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Existence of a mild solution to fractional differential equations with ψ-Caputo derivative, and its ψ-Hölder continuity

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Abstract

This paper is devoted to the study existence of locally/globally mild solutions for fractional differential equations with ψ-Caputo derivative with a nonlocal initial condition. We firstly establish the local existence by making use usual fixed point arguments, where computations and estimates are essentially based on continuous and bounded properties of the Mittag-Leffler functions. Secondly, we establish the called ψ-Hölder continuity of solutions, which shows how $|u(t') - u(t)|$ tends to zero with respect to a small difference $|\psi(t') - \psi(t)|^{\beta}$, $\beta \in (0, 1)$. Finally, by using contradiction arguments, we discuss on the existence of a global solution or maximal mild solution with blowup at finite time.

Keywords: Fractional calculus, Fractional differential equations, ψ-Caputo derivative, Fixed point theorem, Maximal mild solutions, ψ-Hölder continuity.
2010 MSC: 35K05, 35K99, 47J06, 47H10x.

1. Introduction

Fractional calculus is a branch of mathematics that studies integrals and derivatives of non-integer order. It is thought about as a generalization of classical calculus. In fractional calculus, fractional differential equations (FDEs) have an important role in numerous fields of study carried out by mathematicians, physicists, and engineers. In recent years, FDEs have gained much attention from many authors [1, 2, 3, 4, 5, 9, 11]. The

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topic of existence, uniqueness of solutions and their properties is specially studied by making uses of many types of fixed point theorem [6, 7, 8, 10].

In this paper, we consider the problem of finding a function $X = X(t)$, $t \geq 0$, that satisfies the following FDE with ψ-Caputo derivative

$$\begin{cases} {}^{\mathcal{C}}D^{\alpha,\psi}X(t) + \lambda X(t) &= F(t, X(t)), \quad t > 0, \\ X(0) + G(X) &= X_0, \end{cases} \tag{1}$$

where ${}^{\mathcal{C}}D^{\alpha,\psi}$ is the ψ-Caputo derivative of fractional order $\alpha \in (0,1)$, defined by

$${}^{\mathcal{C}}D^{\alpha,\psi}X(t) := \frac{1}{\Gamma(1-\alpha)}\frac{1}{\psi'(t)}\frac{d}{dt}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{-\alpha}X(s)ds, \quad 0 < \alpha < 1,$$

$\lambda > 0$, ψ is continuously differentiable and strictly increasing function on $[0, \infty)$, F, G are functions that will be specified later, and X_0 is a given initial value.

A simple case of the ψ-function is $\psi(t) = t$, $t \geq 0$, where the ψ-Caputo derivative operator ${}^{\mathcal{C}}D^{\alpha,\psi}$ coincides with the usual Caputo's derivative ${}^{\mathcal{C}}D^{\alpha}$. In this case, the existence, regularity, and solution properties for the problem (1) has been studied by many authors, such as [6, 7, 8, 10] and references therein.

Now, let us briefly mention existing results of the problem. In 2018, Almeida-Malinowska-Monteiro [12, Section 3] studied fractional differential equations with a Caputo derivative with respect to a kernel function

$$\begin{cases} {}^{\mathcal{C}}D^{\alpha,\psi}_{a+}X(t) &= F(t, X(t)), \quad t \in [a,b],\, 0 < \alpha \notin \mathbb{N}, \\ X^{[k]}_{\psi}(a) &= X^k_a, \qquad k = 0, 1, 2, ..., n-1, \end{cases} \tag{2}$$

where $n = [\alpha] + 1$, $X^{[k]}_{\psi}(t) := [(1/\psi'(t))d/dt]^k X(t)$, and X^k_a, $k = 0, 1, 2, ..., n-1$, are given numbers. By considering the globally Lipschitz assumption

$$|F(t,x) - F(t,y)| \leq L|x-y|, \ \forall x, y \in \mathbb{R}, \forall t \in [a,b],$$

the authors showed that Problem (2) has a mild solution with a sufficiently small coefficient L. Moreover, by the linear growth assumption

$$|F(t,x)| \leq L_0 + L_1|x|, \ \forall x \in \mathbb{R}, \forall t \in [a,b],$$

they also proved that Problem (2) has a locally mild solution in the interval $[a, a+h]$ (with small length $h > 0$). Samet-Aydi [13] studied Lyapunov-type inequalities for an anti-periodic fractional boundary value problem involving ψ-Caputo fractional derivative. Derbazi-Baitiche [14, Theorem 2] discussed the coupled systems of ψ-Caputo differential equations with initial conditions

$$\begin{cases} {}^{\mathcal{C}}D^{\alpha,\psi}_{a+}X(t) &= F_1(t, X(t), Y(t)), \quad t \in [a,b], \\ D^{\alpha,\psi}_{a+}Y(t) &= F_2(t, X(t), Y(t)), \quad t \in [a,b], \\ (X(a); Y(a)) &= (X_a, Y_a), \end{cases} \tag{3}$$

in Banach spaces, where the fractional order α belongs to $(0,1]$. They constructed the local existence of an integral solution with a small restriction on coefficients of the nonlinearities F_1, F_2 or small length $b-a$ of the time interval, where the MönchâĂŹs fixed point theorem and the technique of measures of noncompactness had been combined. We also refer the reader to the papers [15, 16, 17] and references therein for related problems.

Problems with nonlocal (initial/boundary) condition have developed a rapidly growing area. With this type of problems, investigations are not only motivated by theoretical interest, but also by the fact that life sciences problems. Involving in ψ fractional derivative, we refer to the works [18, 19, 20, 21] and references therein.

In the mentioned results above, considered problems have been studied with the nonlinearities satisfying globally Lipschitz or linear growth assumptions. Besides, these results only gave local existence of mild solutions. In this work, we shall study the existence of a global solution or maximal mild solution with blowup at finite time for Problem (1), where the nonlinearity F is assumed to be locally Lipschitz continuity.

