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Research Article

On Spacelike $(1,3)$-Bertrand Curves in E_2^4

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Abstract
In this paper, it is proved that no special spacelike Frenet curve is a Bertrand curve in E_2^4. Therefore, a generalization of spacelike Bertrand curve is defined and this is called as spacelike $(1,3)$-Bertrand curve in E_2^4. Moreover, the characterizations of spacelike $(1,3)$-Bertrand curves are given in E_2^4.

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1. Introduction

The characterization of a regular curve is one of the important and interesting problems in the theory of curves. The Bertrand curves found by J. Bertrand in 1850 have an important place in differential geometry. These curves are specific examples of parallel (offset) curves that have an important place in computer-aided design and computer-aided manufacturing [1]. In E^3, a C^∞-special Frenet curve γ is called a Bertrand curve if there exists another C^∞-special Frenet curve γ^*, such that the principal normal vector fields of γ and γ^* coincide at the corresponding points [2], [3].

There are many important papers on the Bertrand curves [4], [5]. Izumiya and Takeuchi proved in their work that Bertrand curves can be obtained from spherical curves in E^3 [6]. The Bertrand curves corresponding to constant parameter curves of constant slope surfaces are investigated [7]. When we investigated the properties of the Bertrand curves in E^n, it is easy to see that either κ_2 or κ_3 is zero which means that Bertrand curves in E^n are degenerate curves [3]. This result was restated by Matsuda and Yorozu [2]. They proved that there were not any special Bertrand curves in E^n and defined a new kind, which is called $(1,3)$-Bertrand curves in 4-dimensional Euclidean space. Uçum et al. examined the $(1,3)$-Bertrand curves concerning the casual character of the plane spanned by $\{N(s),\ B_2(s)\}$ in E_1^4 [8].

In this paper, we proved that no special spacelike Frenet curve is a Bertrand curve in E_2^4. Additionally, we gave the characterizations of spacelike $(1,3)$-Bertrand curve in E_2^4.

2. Preliminaries

To meet the requirements in the upcoming sections, the basic elements of the theory of curves in the semi Euclidean space E_2^4 are briefly presented in this section. A more complete elementary information can be found in [9].

The semi-Euclidean space E_2^4 is an Euclidean space provided with standard flat metric given by

$$g = -da_1^2 - da_2^2 + da_3^2 + da_4^2,$$

where (a_1, a_2, a_3, a_4) is a rectangular coordinate system of the E_2^4. A vector w in E_2^4 is called a spacelike, timelike or null (lightlike) if hold $g(w,w) > 0$, $g(w,w) < 0$ or $g(w,w) = 0$ and $w \neq 0$, respectively. The norm of a vector w is given by $\|w\| = \sqrt{|g(w,w)|}$. Therefore, w is a unit vector if $g(w,w) = \pm 1$. Similarly, an arbitrary curve $\gamma = \gamma(s)$ in E_2^4 can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\gamma'(s)$ are spacelike, timelike or null (lightlike), respectively. The velocity of the curve γ is given by $\|\gamma'\|$. Thus, a spacelike curve γ is said to be parametrized by arc length function s if $g(\gamma',\gamma')=1$. Two vectors u and w in E_1^4 are said to be orthogonal if $g(u,w)=0$ [9]. Also,

i. Let us assume that u and w are spacelike vectors, then
   - if they span a spacelike plane, there is a unique number $0\le\theta\le\pi$ such that $g(u,w)=\|u\|\,\|w\|\cos\theta$.
   - if they span a timelike plane, there is a unique number $\theta\ge 0$ such that $g(u,w)=\varepsilon\,\|u\|\,\|w\|\cosh\theta$, where $\varepsilon=+1$ or $\varepsilon=-1$ according to $sgn(u_2)=sgn(w_2)$ or $sgn(u_2)\neq sgn(w_2)$, respectively.

ii. Let us assume that u and w are timelike vectors, then there is a unique number $\theta\ge 0$ such that $g(u,w)=\varepsilon\,\|u\|\,\|w\|\cosh\theta$, where $\varepsilon=+1$ or $\varepsilon=-1$ according to u and w have different time-orientation or the same time-orientation, respectively.

iii. Let us assume that u is spacelike and w is timelike, then there is a unique number $\theta\ge 0$ such that $g(u,w)=\varepsilon\,\|u\|\,\|w\|\sinh\theta$, where $\varepsilon=+1$ or $\varepsilon=-1$ according to $sgn(u_2)=sgn(w_1)$ or $sgn(u_2)\neq sgn(w_1)$, respectively. The corresponding number θ given above will be called simply the angle between u and w [10].

Let $\{T(s),N(s),B_1(s),B_2(s)\}$ denotes the moving Frenet frame along γ in the semi-Euclidean space E_2^4, then $T(s)$, $N(s)$, $B_1(s)$ and $B_2(s)$ are called the tangent, the principal normal, the first binormal, and the second binormal vector fields of γ, respectively.

