Regularization method for the problem of determining the source function using integral conditions

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Abstract

In this article, we deal with the inverse problem of identifying the unknown source of the time-fractional diffusion equation in a cylinder equation by A fractional Landweber method. This problem is ill-posed. Therefore, the regularization is required. The main result of this article is the error between the sought solution and its regularized under the selection of a priori parameter choice rule.

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1. Introduction

According to the history of mathematical research, it has been found that the standard diffusion equation has been used to represent the particle motion Gaussian process. To describe anomalous diffusion phenomena, the classical derivative will be replaced with a non-integer derivative. Therefore, it leads to great applications of differential equations with non-integer derivatives. Fractional derivatives and fractional calculus was also considered by many scientists because of applications in potential theory, physics, electrochemistry, viscoelasticity, biomedicine, control theory, and signal processing, see e.g. [1, 2] and the references therein. Nigmatullin [3] first applied the fractional diffusion equation to describe diffusion in a medium shaped

fractal. Metzler and Klafter [4] gave a proof that a fractional diffusion equation is possible governs a non-Markovian propagation process that has a memory. Among many different interesting topics about the fractional diffusion equation, some types of inverse problems in this genre attract the community interested in research. T. Wei and her group [5, 6, 7, 8] investigated some regularization methods for homogeneous backward problem. Y. Hang and his coauthors [9] used fractional Landweber method for solving backward time-fractional diffusion problem. The diffusion process inverse source problem is intended to detect the source function of the physical field from some indirect measurement (such as last time information or boundary measurement). As we all know, the problem of determining the source function has attracted a lot of mathematicians interested in research because of its applications in practice. Some interesting works on this topic can be found in some previous paper, for example [10]-[38]. In general, the inverse source problem is often ill-posed in the sense of Hadamard. In this work, we focus on the following equation in an axis-symmetric cylinder

$$\begin{cases} \dfrac{\partial^{\beta}}{\partial t^{\beta}} u(r,z,t) = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{\partial^2 u}{\partial z^2} + \Phi(t)f(r,z), \\[2mm] u(r,z,0) = a(r,z), \qquad\qquad\quad 0 < r \le R_0, \\[1mm] u(R_0,z,t) = u(0,z,t) = 0, \qquad 0 < t, 0 < z \le L_0 \\[1mm] u(r,0,t) = u(r,L_0,t) = 0, \qquad 0 < t, 0 < r \le R_0 \\[1mm] \lim_{r \to 0} u(r,z,t) \text{ bounded }, \qquad\qquad 0 \le t \le T, \\[1mm] u \text{ is finite} \qquad t > 0, 0 < r \le R_0, \ 0 < z \le L_0, \end{cases} \tag{1.1}$$

where the Caputo fractional derivative $\dfrac{\partial^{\beta}}{\partial t^{\beta}}$ is defined as follows:

$$D_t^{\beta} u(r,z,t) = \frac{1}{\Gamma(1-\beta)} \int\limits_0^t \frac{u_\tau(r,z,\tau)}{(t-\tau)^\beta} d\tau, \quad 0 < \beta < 1, \tag{1.2}$$

a_ϵ and Φ_ϵ and satisfy

$$\|a_\epsilon - a\|_{L_r^2(\Omega)} + \|\Phi_\epsilon - \Phi\|_{L^\infty(0,T)} \le \epsilon. \tag{1.3}$$

with the following condition on the final time data

$$\theta_1 u(r,z,T) + \theta_2 \int\limits_0^T u(r,z,t)dt = g(r,z). \tag{1.4}$$

The main purpose of this paper is to apply a fractional Landweber method to regularized our inverse source problem. We will demonstrate that the regularized solution will converge on the sought solution. There are two challenges that we need to overcome. The first difficulty is that the problem is considered in the domain of axis-symmetric cylinder making the assessment techniques complicated. The second difficulty is the presence of integral conditions that make estimates of errors cumbersome. It can be said that our result is one of the first results about the source function for the problem (1.1)-(1.4).

The outline of the paper is given as follows: In Section 2, we give some preliminary theoretical results. Ill-posed analysis and conditional stability are obtained in Section 3. In Section 4, we propose the Fractional Landweber regularization method and give a convergence estimate under an a-priori regularization parameter choice rule.