Our contributions are explained as follows:

- We firstly establish the local existence of a mild solution $u \in C([0,T_0];\mathbb{R})$ by making use usual fixed point arguments, where computations and estimates are essentially based on continuous and bounded properties of the Mittag-Leffler functions. This result is local since we need a sufficiently small condition on T_0.
- Secondly, we establish the called ψ-Hölder continuity of solutions, which answer the question: How does $|u(t') - u(t)|$ tend to zero with respect to $|\psi(t') - \psi(t)|^\beta$, $\beta \in (0,1)$? Here, the essential tool is to use differentiation of the Mittag-Leffler functions.
- Finally, by using contradiction arguments, we discuss on the existence of a global solution or maximal mild solution with blowup at finite time. More specifically, we show that Problem (13) has a globally mild solution $u \in C([0,\infty);\mathbb{R})$, or there exists $T_{\max} < \infty$ such that Problem (13) has a maximal mild solution $u \in C([0,T_{\max});\mathbb{R})$ with $\limsup_{t\to T_{\max}^-} |X(t)| = \infty$.

The organization of this paper is divided into five sections. In Section 2, we recall some basic preliminaries, contained functional spaces, Mittag-Leffler functions, and integral representation of solutions. In Section 3, the local existence of a mild solution will be presented. Thereafter, we establish ψ-Hölder continuity of solutions in Section 4. Finally, maximal mild solution will be discussed in Section 5.

2. Preliminaries

2.1. Functional spaces and Mittag-Leffler functions

In this part, we present some basic functional spaces, where we will find solutions of the considered problem. Firstly, let $C([0,T];\mathbb{R})$ be the space of all continuous functions from $[0,T]$ to $\mathbb{R}$, which is endowed with the norm

$$\|X\|_{C([0,T];\mathbb{R})} := \sup_{0\le t\le T} |X(t)|, \quad \forall X \in C([0,T];\mathbb{R}).$$

For $\sigma \in (0,1)$, we define the called ψ-Hölder continuous space

$$C_\psi^\sigma([0,T];\mathbb{R}) = \left\{ X \in C([0,T];\mathbb{R}) \,\Big|\, \sup_{0\le t\ne t'\le T} \frac{|X(t') - X(t)|}{|\psi(t') - \psi(t)|^\sigma} < \infty \right\}$$

corresponding to the norm

$$\|X\|_{C_\psi^\sigma([0,T];\mathbb{R})} := \sup_{0\le t,t'\le T,\, \psi(t')\ne\psi(t)} \frac{|X(t') - X(t)|}{|\psi(t') - \psi(t)|^\sigma},$$

for all $X \in C_\psi^\sigma([0,T];\mathbb{R})$.

In the theory of FDEs, the called Mittag-Leffler functions naturally appear, and they are important in establishing the existence of solutions. It is thought about as a generalization of the usually exponential function. These functions are defined by

$$E_{\alpha,\beta}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},\ \alpha > 0,\ \beta \in \mathbb{R}.$$

Let us recall the following basic results of Mittag-Leffler functions, where their proofs can be found in many literature of fractional calculus.

Lemma 2.1 ([1, 2, 3])**.** *Assume that* $0 < \alpha < 1$. *Then, there exists a positive constant* $C_0 = C_0(\alpha)$ *such that*

$$E_{\alpha,\beta}(-z) \geq 0,\ \beta \in \{1; \alpha\}, \quad E_{\alpha,1}(-z) + E_{\alpha,\alpha}(-z) \leq \frac{C_0}{1+z}, \quad \forall z \geq 0.$$

Lemma 2.2 ([1, 2, 3])**.** *Assume that* $0 < \alpha < 1$. *Then, for* $\lambda > 0$, *and* $z > 0$, *the below differentiation hold*

$$\partial_z E_{\alpha,1}(-\lambda z^\alpha) = -\lambda z^{\alpha-1} E_{\alpha,\alpha}(-\lambda z^\alpha),$$
$$\partial_z (z^{\alpha-1} E_{\alpha,\alpha}(-\lambda z^\alpha)) = z^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda z^\alpha).$$

2.2. Integral representation for solutions

In this part, we shall present some basic definitions of mild solution, its continuation, and maximal solution, which are based on the following integral representation for solutions of Problem (1)

$$\begin{aligned} X(t) =& E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)X_0 + E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)G(X) \\ &+ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha) F(s, X(s)) ds. \end{aligned} \tag{4}$$

Here, this representation can be directly obtained by using the Laplace transform and its inverse, e.g. see [22, Section 5] for more details. Since this work focuses on studying mild solutions (see below for its definition) of Problem (1), we skip the proof of (4).

Definition 2.3 (Mild solutions)**.** A function X in $C([0,T];\mathbb{R})$ is said to be a locally mild solution of Problem (1) in $[0,T]$ if it satisfies the integral equation (4). Morevoer, if a continuous function $X : [0,\infty) \to \mathbb{R}$ satisfies (4), then it is called a globally mild solution of Problem (1).

Definition 2.4 (Continuation of mild solutions)**.** Let $X \in C([0,T];\mathbb{R})$ be a mild solution of Problem (1) in $[0,T]$. If there exists a time $\widetilde{T} > T$, and a function $\widetilde{X} \in C([0,\widetilde{T}];\mathbb{R})$ such that $\widetilde{X}|_{[0,T]} = X$ and $\widetilde{X}$ is a mild solution of Problem (1) in $[0,\widetilde{T}]$, then $\widetilde{X}$ is called a continuation of X.

Definition 2.5 (Maximal solution)**.** If a continuous function $X : [0,T_*) \to \mathbb{R}$ satisfies that

- $X|_{[0,T]}$ is a mild solution of Problem (1) in $[0,T]$ for all $T \in (0,T_*)$,
- X has no continuation,

then it is said to be a maximal mild solution Problem (1).

3. Existence of locally mild solution

In this section, we present our main results, which are the existence of locally mild solutions. We consider the following assumptions on the nonlinearities F, G.

- (H1) The function $F : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist constants $L_F > 0$, $p > 1$ such that

$$|F(t,x) - F(t,y)| \leq L_F(|x|^{p-1} + |y|^{p-1})|x-y|, \quad |F(t,x)| \leq L_F|x|^p,$$

  for all $x, y \in \mathbb{R}$ and $t \geq 0$.

- (H2) The function $G : C([0,T];\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous and there exist a constant $L_G > 0$ such that
$$|G(X) - G(Y)| \leq L_G \|X - Y\|_{C([0,T];\mathbb{R})}, \quad |G(X)| \leq L_G \|X\|_{C([0,T];\mathbb{R})},$$
for all $X, Y \in C([0,T];\mathbb{R})$ and $T \in (0,\infty)$.