A unit speed curve γ is said to be a Frenet curve if $g(\gamma'',\gamma'')\neq 0$. Let γ be a C^∞ special spacelike Frenet curve with spacelike principal normal, timelike both first binormal and second binormal vector fields in E_2^4, parametrized by arc length function s. Moreover, non-zero C^∞ scalar functions κ_1, κ_2 and κ_3 be the first, second, and third curvatures of γ, respectively. Then for the C^∞ special spacelike Frenet curve γ, the Frenet formula is given by

$$\begin{aligned} T' &= \kappa_1 N \\ N' &= -\kappa_1 T+\kappa_2 B_1 \\ B_1' &= \kappa_2 N+\kappa_3 B_2 \\ B_2' &= -\kappa_3 B_1, \end{aligned} \tag{1}$$

where T, N, B_1 and B_2 mutually orthogonal vector fields satisfying

$$g(T,T)=g(N,N)=1,\ g(B_1,B_1)=g(B_2,B_2)=-1. \tag{2}$$

Let γ be a C^∞ special timelike Frenet curve with timelike principal normal, spacelike both first binormal and second binormal vector fields in E_2^4, parametrized by arc length function s. Then for the C^∞ special timelike Frenet curve γ, the Frenet formula is given by

$$\begin{aligned} T' &= -\kappa_1 N \\ N' &= \kappa_1 T+\kappa_2 B_1 \\ B_1' &= \kappa_2 N+\kappa_3 B_2 \\ B_2' &= -\kappa_3 B_1, \end{aligned} \tag{3}$$

where T, N, B_1 and B_2 mutually orthogonal vector fields satisfying

$$g(T,T)=g(N,N)=-1,\ g(B_1,B_1)=g(B_2,B_2)=1 \tag{4}$$

(for the semi-Euclidean space E_v^{n+1}, see [11], [12]).

Definition 1. *Let (I,α), (I,β) be coordinate neighbourhoods of the curves α, $\beta\in E^n$. Let $\{V_1(s),\ldots,V_r(s)\}$, $\{V_1^*(s),\ldots,V_r^*(s)\}$ be the Frenet r-frame at the points $\alpha(s)$ and $\beta(s)$ $(s\in I)$. If $V_2(s)$, $V_2^*(s)$ are linearly dependent for $\forall s\in I$, the curve pair (α,β) is called a Bertrand curve pair [13].*

3. Spacelike Bertrand curves in E_2^4

The following two theorems related to Bertrand curves in E_1^2 and E_1^3 are well known.

Theorem 2. *In E_1^2, every spacelike C^∞-planar curve is a Bertrand curve [14].*

Theorem 3. *In E_1^3, a C^∞-special spacelike Frenet curve with first and second curvatures κ_1 and κ_2 is a spacelike Bertrand curve if and only if there exists a linear relation $a\kappa_1 + b\kappa_2 = 1$, for all $s \in L$, where a, b are nonzero constant real numbers [14], [15].*

Now, let us investigate Bertrand curves in E_2^4.

Definition 4. *A C^∞-special spacelike Frenet curve $\gamma : L \to E_2^4$ is called spacelike Bertrand curve if there exists an another C^∞-special Frenet curve $\gamma^* : L^* \to E_2^4$, distinct from γ, and a regular C^∞-map $\varphi : L \to L^*$, ($s^* = \varphi(s)$, $\frac{d\varphi}{ds} \neq 0$, for all $s \in L$), such that curve has the same 1-normal line at each pair of corresponding points $\gamma(s)$ and $\gamma^*(s^*) = \gamma^*(\varphi(s))$ under φ. Here, s and s^* are arc length parameters of the curves γ and γ^*, respectively. In this case, γ^* is called a Bertrand mate of the spacelike curve γ.*

Definition 5. *Let the curve γ^* with the Frenet vector fields $\{T^*, N^*, B_1^*, B_2^*\}$ be a Bertrand mate of curve γ with the Frenet vector fields $\{T, N, B_1, B_2\}$. There are two possibilities for the Bertrand mate γ^* of the spacelike curve γ;*

1. The Bertrand mate γ^* of the spacelike curve γ is also spacelike. Thus the vector fields are related by

   $$T^*(s^*) = \varepsilon(T(s)\cosh\theta + B_1 \sinh\theta),$$

   since the plane spanned by T and T^* will be timelike according to the frame (1).

2. The Bertrand mate γ^* of the spacelike curve γ is timelike. Thus the vector fields are related by

   $$T^*(s^*) = \varepsilon(T(s)\sinh\theta + B_1 \cosh\theta).$$

Theorem 6. *In E_2^4, no C^∞-special spacelike Frenet curve is a Bertrand curve.*

Proof. Let γ^* be a mate of Bertrand curve γ in E_2^4. Also, the pair of $\gamma(s)$ and $\gamma^*(s^*)$ be the corresponding points of γ and γ^*, respectively. Then for all $s \in L$ the curve γ^* is given by

$$\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + \alpha(s)N(s), \tag{5}$$

where α is C^∞-function on L. By differentiating the equation (5) with respect to s, then

$$\varphi'(s)\frac{d(\gamma^*(s^*))}{ds^*} = \gamma'(s) + \alpha'(s)N(s) + \alpha(s)N'(s)$$

is obtained. Here and hereafter, the subscript prime denotes the differentiation with respect to s. By using the Frenet formulas, it is seen that

$$\varphi'(s)T^*(s^*) = [1 - \alpha(s)\kappa_1(s)]T(s) + \alpha'(s)N(s) + \alpha(s)\kappa_2(s)B_1(s).$$

