lemma 3.8. *(Proposition 3.1, [7]) Let $s > 0$, and let $\delta > 0$, $\sigma \geq 1$, $u \in C\left([0,T]; G^{\sigma,\delta,s}(\mathbb{T})\right)$ be the solution of (1.1). Then $u \in G^{\sigma}$ in $x, \forall t \in [0,T]$, i.e., for some $C > 0$, we have*

$$|\partial_x^l u| \leqslant C^{l+1}(l!)^{\sigma}, l \in \{0,1,...\}, \quad \forall x \in \mathbb{T},\ t \in [0,T]. \tag{3.10}$$

In this section, we shall prove the time regularity of the solution as stated in Theorem 2.1 on the circle. The proof on the line is analogous.

Let us consider as in [3], for $\epsilon > 0$, the sequences

$$m_q = \frac{c(q!)^{\sigma}}{(q+1)^2}, (q = 0,1,2,...), \tag{3.11}$$

and

$$M_q = \epsilon^{1-q} m_q, \epsilon > 0 \ \ and\ (q = 1,2,3,...), \tag{3.12}$$

where c will be chosen (see [1]) so that the next inequality holds

$$\sum_{0 \leq l \leq k} \binom{k}{l} m_l m_{k-l} \leqslant m_k. \tag{3.13}$$

Removing the endpoints 0 and k in the left hand side of (3.13) and using the sequence M_q, we obtain

$$\sum_{0 < l < k} \binom{k}{l} M_l M_{k-l} \leq M_k, \forall \epsilon > 0. \tag{3.14}$$

Next, one can check that for any $\epsilon > 0$ the sequence M_q satisfies the following inequality

$$M_j \leqslant \epsilon M_{j+1},\ for\ j \geq 2. \tag{3.15}$$

Also, one can check that for a given $C > 1$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ we have

$$C^{j+1} j! \leqslant M_j,\ for\ j \geq 2. \tag{3.16}$$

By the definition of M_1 and M_2 in (3.12), we have for $j = 1$, that

$$M_1 = a\epsilon M_2, \quad where \quad a = \frac{9}{4(2!)^{\sigma}},$$

for some $C > 0$. Also, we define the following constants

$$M_0 = \frac{c}{8}\ and\ M = max\{2, \frac{8C}{c}, \frac{4C^2}{c}\}. \tag{3.17}$$

The next lemma is the main idea for the proof of Theorem 2.2.

Lemma 3.9. *Let u be the solution of (1.1) satisfying (3.10), then there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$ we have*

$$|\partial_t^j \partial_x^l u| \leqslant M^{2j+1} M_{l+5j}, j \in \{0,1,2,...\}, l \in \{0,1,2,...\}, \tag{3.18}$$

for all $x \in \mathbb{T},\ t \in [0,T]$.

Proof. (Of Lemma 3.9)
We will prove (3.18) by induction. Let $j = 0$, for $l = 0$, it follows from (3.10) and the definition of M in (3.17) that

$$|u| \leq C \leq M M_0, \quad \forall x \in \mathbb{T},\ t \in [0,T].$$

Similarly, for $l = 1$, we have

$$|\partial_x u| \leq C^2 \leqslant MM_1, \ \forall x \in \mathbb{T}, \ t \in [0,T].$$

By (3.10) and (3.16), for $l \geq 2$, there exists $\epsilon_0 > 0$ such that for any $0 < \epsilon \leqslant \epsilon_0$, we have

$$|\partial_x^l u| \leq C^{l+1}(l!)^\sigma \leq M_l \leq MM_l, \ \forall x \in \mathbb{T}, \ t \in [0,T].$$

This completes the proof of (3.18) for $j = 0$ and $l \in \{0, 1, ...\}$.
Next, we will assume that (3.18) is true for $0 \leq q \leq j$ and $l \in \{0, 1, ...\}$ and we will prove it for $q = j + 1$ and $l \in \{0, 1, ...\}$.
We begin by noting that

$$|\partial_t^{j+1}\partial_x^l u| = |\partial_t^j\partial_x^l(\partial_t u)| \quad \leqslant |\partial_t^j\partial_x^{l+5}u| + |\partial_t^j\partial_x^{l+3}u| + |\partial_t^j\partial_x^{l+1}u| + |\partial_t^j\partial_x^l(\partial_x u^2)|.$$

Using the induction hypotheses and the condition $M > 2$, we estimate the second term $\partial_t^j\partial_x^{l+5}u$, $\partial_t^j\partial_x^{l+3}u$ and $\partial_t^j\partial_x^{l+1}u$ as follows

$$\begin{aligned} |\partial_t^j\partial_x^{l+5}u| &\leqslant M^{2j+1}M_{l+5+5j} = M^{-2}M^{2(j+1)+1}M_{l+5(j+1)}, \\ &\leqslant \frac{1}{4}M^{2(j+1)+1}M_{l+5(j+1)}, \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} |\partial_t^j\partial_x^{l+3}u| &\leqslant M^{2j+1}M_{l+3+5j} = M^{-2}M^{2(j+1)+1}M_{l+5j+3)}, \\ &\leqslant \frac{\epsilon^2}{4}M^{2(j+1)+1}M_{l+5(j+1)}, \end{aligned} \tag{3.20}$$

and

$$|\partial_t^j\partial_x^{l+1}u| \quad \leqslant M^{2j+1}M_{l+1+5j} \leqslant \frac{\epsilon^4}{4}M^{2(j+1)+1}M_{l+5(j+1)}. \tag{3.21}$$

All this estimates are taken for the linear terms. For the nonlinear term $(\partial_x u^2)$, using Leibniz's rule twice and the induction hypothesis, we have a different cases. We need the next results.