In the following theorem, we discuss the existence of a locally mild solution of Problem (1), where the method has been built from the usual fixed point argument, bounded properties of the Mittag-Leffler functions, and the Lebesgue's dominated convergence theorem.

Lemma 3.1. *Let F be defined by the locally Lipschitz assumption (H1), and $X \in C([0,T];\mathbb{R})$ with $T \in (0,\infty)$. Then, the following function*
$$t \mapsto \mathcal{T}_F X(t) := \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} E_{\alpha,\alpha}(-\lambda(\psi(t) - \psi(s))^\alpha) F(s, X(s)) ds$$
is continuous on $[0,T]$.

Proof. Let t, h be satisfied that $0 \leq t < t + h \leq T$. Then, by making some direct computations and using the triangle inequality, one has
$$\begin{aligned} |\mathcal{T}_F X(t+h) - \mathcal{T}_F X(t)| \leq & \int_0^t I^{t,h}_{F,1}(s) |F(s, X(s))| ds \\ & + \int_0^t I^{t,h}_{F,2}(s) |F(s, X(s))| ds \\ & + \int_t^{t+h} I^{t,h}_{F,3}(s) |F(s, X(s))| ds, \end{aligned} \tag{5}$$
where the integrands are given by
$$\begin{aligned} I^{t,h}_{F,1}(s) &:= \psi'(s)\big|(\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1}\big| E_{\alpha,\alpha}(-\lambda(\psi(t+h) - \psi(s))^\alpha), \\ I^{t,h}_{F,2}(s) &:= \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}\big|E_{\alpha,\alpha}(-\lambda(\psi(t+h) - \psi(s))^\alpha) - E_{\alpha,\alpha}(-\lambda(\psi(t) - \psi(s))^\alpha)\big|, \\ I^{t,h}_{F,3}(s) &:= \psi'(s)(\psi(t+h) - \psi(s))^{\alpha-1} E_{\alpha,\alpha}(-\lambda(\psi(t+h) - \psi(s))^\alpha). \end{aligned}$$

It is sufficient to prove that the right hand side of (5) tends to zero as h approaches zero from the right. The first term $\int_0^t I^{t,h}_{F,1}(s)|F(s, X(s))|ds$ can be estimated as follows. It is useful to recall that ψ is an increasing function of t. Therefore, $(\psi(t+h) - \psi(s))^{\alpha-1} < (\psi(t) - \psi(s))^{\alpha-1}$ for all $0 < s < t$. By the bounded property (2.1) of the Mittag-Leffler function $E_{\alpha,\alpha}$, we deduce that
$$\begin{aligned} &\big|(\psi(t+h) - \psi(s))^{\alpha-1} - (\psi(t) - \psi(s))^{\alpha-1}\big| E_{\alpha,\alpha}(-\lambda(\psi(t+h) - \psi(s))^\alpha) \\ &\quad = \big[(\psi(t) - \psi(s))^{\alpha-1} - (\psi(t+h) - \psi(s))^{\alpha-1}\big] E_{\alpha,\alpha}(-\lambda(\psi(t+h) - \psi(s))^\alpha) \\ &\quad \leq C_0 \big[(\psi(t) - \psi(s))^{\alpha-1} - (\psi(t+h) - \psi(s))^{\alpha-1}\big]. \end{aligned}$$

Consequently, by the locally Lipschitz continuity of the nonlinearity F,
$$\begin{aligned} &\int_0^t I^{t,h}_{F,1}(s)|F(s, X(s))|ds \\ &\quad \leq C_0 \int_0^t \psi'(s)\big[(\psi(t) - \psi(s))^{\alpha-1} - (\psi(t+h) - \psi(s))^{\alpha-1}\big]|F(s, X(s))|ds \\ &\quad \leq C_0 L_F \int_0^t \psi'(s)\big[(\psi(t) - \psi(s))^{\alpha-1} - (\psi(t+h) - \psi(s))^{\alpha-1}\big]|X(s)|^p ds \\ &\quad \leq C_0 L_F \|X\|^p_{C([0,T];\mathbb{R})} \int_0^t \psi'(s)\big[(\psi(t) - \psi(s))^{\alpha-1} - (\psi(t+h) - \psi(s))^{\alpha-1}\big] ds \\ &\quad \leq 2 C_0 L_F \|X\|^p_{C([0,T];\mathbb{R})} \frac{(\psi(t+h) - \psi(t))^\alpha}{\alpha} \xrightarrow{h \to 0^+} 0, \end{aligned}$$

where the lateral limit holds by the continuity of ψ function.

Now, we proceed to consider the second integral on the right hand side of (5). Since the composite function $s \mapsto E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)$ is continuous on $[0,t]$, the integrand tends to zero as $h \to 0^+$. In order to prove that $\int_0^t I_{F,2}^{t,h}(s)|F(s,X(s))|ds \to 0$ as $h \to 0^+$, it is necessary to show the integrand is integrable on $[0,t]$ due to the Lebesgue's dominated convergence theorem. Indeed, the bounded property of the Mittag-Leffler function $E_{\alpha,\alpha}(-z)$ reads that

$$\left|E_{\alpha,\alpha}(-\lambda(\psi(t+h)-\psi(s))^\alpha) - E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)\right| \le 2C_0,$$

which subsequently implies

$$I_{F,2}^{t,h}(s)|F(s,X(s))| \le 2C_0 L_F \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|X\|^p_{C([0,T];\mathbb{R})},$$

where we have used the assumption (H1). It is useful to note that the function $s \mapsto \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$ is obviously integrable on the interval $[0,t]$. Summarily, we conclude that the limit $\int_0^t I_{F,2}^{t,h}(s)|F(s,X(s))|ds \to 0$ holds as $h \to 0^+$.