Considering $N(s)$ and $N^*(\varphi(s))$ are coincident and $g(T^*(\varphi(s)), N(s)) = 0$ for all $s \in L$, we get

$$\alpha'(s) = 0$$

that is, α is a constant function on L. Thus, the differentiation of the equation (5) with respect to s is

$$\varphi'(s)T^*(s^*) = [1 - \alpha\kappa_1(s)]T(s) + \alpha\kappa_2(s)B_1(s). \tag{6}$$

By the fact that γ and γ^* are spacelike curves, the tangent vector field of Bertrand mate of γ can be given by

$$T^*(\varphi(s)) = T^*(s^*) = \varepsilon(T(s)\cosh\theta + B_1 \sinh\theta), \tag{7}$$

where θ is a hyperbolic angle between the spacelike tangent vector fields $T^*(s^*)$ and $T(s)$. According to the equations (6) and (7), the hyperbolic functions are defined by

$$\cosh\theta = \frac{1 - \alpha\kappa_1(s)}{\varepsilon\varphi'(s)}, \ \sinh(\theta) = \frac{\alpha\kappa_2(s)}{\varepsilon\varphi'(s)}. \tag{8}$$

By differentiating the equation (7) and applying Frenet formulas,

$$\varphi'(s)\frac{d(T^*(s^*))}{ds^*} \quad = \quad \varepsilon[\frac{d(\cosh\theta(s))}{ds}T + (\kappa_1 \cosh\theta(s) + \kappa_2 \sinh\theta(s))N + \frac{d(\sinh\theta(s))}{ds}B_1 + \kappa_3 \sinh\theta(s)B_2]$$

is obtained. Since $N(s)$ is coincident with $N^*(s^*)$, from the above equation, it is seen that

$$\kappa_3 \sinh\theta(s) = 0.$$

If we notice that κ_3 is different from zero, then $\sinh\theta(s) = 0$. Considering the equations (8) and $\kappa_2(s) \neq 0$, then $\alpha(s) = 0$. In that time the equation (5) implies that γ^* is coincident with γ. This is a contradiction. ■

By the fact that γ is spacelike curve and γ^* is timelike curve, the tangent vector field of Bertrand mate of γ can be given by

$$T^*(s^*) = \varepsilon(T(s)\sinh\theta + B_1\cosh\theta). \tag{9}$$

According to the equations (6) and (9), the hyperbolic functions are defined by

$$\sinh\theta = \frac{1-\alpha\kappa_1(s)}{\varepsilon\varphi'(s)}, \cosh(\theta) = \frac{\alpha\kappa_2(s)}{\varepsilon\varphi'(s)}. \tag{10}$$

By differentiating the equation (9) and applying Frenet formulas,

$$\varphi'(s)\frac{d(T^*(s^*))}{ds^*} = \varepsilon[\frac{d(\sinh\theta(s))}{ds}T + (\kappa_1\sinh\theta(s) + \kappa_2\cosh\theta(s))N + \frac{d(\cosh\theta(s))}{ds}B_1 + \kappa_3\cosh\theta(s)B_2] \tag{11}$$

is obtained. Since $N(s)$ is coincident with $N^*(s^*)$, from the equation (11), it is seen that

$$\kappa_3\cosh\theta(s) = 0.$$

If we notice that κ_3 is different from zero, then $\cosh\theta(s) = 0$. Considering the equation (10) and $\kappa_2(s) \neq 0$, then $\alpha(s) = 0$. In that time the equation (5) implies that γ^* is coincident with γ. This is a contradiction. So, the proof is completed.

4. Spacelike $(1,3)$-Bertrand curves in E_2^4

In this section, we introduce the concept of spacelike $(1,3)$-Bertrand curve in E_2^4.

Definition 7. *Let $\gamma : I \subseteq R \to E_2^4$ be a C^∞-special Frenet curve. The plane spanned by the principal normal vector $N(s)$ and the second binormal vector $B_2(s)$ is called the $(1,3)$-normal plane of γ at the point $s \in I$.*

Definition 8. *Let $\gamma : I \subseteq R \to E_2^4$ and $\gamma^* : I^* \subseteq R \to E_2^4$ be C^∞-special Frenet curves. If the Frenet $(1,3)$-normal plane of γ coincides with the $(1,3)$-normal plane of γ^* at corresponding points, then γ is called a $(1,3)$-Bertrand curve and γ^* is called the $(1,3)$-Bertrand mate curve of γ.*

Let $\gamma : I \subseteq R \to E_2^4$ be a spacelike $(1,3)$-Bertrand curve with the Frenet frame $\{T,N,B_1,B_2\}$ and the curvatures $\kappa_1,\kappa_2,\kappa_3$ and $\gamma^* : I^* \subseteq R \to E_2^4$ be a $(1,3)$-Bertrand mate curve of γ with the Frenet frame $\{T^*,N^*,B_1^*,B_2^*\}$ and the curvatures $\kappa_1^*,\kappa_2^*,\kappa_3^*$.