2. Statement of the problem

We introduced the Lesbesgue space associated with the measure rdr, i.e

$$L_r^2(\Omega) = \left\{ \nu : \Omega \to \mathbb{R} \text{ measurable } \int_\Omega \nu^2(r,z)rdrdz < \infty \right\}, \tag{2.5}$$

which is a Hilbert space with the scalar product $\langle u, \nu \rangle_r = \int_\Omega u(r,z)\nu(r,z)rdrdz$, and norm is given by

$$\|v\|_{L_r^2(\Omega)} = \left(\int_\Omega \nu^2(r,z)rdrdz \right)^{\frac{1}{2}} \cdot \text{Throughout this paper, for the convenience of writing,}$$

Definition 2.1. *(See [19]) For any constant γ and $r \in \mathbb{R}$, the Mittag-Leffler function is defined as:*

$$E_{\gamma,\alpha}(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\gamma j + \alpha)}, \quad z \in \mathbb{C}, \tag{2.6}$$

where $\gamma > 0$ and $\alpha \in \mathbb{R}$ are arbitrary constant.

Lemma 2.1. *[12] Assuming that $0 < \beta_0 < \beta_1 < 1$, then there exist constants $\mathcal{C}_1$ and $\mathcal{C}_2$ depending only on β, β_1 such that*

$$\frac{\mathcal{C}_1}{\Gamma(1-\beta)} \frac{1}{1-z} \leq E_{\beta,1}(z) \leq \frac{\mathcal{C}_2}{\Gamma(1-z)} \frac{1}{1-z}, \quad z \geq 0. \tag{2.7}$$

Lemma 2.2. *[13] For $\lambda_{mn} \geq \lambda_{11} > 0$, then there exists constant $\mathcal{C}_3$ and $\mathcal{C}_4$ depending only on β, T, λ_{11} such that*

$$\frac{\mathcal{C}_3}{\lambda_j^2} \leq E_{\beta,1}(-\lambda_{mn}T^\beta) \leq \frac{\mathcal{C}_4}{\lambda_j^2}.$$

Proof. This proof can be found in [13]. □

Lemma 2.3. *Let $\mathcal{C}_5, \mathcal{C}_6 \geq 0$ satisfy $\mathcal{C}_5 \leq |\Phi(t)| \leq \mathcal{C}_6$, $\forall t \in [0,T]$, let choose $\epsilon \in \left(0, \frac{\mathcal{C}_5}{2}\right)$, by denoting $\mathcal{B}(\mathcal{C}_5, \mathcal{C}_6) = \mathcal{C}_6 + \frac{\mathcal{C}_5}{2}$, we get $\frac{\mathcal{C}_5}{2} \leq |\Phi_\epsilon(t)| \leq \mathcal{B}(\mathcal{C}_5, \mathcal{C}_6)$.*

Proof. This proof can be found at [14]. □

Lemma 2.4. *[13] For $\lambda_{mn} > 0$, $\beta > 0$, and positive integer $j \in \mathbb{N}$, we have:*

$$\frac{d}{dt}\left(tE_{\beta,2}(-\lambda_{mn}t^\beta)\right) = E_{\beta,1}(-\lambda_{mn}t^\beta), \quad \frac{d}{dt}\left(E_{\beta,1}(-\lambda_{mn}t^\beta)\right) = -\lambda_{mn}t^{\beta-1}E_{\beta,\beta}(-\lambda_j t^\beta). \tag{2.8}$$

Lemma 2.5. *For any $0 < \beta < 1$, $E_{\beta,1}(-t^\beta)$ is completely monotonic, see [] with $\mathcal{A}_{mn}(\beta,\tau) = (t - \tau)^{\beta-1}E_{\beta,\beta}\left(-\lambda_{mn}(t-\tau)^\beta\right)$, we get*

$$a) \quad \frac{1}{\lambda_{mn}}\left(1 - E_{\beta,1}\left(-\lambda_{11}T^\beta\right)\right) \leq \int_0^T \mathcal{A}_{mn}(\beta, T-\tau)d\tau \leq \frac{1}{\lambda_{mn}}, \tag{2.9}$$

$$b) \quad \frac{T}{\lambda_{mn}}\left(1 - E_{\beta,2}\left(-\lambda_{11}T^\beta\right)\right) \leq \int_0^T \left(\int_0^t \mathcal{A}_{mn}(\beta, t-\tau)d\tau\right)dt \leq \frac{T}{\lambda_{mn}}. \tag{2.10}$$

Proof. Please see the proof in [20]. □

3. Ill-posed analysis and conditional stability

Theorem 3.1. *The solution of problem* (1.1) *represented by the formula* (3.20).