Next, we will consider the the last integral on the right hand side of (5). By employing the same techniques as above estimates, one can check the following chain

$$\begin{aligned}\int_t^{t+h} I_{F,3}^{t,h}(s)|F(s,X(s))|ds &\le C_0 \int_t^{t+h} \psi'(s)(\psi(t+h)-\psi(s))^{\alpha-1}|F(s,X(s))|ds \\ &\le C_0 L_F \|X\|^p_{C([0,T];\mathbb{R})} \int_t^{t+h} \psi'(s)(\psi(t+h)-\psi(s))^{\alpha-1}ds \\ &= C_0 L_F \|X\|^p_{C([0,T];\mathbb{R})} \frac{(\psi(t+h)-\psi(t))^\alpha}{\alpha} \xrightarrow{h\to 0^+} 0, \end{aligned} \tag{6}$$

which concludes the proof. □

Theorem 3.2. *Let F, G be defined by the assumptions (H1)-(H2). If $C_0L_G < 1$, then Problem (1) has only a locally mild solution.*

Proof. Firstly, it follows from $C_0L_G < 1$ that one can find $0 < K < 1$ satisfying $C_0L_G < K$. We also observe that the function $T \mapsto (\psi(T)-\psi(0))^\alpha$ is increasing of T. Moreover, it tends to zero as $T \to 0^+$ by its continuity. Hence, there exists $T_0 > 0$ such that

$$\psi(T_0) < \left(\frac{1}{L_F\alpha^{-1}}\right)^{\frac{1}{\alpha}} \left(\frac{2C_0}{K - C_0L_G}\right)^{-\frac{p}{\alpha}} |X_0|^{\frac{1}{\alpha}-\frac{p}{\alpha}} + \psi(0), \tag{7}$$

which subsequently ensures

$$\frac{2C_0|X_0|}{K-C_0L_G} < \left(\frac{\alpha(K-C_0L_G)}{2C_0L_F(\psi(T_0)-\psi(0))^\alpha}\right)^{1/(p-1)}.$$

So, one can find a positive number R_0 such that

$$\frac{2C_0|X_0|}{K-C_0L_G} < R_0 < \left(\frac{\alpha(K-C_0L_G)}{2C_0L_F(\psi(T_0)-\psi(0))^\alpha}\right)^{1/(p-1)}. \tag{8}$$

Let us denote the ball $B_{R_0} := \{u \in C([0,T_0];\mathbb{R}) \,|\, \|u\|_{C([0,T_0];\mathbb{R})} \le R_0\}$. Then, B_{R_0} is a closed convex, nonempty subset of $C([0,T_0];\mathbb{R})$. On this ball, we define the mapping

$$\begin{aligned}\mathcal{T}X(t) :=& E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)X_0 + E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)G(X) \\ &+ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)F(s,X(s))ds \\ =:& \mathcal{T}_{X_0}(t) + \mathcal{T}_G X(t) + \mathcal{T}_F X(t), \quad t \in [0,T_0]. \end{aligned} \tag{9}$$

where $\mathcal{T}_F X$ is defined by Lemma 3.1. We will prove that $\mathcal{T}$ has a unique fixed point, which is a mild solution of Problem (1). The proof will be divided into two steps by using contraction map theorem, bounded properties of the Mittag-Leffler functions, and the Lebesgue's dominated convergence theorem.

Step 1. Proving $\mathcal{T}$ well-defined on B_{R_0}: It is useful to recall that the Mittag-Leffler function $z \mapsto E_{\alpha,1}(-z)$ is a continuous on $[0,\infty)$. Therefore, by the continuity and increasing assumption of ψ-function, the composite function $t \mapsto E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)$ is continuous on $[0,T_0]$. Hence, for t,h such that $0 \le t < t+h \le T_0$, we have

$$|\mathcal{T}_{X_0}(t+h)-\mathcal{T}_{X_0}(t)| \le \big|E_{\alpha,1}(-\lambda(\psi(t+h)-\psi(0))^\alpha)-E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)\big||X_0|,$$

which invokes that $\mathcal{T}_{X_0}(t+h)-\mathcal{T}_{X_0}(t) \to 0$ as $h \to 0^+$. On the other hand, the assumption (H2) guarantees that $|G(X)| \le L_G\|X\|_{C([0,T_0];\mathbb{R})}$. Hence, by similarly argument as above, there hold

$$\begin{aligned}
&|\mathcal{T}_G X(t+h)-\mathcal{T}_G X(t)| \\
&\quad \le \big|E_{\alpha,1}(-\lambda(\psi(t+h)-\psi(0))^\alpha)-E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)\big||G(X)| \\
&\quad \le \big|E_{\alpha,1}(-\lambda(\psi(t+h)-\psi(0))^\alpha)-E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)\big|L_G\|X\|_{C([0,T_0];\mathbb{R})},
\end{aligned}$$

which consequently yields that $\mathcal{T}_G X(t+h)-\mathcal{T}_G X(t) \to 0$ as $h \to 0^+$.

By the above arguments and applying Lemma 3.1, the continuity of $\mathcal{T}X$ is concluded. Hence, it is necessary to prove that $\mathcal{T}B_{R_0} \subset B_{R_0}$. For this purpose, we now let $X \in B_{R_0}$. Then, by the bounded property of the Mittag-Leffler function $E_{\alpha,1}$, it is obvious that $|\mathcal{T}_{X_0}(t)|+|\mathcal{T}_G X(t)| \le C_0|X_0|+C_0L_G\|X\|_{C([0,T_0];\mathbb{R})}$. Moreover, by the assumption (H1) also,

$$\begin{aligned}
|\mathcal{T}_F X(t)| &\le \left|\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)F(s,X(s))ds\right| \\
&\le C_0\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|F(s,X(s))|ds \\
&\le C_0\, L_F\, \alpha^{-1}\,(\psi(T_0)-\psi(0))^\alpha\, \|X\|^p_{C([0,T_0];\mathbb{R})}.
\end{aligned}$$

Summarily, we have

$$\begin{aligned}
|\mathcal{T}X(t)| &\le |\mathcal{T}_{X_0}(t)|+|\mathcal{T}_G X(t)|+|\mathcal{T}_F X(t)| \\
&\le C_0|X_0|+C_0L_G\|X\|_{C([0,T_0];\mathbb{R})}+C_0L_F\alpha^{-1}(\psi(T_0)-\psi(0))^\alpha\|X\|^p_{C([0,T_0];\mathbb{R})} \\
&\le C_0|X_0|+C_0L_GR+C_0L_F\alpha^{-1}(\psi(T_0)-\psi(0))^\alpha R_0^p \qquad (10)
\end{aligned}$$

By employing the conditions (7) and (8) together, we derive

$$\begin{aligned}
&C_0|X_0|+C_0L_GR_0+C_0L_F\alpha^{-1}(\psi(T_0)-\psi(0))^\alpha R_0^p \\
&\qquad \le \frac{K-C_0L_G}{2}R_0+C_0L_GR_0+\frac{K-C_0L_G}{2}R_0 = KR_0, \qquad (11)
\end{aligned}$$

Thus, $|\mathcal{T}X(t)| \le R_0$ for all $t \in [0,T_0]$, namely, $\mathcal{T}B_{R_0} \subset B_{R_0}$.