Theorem 9. *Let γ be a C^∞-special spacelike Frenet curve with non-zero curvatures κ_1, κ_2, κ_3 in E_2^4. Then γ is a spacelike $(1,3)$-Bertrand curve whose the Bertrand mate γ^* is also spacelike, if and only if there exists the constant real numbers α, β, μ, δ satisfying*

1. $\alpha\kappa_2(s) - \beta\kappa_3(s) \neq 0$,
2. $\mu[\alpha\kappa_2(s) - \beta\kappa_3(s)] + \alpha\kappa_1(s) = 1, \mu = \cosh\theta_0(s)(\sinh\theta_0(s))^{-1}$,
3. $\delta\kappa_3(s) = \mu\kappa_1(s) + \kappa_2(s), \delta = \cosh\phi_0(s)(\sinh\phi_0(s))^{-1}$,
4. $(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + \kappa_1\kappa_2(\mu^2+1) \neq 0$, *for all $s \in I$, where the plane $sp(T,T^*)$ and the plane $sp(N,N^*)$ are timelike.*

Proof. Assume that γ is a spacelike $(1,3)$-Bertrand curve parametrized by arc-length s and with nonzero curvature functions $\kappa_1,\kappa_2,\kappa_3$ and the curve γ^* is the $(1,3)$-Bertrand mate curve of the curve γ, with arc-length s^*. Then the timelike plane spanned by $\{N(s),B_2(s)\}$ coincides with the plane spanned by $\{N^*(s^*),B_2^*(s^*)\}$. Since $\{N(s),B_2(s)\} = \{N^*(s^*),B_2^*(s^*)\}$, $\{N^*(s),B_2^*(s)\}$ is a timelike plane and γ^* can be a spacelike or timelike curve with timelike $(1,3)$-normal plane. Then we can write the curve γ^* as follows:

$$\gamma^*(s^*) = \gamma^*(\varphi(s)) = \gamma(s) + \alpha(s)N(s) + \beta(s)B_2(s) \tag{12}$$

for all $s^* \in I^*, s \in I$, where $\alpha(s)$ and $\beta(s)$ are C^∞-functions on I. Differentiating (12) with respect to s and using the Frenet formula (1), we get

$$\varphi'(s)T^*(s^*) = [1-\alpha\kappa_1]T(s) + \alpha'(s)N(s) + [\alpha\kappa_2 - \beta\kappa_3]B_1(s) + \beta'(s)B_2(s). \tag{13}$$

Multiplying equation (13) by $N(s)$ and $B_2(s)$, respectively, we have

$$\alpha'(s) = 0, \beta'(s) = 0,$$

that is, α and β are constant functions on I. Then, we find

$$\varphi'(s)T^*(s^*) = [1-\alpha\kappa_1(s)]T(s) + [\alpha\kappa_2(s) - \beta\kappa_3(s)]B_1(s) \tag{14}$$

from the equality (13). Since $T(s)$ is spacelike and $T^*(s^*)$ is spacelike or timelike, then

$$\pm(\varphi'(s))^2 = [1-\alpha\kappa_1(s)]^2 - [\alpha\kappa_2(s) - \beta\kappa_3(s)]^2. \tag{15}$$

Also, if γ^* is a spacelike curve and the plane spanned by $\{T, T^*\}$ is timelike, then we can write

$$\begin{aligned} T^*(s^*) &= \varepsilon(\cosh\theta(s)T(s) + \sinh\theta(s)B_1(s)) \\ \cosh\theta(s) &= \frac{1-\alpha\kappa_1(s)}{\varepsilon\varphi'(s)}, \sinh\theta(s) = \frac{\alpha\kappa_2(s)-\beta\kappa_3(s)}{\varepsilon\varphi'(s)}, \end{aligned} \tag{16}$$

where θ is a hyperbolic angle between the tangent vector fields $T(s)$ and $T^*(s^*)$ of γ and γ^*. By differentiating the equation (16) with respect to and applying Frenet formulas,

$$\varphi'(s)\kappa_1^* N^* = \varepsilon\frac{d(\cosh\theta(s))}{ds}T + \varepsilon[\kappa_1\cosh\theta(s) + \kappa_2\sinh\theta(s)]N + \varepsilon\frac{d(\sinh\theta(s))}{ds}B_1 + \varepsilon\kappa_3\sinh\theta(s)B_2$$

is obtained. Since $N^*(s^*)$ is a linear combination $N(s)$ and $B_2(s)$, it easily seen that

$$\frac{d(\cosh\theta(s))}{ds} = 0, \frac{d(\sinh\theta(s))}{ds} = 0$$

that is, θ is a constant function on I with value θ_0. Thus, we rewrite the equation (16) as

$$T^*(s^*) = \varepsilon(\cosh\theta_0(s)T(s) + \sinh\theta_0(s)B_1(s)) \tag{17}$$

and

$$\varepsilon\varphi'(s)\cosh\theta_0(s) = 1-\alpha\kappa_1(s) \tag{18}$$

$$\varepsilon\varphi'(s)\sinh\theta_0(s) = \alpha\kappa_2(s) - \beta\kappa_3(s) \tag{19}$$

for all $s \in I$. According to these last two equations, it is seen that

$$(1-\alpha\kappa_1(s))\sinh\theta_0(s) = (\alpha\kappa_2(s) - \beta\kappa_3(s))\cosh\theta_0(s) \tag{20}$$