Lemma 3.10. *([3]) Given $n, k \in \{0, 1, 2, ...\}$ we have*

$$\sum_{p=0}^{n}\sum_{q=0}^{k}\binom{n}{p}\binom{k}{q}L_{(n-p)+5(k-q)}L_{p+5q} \leqslant \sum_{r=1}^{m}\binom{m}{r}L_rL_{m-r}, \tag{3.22}$$

where $L_j, j = 0, 1, ..., m$ positive real numbers with $m = n + 5k$

$$\begin{aligned} |\partial_t^j\partial_x^{l+1}(u^2)| &\leqslant \sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}|\partial_t^{j-q}\partial_x^{l+1-p}u||\partial_t^q\partial_x^p u|, \\ &\leqslant \sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M^{2(j-q)+1}M_{l+1-p+5(j-q)}M^{2q+1}M_{p+5q}, \\ &= M^{2(j+1)}\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M_{l+1-p+5(j-q)}M_{p+5q}. \end{aligned}$$

Next, using Lemma 3.10, with $n = l+1, k = j, L_j = M_j, m = l+1+5j$, we obtain

$$\sum_{p=0}^{l+1}\sum_{q=0}^{j}\binom{l+1}{p}\binom{j}{p}M_{l+1-p+5(j-q)}M_{p+5q},$$

$$\leqslant \sum_{r=1}^{m}\binom{m}{r}L_rL_{m-r} \le (M_0+\epsilon)M_m,$$

$$= (M_0+\epsilon)M_{l+5j+1},$$

then

$$\begin{aligned}|\partial_t^j\partial_x^{l+1}(u^2)| &\leqslant M^{2(j+1)}(M_0+\epsilon)M_{l+5j+1},\\ &\leqslant M^{-2}M^{2(j+1)+1}\epsilon^4(M_0+\epsilon)M_{l+5(j+1)},\\ &\leqslant \frac{\epsilon^4(M_0+\epsilon)}{4}M^{2(j+1)+1}M_{l+5(j+1)}.\end{aligned}$$

Noting that in the last inequality we have used the fact that $l+5j+1 \ge 2$, since we are assuming that either $j \neq 0$ or $l \neq 0$.

Now, choosing $\epsilon \le \epsilon_0 = \left(\frac{1}{(M_0+\epsilon)}\right)^{\frac{1}{4}} < 1$ to obtain

$$\epsilon^4(M_0+\epsilon) \le \epsilon^4(M_0+1) \le (M_0+1)\left(\frac{1}{(M_0+1)}\right) = 1.$$

Hence,

$$|\partial_t^j\partial_x^{l+1}(u^2)| \le \frac{1}{4}M^{2(j+1)+1}M_{l+5(j+1)}. \tag{3.23}$$

Which completes the proof. □

Proof. (Of Theorem 2.2) By Lemma 3.9, we have

$$|\partial_t^j\partial_x^l u| \leqslant M^{2j+1}M_{l+5j},\ j \in \{0,1,2,...\},\ l \in \{0,1,2,...\},$$

where

$$M_q = \epsilon^{1-q}\frac{c(q!)^\sigma}{(q+1)^2},\ q = 1,2,....$$

Applying this inequality for $j \in \{1,2,...\}$ and $l = 0$ gives

$$\begin{aligned}|\partial_t^j u| &\leqslant M^{2j+1}M_{5j} = MM^{2j}\epsilon^{1-5j}\frac{c((5j)!)^\sigma}{(5j+1)^2},\\ &\leqslant M\epsilon c\left(\frac{M^2}{\epsilon^5}\right)^j((5j)!)^\sigma,\\ &\leqslant L_0L^j((5j)!)^\sigma,\\ &\leqslant L_0L^jA^{5\sigma j}((j!)^5)^\sigma,\\ &\leqslant A_0^{j+1}(j!)^{5\sigma},\end{aligned} \tag{3.24}$$

where $L_0 = M\epsilon c,\ L = \dfrac{M^2}{\epsilon^5}$ since $(5j)! \leqslant A^{5j}(j!)^5$ for $A > 0$ and $A_0 = max\{L_0, LA^{5\sigma}\}$. We also have from (3.18) for $l = j = 0$, that

$$|u| \leq MM_0 = M\frac{c}{8}, \quad \forall x \in \mathbb{T},\ t \in [0,T]. \tag{3.25}$$

Setting $C = max\{M\frac{c}{8}, A_0\}$, it follows from (3.24) and (3.25) that for $j \in \{0,1,2,...\}$, we have

$$|\partial_t^j u| \leqslant C^{j+1}(j!)^{5\sigma}, \quad \forall x \in \mathbb{T},\ t \in [0,T].$$

Hence, $u \in G^{5\sigma}$ in t.
Which completes the proof of Theorem 2.2. □

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