Step 2. $\mathcal{T}$ is a contraction mapping. Let us arbitrarily take $X,Y \in B_{R_0}$, and make slightly modifying techniques in the previous step. Firstly, by using the locally Lipschitz assumption (H1) on F, one can see that

$$|F(s,X(s))-F(s,Y(s))| \le L_F\big(\|X\|^{p-1}_{C([0,T_0];\mathbb{R})}+\|Y\|^{p-1}_{C([0,T_0];\mathbb{R})}\big)\|X-Y\|_{C([0,T_0];\mathbb{R})}.$$

which according shows

$$\begin{aligned}|\mathcal{T}_F X(t)-\mathcal{T}_F Y(t)| \leq &\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|F(s,X(s))-F(s,Y(s))|ds\\ \leq &C_0L_F\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}(|X(s)|^{p-1}+|Y(s)|^{p-1})|X(s)-Y(s)|ds\\ \leq &2C_0L_FR_0^{p-1}\left(\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}ds\right)\|X-Y\|_{C([0,T_0];\mathbb{R})}\\ = &2C_0L_FR_0^{p-1}\alpha^{-1}(\psi(T_0)-\psi(0))^{\alpha}\|X-Y\|_{C([0,T_0];\mathbb{R})}.\end{aligned}$$

By using the conditions (7) and (8) similarly as (11), one get

$$|\mathcal{T}_F X(t)-\mathcal{T}_F Y(t)| \leq (K-C_0L_G)\|X-Y\|_{C([0,T_0];\mathbb{R})} \tag{12}$$

Finally, taking the above estimate together gives that

$$|\mathcal{T}X(t)-\mathcal{T}Y(t)| \leq |\mathcal{T}_G X(t)-\mathcal{T}_G Y(t)|+|\mathcal{T}_F X(t)-\mathcal{T}_F Y(t)| \leq K\|X-Y\|_{C([0,T_0];\mathbb{R})}.$$

Since $0<K<1$, we conclude that $\mathcal{T}$ is a contraction mapping. Thus, it possesses only a fixed point in B_{R_0}, which is the unique locally mild solution of the problem. □

Corollary 3.3. In Theorem 3.2, it requires the condition C_0L_G to establish the existence of a locally mild solution. Hence, this result holds for the case $G\equiv 0$. Namely, Theorem 3.2 also concludes that the problem

$$\begin{cases} {}^{\mathcal{C}}D^{\alpha,\psi}X(t)+\lambda X(t) &= F(t,X(t)), \quad t>0,\\ X(0) &= X_0.\end{cases} \tag{13}$$

has a locally mild solution in $C([0,T_0];\mathbb{R})$, where F satisfies the assumption (H1).

Remark 3.3.1. *Let us discuss the condition (7). Since the power $1/\alpha-p/\alpha$ is negative, we can observe that: if the initial value X_0 is large enough, then it requires that the local existence time T_0 is sufficiently small.*

Remark 3.3.2. *In the case $\psi(t)=t$, the ψ-Caputo derivative ${}^{\mathcal{C}}D^{\alpha,\psi}$ becomes the well-known Caputo derivative ${}^{\mathcal{C}}D^{\alpha}$. The respective problems for this fractional derivative have been studied by many authors, such as, []...*

4. ψ-Hölder continuity

This section discusses Hölder continuity of the mild solution given in Section 3. More clearly, we will show that the difference $|u(t')-u(t)|$ is bounded by $|\psi(t')-\psi(t)|^{\beta}$ with a parameter $\beta\in(0,1)$. Suitably, we call this by ψ-Hölder continuity of the solution.

In the following theorem, ψ-Hölder continuity of the solution will be obtained, where techniques in Lemma 3.1 can be improved by making use differentiation and bounded properties of the Mittag-Leffler functions.

Theorem 4.1 (ψ-Hölder continuity)**.** *Assume that F,G fulfills the assumptions (H1)-(H2) with $C_0L_G<1$. Let $u\in C([0,T];\mathbb{R})$ be a mild solution of Problem (1). Then, $u\in C_{\psi}^{\alpha(1-\gamma_\alpha)}([0,T];\mathbb{R})$ for $\gamma_\alpha\in[0,1]$, and it satisfies the estimate*

$$|u(t')-u(t)| \lesssim |\psi(t')-\psi(t)|^{\alpha(1-\gamma_\alpha)}, \quad \forall t,t'\in[0,T]. \tag{14}$$

Proof. For the sake of convenience, we assume that $0 \le t < t' \le T$. By differentiation of the Mittag-Leffler function $E_{\alpha,1}$ given in Lemma 2.2, we have the following composite differentiation

$$\partial_s E_{\alpha,1}(-\lambda(\psi(s)-\psi(0))^\alpha) = \psi'(s)(\psi(s)-\psi(0))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(s)-\psi(0))^\alpha).$$