If $\sinh\theta_0(s) = 0$, then it satisfies $\cosh\theta_0 = 1$ and $T^*(s^*) = T(s)$. The differentiation of this equality with respect to s is

$$\varphi'(s)\kappa_1^*(s^*)N^*(s^*) = \kappa_1(s)N(s),$$

that is, $N(s)$ is linear dependence with $N^*(s)$. According to Theorem 6 this is a contradiction. Thus, only the case of $\sinh\theta_0(s) \neq 0$ must be considered. The equation (20) satisfies

$$\alpha\kappa_2(s) - \beta\kappa_3(s) \neq 0$$

that is, the relation given in the first clause of the theorem is proved. ■

Since $\sinh\theta_0(s) \neq 0$, the equation (20) can be rewritten as

$$1 = \frac{\cosh\theta_0(s)}{\sinh\theta_0(s)}(\alpha\kappa_2(s) - \beta\kappa_3(s)) + \alpha\kappa_1(s).$$

Let us denote the constant value $\mu = \cosh\theta_0(s)(\sinh\theta_0(s))^{-1}$ by the constant real number μ, then μ is an element of interval $(-\infty,-1)\cup(1,\infty)$ and

$$\mu(\alpha\kappa_2(s) - \beta\kappa_3(s)) + \alpha\kappa_1(s) = 1.$$

This proves the relation given in the second clause of the theorem.

By differentiating the equation (17) with respect to s and applying Frenet formulas, we have

$$\varphi'(s)\kappa_1^*(s^*)N^*(s^*) = \varepsilon[(\cosh\theta_0(s)\kappa_1 + \sinh\theta_0(s)\kappa_2)N + \sinh\theta_0(s)\kappa_3 B_2]$$

for all $s \in I$. Taking into consideration the equations (18), (19) and the second clause of the theorem, the above equality satisfy

$$(\varphi'(s)\kappa_1^*(s^*))^2 = (\alpha\kappa_2(s) - \beta\kappa_3(s))^2[(\mu\kappa_1(s) + \kappa_2(s))^2 - \kappa_3^2(s)](\varphi'(s))^{-2}.$$

From the equation (15) and the second clause of the theorem, we get

$$(\varphi'(s))^2 = (\mu^2 - 1)[\alpha\kappa_2(s) - \beta\kappa_3(s)]^2.$$

Thus, we obtain

$$(\varphi'(s)\kappa_1^*(s^*))^2 = \frac{1}{\mu^2 - 1}[(\mu\kappa_1(s) + \kappa_2(s))^2 - \kappa_3^2(s)]. \tag{21}$$

On the other hand, since the vector field N^* is spacelike, if the plane spanned by $\{N, N^*\}$ is timelike, we can give

$$N^*(s^*) = \varepsilon(\cosh\phi(s)N(s) + \sinh\phi(s)B_2(s)). \tag{22}$$

From the equations (18), (19) and the second clause of the theorem, we obtain

$$\cosh\phi(s) = \frac{[\alpha\kappa_2(s) - \beta\kappa_3(s)](\mu\kappa_1(s) + \kappa_2(s))}{\varepsilon(\varphi'(s))^2\kappa_1^*(s^*)}, \tag{23}$$

$$\sinh\phi(s) = \frac{[\alpha\kappa_2(s) - \beta\kappa_3(s)]\kappa_3(s)}{\varepsilon(\varphi'(s))^2\kappa_1^*(s^*)}, \tag{24}$$

for all $s \in I$ and $\phi(s) \in C^\infty$-function on I. By differentiating the equation (22) with respect to s and applying Frenet formulas, we have

$$\frac{d(\cosh\phi(s))}{ds} = 0, \frac{d(\sinh\phi(s))}{ds} = 0$$

that is, $\phi(s)$ is a constant function on I with value ϕ_0. Let us denote $\delta = \cosh\phi_0(s)(\sinh\phi_0(s))^{-1}$ by the constant real number δ, then $\delta \in (-\infty, -1) \cup (1, \infty)$. The ratio of (23) and (24) holds

$$\delta = \frac{\mu\kappa_1(s) + \kappa_2(s)}{\kappa_3(s)},$$

that is, $\delta\kappa_3(s) = \mu\kappa_1(s) + \kappa_2(s)$ for all $s \in I$. Thus the third clause of the theorem is obtained. Moreover, we can give

$$\varphi'(s)\kappa_2^*(s^*)B_1^*(s^*) \quad = \quad \varepsilon[-\cosh\phi_0(s)\kappa_1 T + (\cosh\phi_0(s)\kappa_2 - \sinh\phi_0(s)\kappa_3)B_1] + \varphi'(s)\kappa_1^*(s^*)T^*.$$

If we substitute the equations (14), (23) and (24) into the above equality, we obtain

$$\varphi'(s)\kappa_2^*(s^*)B_1^*(s) = (\varphi'(s))^{-2}(\kappa_1^*(s^*))^{-1}[D(s)T + E(s)B_1],$$

where

$$\begin{aligned} D(s) &= (\varphi'(s)\kappa_1^*(s^*))^2(1 - \alpha\kappa_1) - (\alpha\kappa_2 - \beta\kappa_3)(\mu\kappa_1 + \kappa_2)\kappa_1 \\ E(s) &= [(\varphi'(s)\kappa_1^*(s^*))^2 + \mu\kappa_1\kappa_2 + \kappa_2^2 - \kappa_3^2](\alpha\kappa_2 - \beta\kappa_3). \end{aligned}$$

for all $s \in I$. By the second clause of the theorem and the equation (21), $D(s)$ and $E(s)$ can be rewritten as;

$$\begin{aligned} D(s) &= (\mu^2 - 1)^{-1}(\alpha\kappa_2 - \beta\kappa_3)[(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + \kappa_1\kappa_2(\mu^2 + 1)] \\ E(s) &= (\mu^2 - 1)^{-1}(\alpha\kappa_2 - \beta\kappa_3)\mu[(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + \kappa_1\kappa_2(\mu^2 + 1)]. \end{aligned}$$

By the fact that $\varphi'(s)\kappa_2^*(s^*)B_1^*(s) \neq 0$ for all $s \in I$, it is proved that

$$(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + \kappa_1\kappa_2(\mu^2 + 1) \neq 0.$$

This is the last clause of the theorem.

Now, we will prove the sufficient condition of the theorem.

Thus, we assume that γ is a spacelike C^∞-special Frenet curve in E_2^4 with curvatures $\kappa_1, \kappa_2, \kappa_3$ satisfying the all clause of the theorem for the constant real numbers α, β, μ, δ. We define a spacelike curve γ^* by

$$\gamma^*(s^*) = \gamma(s) + \alpha N(s) + \beta B_2(s), \tag{25}$$

where s is the arc length parameter of γ. By differentiating this equation with respect to s and applying Frenet formulas,

$$\varphi'(s)T^*(s^*) = [1 - \alpha\kappa_1(s)]T(s) + [\alpha\kappa_2(s) - \beta\kappa_3(s)]B_1(s)$$

is obtained. Considering the second clause of the theorem, this equation is rewritten as;

$$\varphi'(s)T^*(s^*) = [\alpha\kappa_2(s) - \beta\kappa_3(s)](\mu T(s) + B_1(s))$$

for all $s \in I$. From the first clause of the theorem it is seen that γ^* is regular curve. Thus, arc length parameter of γ^* denoted by s^* can be given by

$$s^* = \varphi(s) = \int_0^s \left\| \frac{d\gamma^*}{dt} \right\| dt,$$

where $\varphi : I \to I$ is a regular map. The differentiation of φ with respect to s is

$$\varphi'(s) = \sqrt{\mu^2 - 1}[\alpha\kappa_2(s) - \beta\kappa_3(s)].$$

Also, here we notice that $\mu \in (-\infty, -1) \cup (1, \infty)$ and $\mu^2 - 1 > 0$. Differentiating the equation (25) with respect to s, we get

$$\varphi'(s)\frac{d\gamma^*(s^*)}{ds^*} = [\alpha\kappa_2(s) - \beta\kappa_3(s)](\mu T(s) + B_1(s)).$$

Now, let us define a unit vector field T^* along γ^* by $\frac{d\gamma^*(s^*)}{ds^*}$, then

$$T^*(\varphi(s)) = (\mu^2 - 1)^{-\frac{1}{2}}(\mu T(s) + B_1(s)). \tag{26}$$

By differentiating this equation with respect to s and using Frenet formulas,

$$\varphi'(s)\frac{dT^*(\varphi(s))}{ds^*} = (\mu^2 - 1)^{-\frac{1}{2}}[(\mu\kappa_1(s) + \kappa_2(s))N(s) + \kappa_3(s)B_2(s)]$$

and

$$\left\| \frac{dT^*(\varphi(s))}{ds^*} \right\| = \frac{\sqrt{\left|(\mu\kappa_1(s) + \kappa_2(s))^2 - \kappa_3^2(s)\right|}}{\varphi'(s)\sqrt{\mu^2 - 1}}.$$

By the third clause of the theorem, it is seen that

$$\sqrt{\left|(\mu\kappa_1(s) + \kappa_2(s))^2 - \kappa_3^2(s)\right|} = \sqrt{|\delta^2 - 1|\,\kappa_3^2}$$

and we notice that $\delta \in (-\infty, -1) \cup (1, \infty)$ and $\mu^2 - 1 > 0$. Thus, we can write

$$\left\| \frac{dT^*(\varphi(s))}{ds^*} \right\| = \frac{\sqrt{(\delta^2 - 1)\kappa_3^2}}{\varphi'(s)\sqrt{\mu^2 - 1}}.$$

Since $\kappa_3 > 0$ and $\varphi(s) > 0$ for all $s \in I$, we obtain

$$\kappa_1^*(s^*) = \left\| \frac{dT^*(\varphi(s))}{ds^*} \right\| > 0. \tag{27}$$

Thus, $N^*(s^*)$ spacelike unit vector field can be defined by

$$\begin{aligned} N^*(s^*) &= N^*(\varphi(s)) = \frac{1}{\kappa_1^*(\varphi(s))} T^{*\prime}(\varphi(s)) \\ &= \frac{(\mu\kappa_1(s)+\kappa_2(s))N(s)+\kappa_3(s)B_2(s)}{\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}}. \end{aligned} \tag{28}$$

Also,

$$N^*(s^*) = \varepsilon(\cosh\xi(s)N(s)+\sinh\xi(s)B_2(s)) \tag{29}$$

$$\cosh\xi(s) = \frac{\mu\kappa_1(s)+\kappa_2(s)}{\varepsilon\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}}$$

$$\sinh\xi(s) = \frac{\kappa_3(s)}{\varepsilon\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}}$$

for all $s \in I$. Here ξ is a C^∞-function on I. By differentiating the equation (29) with respect to s and using the Frenet formulas, we get

$$\varphi'(s)\frac{d(N^*(\varphi(s))}{ds^*} = \varepsilon[-\kappa_1\cosh\xi(s)T + \frac{d(\cosh\xi(s))}{ds}N + ((\kappa_2\cosh\xi(s)-\kappa_3\sinh\xi(s))B_1 + \frac{d(\sinh\xi(s))}{ds}B_2].$$