Therefore, upon the fundamental theorem of Calculus, there hold

$$\begin{aligned}
|\mathcal{T}_{X_0}(t') - \mathcal{T}_{X_0}(t)| &\le \left|E_{\alpha,1}(-\lambda(\psi(t')-\psi(0))^\alpha) - E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)\right||X_0| \\
&= \left|\int_t^{t'} \partial_s E_{\alpha,1}(-\lambda(\psi(s)-\psi(0))^\alpha)ds\right||X_0| \\
&= \lambda|X_0|\int_t^{t'} \psi'(s)(\psi(s)-\psi(0))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(s)-\psi(0))^\alpha)ds \\
&\le \lambda|X_0|\int_t^{t'} \psi'(s)(\psi(s)-\psi(0))^{\alpha-1}\frac{C_0}{1+\lambda(\psi(s)-\psi(0))^\alpha}ds \\
&\le \lambda|X_0|\int_t^{t'} \psi'(s)(\psi(s)-\psi(0))^{\alpha-1}(\lambda(\psi(s)-\psi(0)))^{-\alpha\gamma_\alpha}ds,
\end{aligned}$$

where we have used the boundedness of Mittag-Leffler functions given in Lemma 2.1 and note that $1 + \lambda(\psi(s)-\psi(0))^\alpha \ge (\lambda(\psi(s)-\psi(0)))^{-\alpha\gamma_\alpha}$ for $\gamma_\alpha \in [0,1]$. Therefore, by making some direct computations, one has

$$\begin{aligned}
|\mathcal{T}_{X_0}(t') - \mathcal{T}_{X_0}(t)| &\le \lambda|X_0|\int_t^{t'} \psi'(s)(\psi(s)-\psi(0))^{\alpha(1-\gamma_\alpha)-1}ds \\
&= \lambda|X_0|\frac{(\psi(t')-\psi(0))^{\alpha(1-\gamma_\alpha)} - (\psi(t)-\psi(0))^{\alpha(1-\gamma_\alpha)}}{\alpha(1-\gamma_\alpha)}.
\end{aligned}$$

Since $\alpha(1-\gamma_\alpha) \in (0,1)$, the difference $(\psi(t')-\psi(0))^{\alpha(1-\gamma_\alpha)} - (\psi(t)-\psi(0))^{\alpha(1-\gamma_\alpha)}$ is less than or equal to $(\psi(t')-\psi(t))^{\alpha(1-\gamma_\alpha)}$. This implies

$$|\mathcal{T}_{X_0}(t') - \mathcal{T}_{X_0}(t)| \le \frac{\lambda|X_0|}{\alpha(1-\gamma_\alpha)}(\psi(t')-\psi(t))^{\alpha(1-\gamma_\alpha)}.$$

Similarly, by using the assumption (H2), we can derive the estimate

$$|\mathcal{T}_G X(t') - \mathcal{T}_G X(t)| \le \frac{\lambda L_G\|X\|_{C([0,T];\mathbb{R})}}{\alpha(1-\gamma_\alpha)}(\psi(t')-\psi(t))^{\alpha(1-\gamma_\alpha)}.$$

Let us establish the ψ-Hölder continuity of the term $\mathcal{T}_F X$. For $0 < s < r$, applying differentiation of the Mittag-Leffler function $E_{\alpha,\alpha}$ in Lemma 2.2 yields that

$$\begin{aligned}
&\partial_r\left[(\psi(r)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(r)-\psi(s))^\alpha)\right] \\
&\quad = \psi'(r)(\psi(r)-\psi(s))^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda(\psi(r)-\psi(s))^\alpha).
\end{aligned}$$

We then use the fundamental theorem of Calculus to obtain

$$\begin{aligned}
&\mathcal{T}_F X(t+h) - \mathcal{T}_F X(t) \\
&= \int_0^t \int_t^{t'} \psi'(s)\partial_r\big[(\psi(r)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(r)-\psi(s))^\alpha)\big]F(s,X(s))drds \\
&+ \int_t^{t'} \psi'(s)(\psi(t')-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t')-\psi(s))^\alpha)|F(s,X(s))|ds \\
&= \int_0^t \int_t^{t'} \psi'(s)\psi'(r)(\psi(r)-\psi(s))^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda(\psi(r)-\psi(s))^\alpha)F(s,X(s))drds \\
&+ \int_t^{t'} \psi'(s)(\psi(t')-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t')-\psi(s))^\alpha)F(s,X(s))ds \\
&=: \int_0^t \int_t^{t'} I_{F,4}(r,s)F(s,X(s))drds + \int_t^{t'} I_{F,5}(t',s)F(s,X(s))ds. \qquad (15)
\end{aligned}$$

The second term above can be estimated similarly as (6), which helps to read that

$$\left|\int_t^{t'} I_{F,5}(t',s)F(s,X(s))ds\right| \le \frac{C_0L_F}{\alpha}\|X\|^p_{C([0,T];\mathbb{R})}(\psi(t')-\psi(t))^{\alpha(1-\gamma_\alpha)},$$

where we have used the assumption (H1). By using this assumption and making some direct computations, we estimate the first term of (15) as follows

$$\begin{aligned}
&\left|\int_0^t \int_t^{t'} I_{F,4}(r,s)F(s,X(s))drds\right| \\
&\le C_0L_F\|X\|^p_{C([0,T];\mathbb{R})}\int_0^t\int_t^{t'} \psi'(s)\psi'(r)(\psi(r)-\psi(s))^{\alpha(1-\gamma_\alpha)-2}drds \\
&= C_0L_F\|X\|^p_{C([0,T];\mathbb{R})}\int_0^t \psi'(s)\frac{(\psi(t)-\psi(s))^{\alpha(1-\gamma_\alpha)-1}-(\psi(t')-\psi(s))^{\alpha(1-\gamma_\alpha)-1}}{1-\alpha(1-\gamma_\alpha)}ds \\
&= C_0L_F\|X\|^p_{C([0,T];\mathbb{R})}\frac{(\psi(t)-\psi(s))^{\alpha(1-\gamma_\alpha)}-(\psi(t')-\psi(s))^{\alpha(1-\gamma_\alpha)}+(\psi(t')-\psi(t))^{\alpha(1-\gamma_\alpha)}}{\alpha(1-\gamma_\alpha)[1-\alpha(1-\gamma_\alpha)]},
\end{aligned}$$

where it is useful to note that $\alpha(1-\gamma_\alpha)\in(0,1)$. Since the function ψ is increasing, the difference $(\psi(t)-\psi(s))^{\alpha(1-\gamma_\alpha)}-(\psi(t')-\psi(s))^{\alpha(1-\gamma_\alpha)}$ is obviously negative. By skipping this difference, we derive the below estimate

$$\begin{aligned}
&\left|\int_0^t \int_t^{t'} I_{F,4}(r,s)F(s,X(s))drds\right| \\
&\qquad\le \frac{C_0L_F}{\alpha(1-\gamma_\alpha)[1-\alpha(1-\gamma_\alpha)]}\|X\|^p_{C([0,T];\mathbb{R})}(\psi(t')-\psi(t))^{\alpha(1-\gamma_\alpha)}.
\end{aligned}$$

The conclusion of the theorem and the desired inequality are immediately obtained by taking the above estimates together. □

Remark 4.1.1. *In the case $\psi(t)=t$, the ψ-Hölder continuity becomes the usual Hölder continuity. If the ψ-function is already Hölder continuous of exponent $\sigma\in(0,1)$, then we can deduce from the inequality (14) that*

$$|u(t')-u(t)| \lesssim |\psi(t')-\psi(t)|^{\alpha(1-\gamma_\alpha)} \lesssim |t'-t|^{\sigma\alpha(1-\gamma_\alpha)},$$

for all $t,t'\in[0,T]$. Then, u is Hölder continuous with the exponent $\sigma\alpha(1-\gamma_\alpha)$.