The differentiation of the third clause of the theorem with respect to s is

$$(\mu\kappa_1'(s)+\kappa_2'(s))\kappa_3(s)-(\mu\kappa_1(s)+\kappa_2(s))\kappa_3'(s)=0.$$

Substituting this equation, we get

$$\frac{d(\cosh\xi(s))}{ds}=0,\ \frac{d(\sinh\xi(s))}{ds}=0$$

that is, ξ is a constant function on I with value ξ_0. Thus, we write

$$\cosh\xi_0(s) = \frac{\mu\kappa_1(s)+\kappa_2(s)}{\varepsilon\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}}, \tag{30}$$

$$\sinh\xi_0(s) = \frac{\kappa_3(s)}{\varepsilon\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}}. \tag{31}$$

Also, we get

$$\varphi'(s)\frac{d(N^*(\varphi(s))}{ds^*} = \varepsilon[-\kappa_1\cosh\xi_0(s)T + (\kappa_2\cosh\xi_0(s)-\kappa_3\sinh\xi_0(s))B_1]. \tag{32}$$

Then, from the equation (29), it satisfies

$$N^*(s^*) = \varepsilon(\cosh\xi_0(s)N(s)+\sinh\xi_0(s)B_2(s)). \tag{33}$$

By considering the equations (26) and (27), we obtain

$$\kappa_1^*(s^*)T^*(\varphi(s)) = (\mu T(s)+B_1(s))\frac{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}{\varphi'(s)(\mu^2-1)\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2-\kappa_3^2(s)}}.$$

Also, by substituting the equations (30) and (31) into equation (32), we get

$$\frac{d(N^*(\varphi(s))}{ds^*} = -\frac{(\mu\kappa_1+\kappa_2)\kappa_1}{\varphi'(s)\sqrt{(\mu\kappa_1+\kappa_2)^2-\kappa_3^2}}T + \frac{(\mu\kappa_1+\kappa_2)\kappa_2-\kappa_3^2}{\varphi'(s)\sqrt{(\mu\kappa_1+\kappa_2)^2-\kappa_3^2}}B_1,$$

for $s \in I$. By the last two equations, we obtain

$$\frac{d(N^*(\varphi(s))}{ds^*} + \kappa_1^*(s^*)T^*(\varphi(s)) = \frac{A(s)}{C(s)}T(s) + \frac{B(s)}{C(s)}B_1(s),$$

where

$$\begin{aligned} A(s) &= (\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + (\mu^2+1)\kappa_1\kappa_2 \\ B(s) &= \mu[(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + (\mu^2+1)\kappa_1\kappa_2] \\ C(s) &= \varphi'(s)(\mu^2-1)\sqrt{(\mu\kappa_1+\kappa_2)^2 - \kappa_3^2} \neq 0. \end{aligned}$$

By the fact that

$$\kappa_2^*(\varphi(s)) = \left\| \frac{d(N^*(\varphi(s))}{ds^*} + \kappa_1^*(s^*)T^*(\varphi(s)) \right\| > 0,$$

for all $s \in I$, we see

$$\kappa_2^*(\varphi(s)) = \frac{\left|(\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu + (\mu^2+1)\kappa_1\kappa_2\right|}{\varphi'(s)\sqrt{[(\mu\kappa_1+\kappa_2)^2 - \kappa_3^2](\mu^2-1)}}.$$

Thus, we can define a unit vector field $B_1^*(s^*)$ along γ by

$$B_1^*(s^*) = B_1^*(\varphi(s)) = \frac{1}{\kappa_2^*(\varphi(s))}[\frac{d(N^*(\varphi(s))}{ds^*} + \kappa_1^*(s^*)T^*(\varphi(s)],$$

such that

$$B_1^*(s^*) = \frac{1}{\sqrt{\mu^2-1}}(T(s) + \mu B_1(s)). \tag{34}$$

Also, since $B_2^*(s^*) = \varepsilon(\sinh\xi_0(s)N(s) + \cosh\xi_0(s)B_2(s))$ for all $s \in I$, another unit vector field $B_2^*(s^*)$ along γ can

$$B_2^*(s^*) = \frac{\kappa_3(s)}{\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2 - \kappa_3^2(s)}}N(s) + \frac{\mu\kappa_1(s)+\kappa_2(s)}{\sqrt{(\mu\kappa_1(s)+\kappa_2(s))^2 - \kappa_3^2(s)}}B_2(s). \tag{35}$$

Now, from the equations (26), (28), (34) and (35), it is seen that

$$\det[T^*(s^*), N^*(s^*), B_1^*(s^*), B_2^*(s^*)] = \det[T(s), N(s), B_1(s), B_2(s)] = 1.$$

$T^*(s^*)$, $N^*(s^*)$, $B_1^*(s^*)$, $B_2^*(s^*)$ are mutually orthogonal vector fields satisfying

$$\begin{aligned} g(T^*(s^*), T^*(s^*)) &= g(N^*(s^*), N^*(s^*)) = 1 \\ g(B_1^*(s^*), B_1^*(s^*)) &= g(B_2^*(s^*), B_2^*(s^*)) = -1. \end{aligned}$$

Thus the tetrahedron $\{T^*(s^*), N^*(s^*), B_1^*(s^*), B_2^*(s^*)\}$ along γ^* is an orthonormal frame where $T^*(s^*)$ and $N^*(s^*)$ are spacelike vector fields, $B_1^*(s^*)$ and $B_2^*(s^*)$ are timelike vector fields. On the other hand, by considering the equation (34) and the differentiation of the equation (35), we obtain

$$\begin{aligned} \kappa_3^*(s^*) &= \left\langle \frac{dB_2^*(s^*)}{ds^*}, B_1^*(s^*) \right\rangle \\ &= \frac{\kappa_1(s)\kappa_3(s)(\mu-1)}{\varphi'(s)\sqrt{[(\mu\kappa_1+\kappa_2)^2 - \kappa_3^2](\mu^2-1)}} > 0, \end{aligned}$$

for all $s \in I$. Therefore, γ is a C^∞-special curve in E_2^4 and the Frenet $(1,3)$-normal plane at the corresponding point $\gamma^*(s^*) = \gamma^*(\varphi(s))$ of γ^*. Thus, (γ, γ^*) is a mate of $(1,3)$-Bertrand curve in E_2^4. Finally, the proof of the theorem is completed.

Example 10. *(The spacelike curve equation given in [16]) Let $\gamma(s)$ be a unit speed spacelike curve in E_2^4 given by*

$$\gamma(s) = \frac{1}{15\sqrt{2}}(\sinh(3\sqrt{5}s), 9\cosh(\sqrt{5}s), 9\sinh(\sqrt{5}s), \cosh(3\sqrt{5}s)).$$

We easily obtain the Frenet vectors and curvatures as follows:

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{10}}(\cosh(3\sqrt{5}s), 3\sinh(\sqrt{5}s), 3\cosh(\sqrt{5}s), \sinh(3\sqrt{5}s)), \\ N(s) &= \frac{1}{\sqrt{2}}(\sinh(3\sqrt{5}s), \cosh(\sqrt{5}s), \sinh(\sqrt{5}s), \cosh(3\sqrt{5}s)), \\ B_1(s) &= \frac{1}{\sqrt{10}}(3\cosh(3\sqrt{5}s), \frac{1}{4}\sinh(\sqrt{5}s), \frac{1}{4}\cosh(\sqrt{5}s), 3\sinh(3\sqrt{5}s)), \\ B_2(s) &= \frac{1}{\sqrt{2}}(\sinh(3\sqrt{5}s), -\frac{3}{4}\cosh(\sqrt{5}s), -\frac{3}{4}\sinh(\sqrt{5}s), \cosh(3\sqrt{5}s)). \end{aligned}$$

The curvatures of γ are $\kappa_1(s) = 3, \kappa_2(s) = 4$ and $\kappa_3(s) = 5$. Let us take $\alpha = \beta = -\frac{4}{15}, \mu = \frac{27}{4}$ and $\delta = \frac{97}{20}$. Then, it is obvious that the relations given in Theorem 9 are hold. Therefore the curve γ is a $(1,3)$-Bertrand curve in E_2^4 and the $(1,3)$-Bertrand mate curve γ^* of the curve γ is a spacelike curve given as follows:

$$\gamma^*(s) = \frac{1}{15\sqrt{2}}(-7\sinh(3\sqrt{5}s), 8\cosh(\sqrt{5}s), 8\sinh(\sqrt{5}s), -7\cosh(3\sqrt{5}s)).$$

Theorem 11. *Let γ be a C^∞-special spacelike Frenet curve with non-zero curvatures κ_1, κ_2, κ_3 in E_2^4. Then γ is a spacelike $(1,3)$-Bertrand curve whose the Bertrand mate γ^* is timelike, if and only if there exist the constant real numbers α, β, μ, δ satisfying*

1. $\alpha\kappa_2(s) - \beta\kappa_3(s) \neq 0$
2. $\mu(\alpha\kappa_2(s) - \beta\kappa_3(s)) + \alpha\kappa_1(s) = 1, \mu = \sinh\theta_0(s)(\cosh\theta_0(s))^{-1}$
3. $\delta\kappa_3(s) = \mu\kappa_1(s) + \kappa_2(s), \delta = \sinh\phi_0(s)(\cosh\phi_0(s))^{-1}$
4. $\kappa_1\kappa_2(\mu^2+1) - (\kappa_1^2 + \kappa_2^2 - \kappa_3^2)\mu \neq 0$

for all $s \in I$.

Proof. If γ^* is a timelike curve, the proof is made similarly to the proof of Theorem 9, taking into account the equations (3), (4) and the equality

$$T^*(s^*) = \varepsilon(\sinh\theta(s)T(s) + \cosh\theta(s)B_1(s)).$$

■

5. Conclusions

In this paper, we proved that, no special spacelike Frenet curve is a Bertrand curve in E_2^4. Therefore, we defined a generalization of spacelike Bertrand curve and we called it as spacelike $(1,3)$-Bertrand curve in E_2^4. Moreover, we gave the characterizations of spacelike $(1,3)$-Bertrand curve in E_2^4.

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