5. Maximal solution

In this section, we will study existence of the maximal mild solution to our problem with the initial value condition instead of nonlocal condition. Explicitly, we will study Problem (13). We will show that Problem (13) has a globally mild solution $u \in C([0,\infty);\mathbb{R})$, or there exists a time $T_{\max} < \infty$ such that Problem (13) has a maximal mild solution $u \in C([0,T_{\max});\mathbb{R})$ with blowup at the finite time $T_{\max}$.

For the purpose, we firstly present the following lemma, where we show that any mild solution $X \in C([0,T];\mathbb{R})$ always has a continuation.

Lemma 5.1. *(Continuation) Assume that F satisfies (H1). Let $X \in C([0,T];\mathbb{R})$ be a mild solution of Problem (13). Then, it has a unique continuation.*

Proof. We will prove this lemma by making uses of usual fixed point arguments also. According to the proof of Theorem 3.2, we recall that X is the unique solution of the following integral equation in the interval $[0,T]$

$$\begin{aligned} X(t) =& E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^{\alpha})X_0 \\ &+ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^{\alpha})F(s,X(s))ds. \end{aligned} \tag{16}$$

Let us denote $\mathcal{M} := \sup\{|X(t)|\,|\,0 \le t \le T\}$, $\mathcal{N} := \mathcal{M} + X(T)$, and take a real number ρ such that $0 < \rho < \mathcal{M}$. Then, there always exists a sufficiently small number $\epsilon > 0$ such that the following conditions hold for all $t \in [T, T+\epsilon]$

$$\begin{cases} |X_0|\big|E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^{\alpha}) - E_{\alpha,1}(-\lambda(\psi(T)-\psi(0))^{\alpha})\big| \le \dfrac{\rho}{8}, \\ \displaystyle\int_T^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^{\alpha})ds \le \dfrac{\rho}{8\mathcal{N}^p}, \\ \displaystyle\int_0^T \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\big|E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^{\alpha}) - E_{\alpha,\alpha}(-\lambda(\psi(T)-\psi(s))^{\alpha})\big|ds \le \dfrac{\rho}{8\mathcal{N}^p}, \\ \displaystyle\int_0^T \psi'(s)\big|(\psi(t)-\psi(s))^{\alpha-1} - (\psi(T)-\psi(s))^{\alpha-1}\big|E_{\alpha,\alpha}(-\lambda(\psi(T)-\psi(s))^{\alpha})ds \le \dfrac{\rho}{8\mathcal{N}^p}, \end{cases}$$

where the integrands are obviously integrable. With the parameters ρ, ϵ as above, we now define the following space

$$\mathbb{V}_{\rho,\epsilon} = \Big\{\widetilde{X} \in C([0,T+\epsilon];\mathbb{R}) \,\Big|\, \widetilde{X}|_{[0,T]} = X \text{ and } |X(t)-X(T)| \le \rho,\ \forall t \in [T,T+\epsilon]\Big\},$$

and the mapping $\widetilde{\mathcal{T}} : \mathbb{V}_{\rho,\epsilon} \to C([0,T+\epsilon];\mathbb{R})$ as follows

$$\begin{aligned} \widetilde{\mathcal{T}}\widetilde{X}(t) :=& E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^{\alpha})X_0 \\ &+ \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^{\alpha})F(s,\widetilde{X}(s))ds, \end{aligned} \tag{17}$$

for all $t \in [0,T+\epsilon]$.

By Lemma 3.1, $\widetilde{\mathcal{T}}\widetilde{X}$ is a continuous function of t on the interval $[0,T+\epsilon]$. Moreover, since $\widetilde{X}|_{[0,T]} = X$, we can observe from (16) and (17) that $\widetilde{\mathcal{T}}\widetilde{X}|_{[0,T]} = X$. Hence, in order to show that the mapping $\widetilde{\mathcal{T}}$ is well-defined, we need to prove the below inequality

$$|\widetilde{\mathcal{T}}\widetilde{X}(t) - X(T)| \le \rho,\ \forall t \in [T,T+\epsilon]. \tag{18}$$

Let us consider $t \in [T, T+\epsilon]$. Then, we also imply from the equations (16)-(17), and the identity $X(s) = \widetilde{X}|_{[0,T]}(s)$ for all $s \in [0,T]$ (since $\widetilde{X} \in \mathbb{V}_{\rho,\epsilon}$) that

$$\begin{aligned}
\widetilde{\mathcal{T}}\widetilde{X}(t) - X(T) &:= \big(E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)X_0 - E_{\alpha,1}(-\lambda(\psi(T)-\psi(0))^\alpha)X_0\big) \\
&+ \bigg(\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)F(s,\widetilde{X}(s))ds \\
&- \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(T)-\psi(s))^\alpha)F(s,X(s))ds\bigg) \\
&= \big(E_{\alpha,1}(-\lambda(\psi(t)-\psi(0))^\alpha)X_0 - E_{\alpha,1}(-\lambda(\psi(T)-\psi(0))^\alpha)X_0\big) \\
&+ \bigg(\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)F(s,\widetilde{X}(s))ds \\
&- \int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(T)-\psi(s))^\alpha)F(s,\widetilde{X}(s))ds\bigg) \\
&=: J_{X_0}(t,T) + J_F(t,T).
\end{aligned}$$

By choosing parameter ϵ as the beginning of this proof, we have $|J_{X_0}(t,T)| \le \rho/8$. Besides, upon the assumption (H1), we can bound the nonlinear term $|F(s,\widetilde{X}(s))|$ by $\mathcal{N}^p$. Henceforth, by some direct computations, one can derive

$$\begin{aligned}
&|J_F(t,T)| \le \mathcal{N}^p \int_T^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha)ds \\
&+ \mathcal{N}^p \int_0^T \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\big|E_{\alpha,\alpha}(-\lambda(\psi(t)-\psi(s))^\alpha) - E_{\alpha,\alpha}(-\lambda(\psi(T)-\psi(s))^\alpha)\big|ds \\
&+ \mathcal{N}^p \int_0^T \psi'(s)\big|(\psi(t)-\psi(s))^{\alpha-1} - (\psi(T)-\psi(s))^{\alpha-1}\big|E_{\alpha,\alpha}(-\lambda(\psi(T)-\psi(s))^\alpha)ds,
\end{aligned}$$

which consequently yields $|J_F(t,T)| \le 3\rho/8$. Summarily, we obtain (18), namely, the mapping $\widetilde{\mathcal{T}}$ is well-defined on $\mathbb{V}_{\rho,\epsilon}$. Finally, by slightly modifying the above estimates, we can show that $\widetilde{\mathcal{T}}$ is a contraction mapping, and so it possesses a unique fixed point in $\mathbb{V}_{\rho,\epsilon}$. This is the unique continuation of X. We complete the proof. □

Thanks to the above lemma, we shall establish the existence of a globally mild solution or a maximal mild solution with blowup at the finite time $T_{\max}$ in the following lemma.

Theorem 5.2 (Maximal solution)**.** *Assume that F satisfies the assumption (H1). Then,*

i) Problem (13) has a globally mild solution $X \in C([0,\infty);\mathbb{R})$; or

ii) There exists $T_{\max} = T_{\max}(X_0) < \infty$ such that Problem (13) has a maximal local mild solution $X \in C([0,T_{\max});\mathbb{R})$ with $\limsup_{t\to T_{\max}^-}|X(t)| = \infty$.

Proof. We will prove this theorem by contradiction. Firstly, we denote

$$\mathbb{W} := \Big\{T \in (0,\infty) \,\Big|\, \text{Problem (13) has a unique mild solution} X \in C([0,T];\mathbb{R})\Big\}, \tag{19}$$

and $T_{\max} := \sup \mathbb{W}$. Here, we note that $T_{\max}$ can be equaled to positive infinity. If $T_{\max} = \infty$, then the part i occurs. Conversely, if the time $T_{\max}$ is finite, namely $T_{\max} < \infty$, then we shall prove $\limsup_{t\to T_{\max}^-}|X(t)| = \infty$. By contradiction, let us assume that there really exists a positive constant $\mathcal{M}$ such that $|X(t)| \le \mathcal{M}$ for all $t \in [0, T_{\max})$.

Let $\{T_n | n = 1,2,...\}$ be a non-negative, non-decreasing sequence such that $\lim_{n\to\infty} T_n = T_{\max}$. For $n \ge 1$, $k \ge 1$, by making same computations and arguments as the proof of Lemma 3.1, one can check the below chain

$$
\begin{aligned}
&|X(T_{n+k}) - X(T_n)| \\
&\quad \le \left|E_{\alpha,1}(-\lambda(\psi(T_{n+k}) - \psi(0))^\alpha) - E_{\alpha,1}(-\lambda(\psi(T_n) - \psi(0))^\alpha)\right||X_0| \\
&\quad + \sum_{j=1}^{2} \int_0^{T_n} \mathcal{R}_{F,j}^{n,k}(s)|F(s,X(s))|ds + \int_{T_n}^{T_{n+k}} \mathcal{R}_{F,3}^{n,k}(s)|F(s,X(s))|ds \\
&\quad \le \left|E_{\alpha,1}(-\lambda(\psi(T_{n+k}) - \psi(0))^\alpha) - E_{\alpha,1}(-\lambda(\psi(T_n) - \psi(0))^\alpha)\right||X_0| \\
&\quad + \sum_{j=1}^{2} L_F \mathcal{M}^p \int_0^{T_n} \mathcal{R}_{F,j}^{n,k}(s)ds + L_F \mathcal{M}^p \int_{T_n}^{T_{n+k}} \mathcal{R}_{F,3}^{n,k}(s)ds \xrightarrow{n,k\to\infty} 0,
\end{aligned}
$$

where the kernels $\mathcal{R}_{F,j}^{n,k}$, $j = 1, 2, 3$, are given by

$$
\begin{aligned}
\mathcal{R}_{F,1}^{n,k}(s) &:= \psi'(s)\left|(\psi(T_{n+k}) - \psi(s))^{\alpha-1} - (\psi(T_n) - \psi(s))^{\alpha-1}\right|E_{\alpha,\alpha}(-\lambda(\psi(T_{n+k}) - \psi(s))^\alpha), \\
\mathcal{R}_{F,2}^{n,k}(s) &:= \psi'(s)(\psi(T_n) - \psi(s))^{\alpha-1}\left|E_{\alpha,\alpha}(-\lambda(\psi(T_{n+k}) - \psi(s))^\alpha) - E_{\alpha,\alpha}(-\lambda(\psi(T_n) - \psi(s))^\alpha)\right|, \\
\mathcal{R}_{F,3}^{n,k}(s) &:= \psi'(s)(\psi(T_{n+k}) - \psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(T_{n+k}) - \psi(s))^\alpha).
\end{aligned}
$$

Hence, $\{X(T_n)|n \ge 1\}$ is a Cauchy sequence. Consequently, the limit $\lim_{n\to\infty} X(T_n) =$ finitely exists, which allows to extend the function X to $[0, T_{\max}]$ such that

$$
\begin{aligned}
X(t) =& E_{\alpha,1}(-\lambda(\psi(t) - \psi(0))^\alpha)X_0 \\
&+ \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1}E_{\alpha,\alpha}(-\lambda(\psi(t) - \psi(s))^\alpha)F(s, X(s))ds,
\end{aligned}
$$

for all $t \in [0, T_{\max}]$. Now, due to Lemma 5.1, this solution has a continuation to an interval $[0, T_{\max} + \epsilon]$ with some $\epsilon > 0$. This is a contradiction with (19). Therefore, by contraction, if $T_{\max} < \infty$, then $\limsup_{t\to T_{\max}^-} |X(t)| = \infty$. □

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