Proof. The solution of problem (1.1) is as follows:

$$
u(r, z, t) = \sum_{m,n=1}^{+\infty} \left(u_{0,mn} \mathrm{E}_{\beta,1}\left(-\lambda_{mn} t^{\beta} \right) J_0\left(\frac{\zeta_m}{R_0}r\right) \sin\left(\frac{n\pi}{L_0}z\right) \right.
$$

$$
+ \sum_{m,n=1}^{+\infty} \left(f_{mn} \int_0^t (t-\tau)^{\beta-1} \mathrm{E}_{\beta,\beta}(-\lambda_{mn}(t-\tau)^{\beta})\Phi(\tau)d\tau \right) J_0\left(\frac{\zeta_m}{R_0}r\right) \sin\left(\frac{n\pi}{L_0}z\right). \tag{3.11}
$$

whereby

$$
\lambda_{mn} = \left(\frac{\zeta_m}{R_0}\right)^2 + \left(\frac{n\pi}{L_0}\right)^2, \quad m, n = 1, 2, ..., \infty,
$$

$$
f_{mn} = \frac{4}{L_0 R_0^2 J_1^2(\zeta_m)} \int_0^{R_0}\int_0^{L_0} J_0\left(\frac{\zeta_m}{R_0}r\right) \sin\left(\frac{n\pi}{L_0}z\right) f(r,z) r\, dr\, dz,
$$

$$
u_{0,mn} = \frac{4}{L_0 R_0^2 J_1^2(\zeta_m)} \int_0^{R_0}\int_0^{L_0} J_0\left(\frac{\zeta_m}{R_0}r\right) \sin\left(\frac{n\pi}{L_0}z\right) u(r,z) r\, dr\, dz. \tag{3.12}
$$

where $J_0(z)$ and $J_1(z)$ denote the 0th order and 1st order Bessel function, and ζ_m are the sequence of solution of the equation $J_0(z) = 0$ which satisfy

$$
0 < \zeta_1 < \zeta_2 < \cdots < \zeta_m < \cdots, \quad \lim_{m\to\infty} \zeta_m = \infty. \tag{3.13}
$$

Defining

$$
\omega_m(z) = \frac{\sqrt{2}}{R_0 J_1(\zeta_m)} J_0\left(\frac{\zeta_m}{R_0}r\right), \quad e_n(z) = \sqrt{\frac{2}{L_0}} \sin\left(\frac{n\pi}{L_0}z\right), \quad \Psi_{m,n}(r,z) = \omega_m(r)e_n(z). \tag{3.14}
$$

then it is easy to check that the eigenfunctions $\left\{\Psi_{m,n}(r,z)\right\}_{m,n\geq 1}$ from an orthonormal basis in $L_r^2(\Omega)$. Using the eigenfunctions $\Psi_{m,n}(r,z)$ as a basic, formula (3.11) can be written for a shorter as follows

$$
u_{m^\dagger,n^\dagger}(t) = u_{0,m^\dagger n^\dagger} \mathrm{E}_{\beta,1}\left(-\lambda_{mn} t^{\beta} \right) + f_{m^\dagger n^\dagger} \left(\int_0^t (t-\tau)^{\beta-1} \mathrm{E}_{\beta,\beta}(-\lambda_{mn}(t-\tau)^{\beta})\Phi(\tau)d\tau \right). \tag{3.15}
$$

From the fact that $\theta_1 u(r,z,T) + \theta_2 \int_0^T u(r,z,t)dt = g(r,z)$, we find that

$$
\theta_1 \sum_{m,n=1}^{+\infty} u_{m,n}(T)\Psi_{m,n}(r,z) + \theta_2 \int_0^T \left(\sum_{m,n=1}^{+\infty} u_{m,n}(t)\Psi_{m,n}(r,z)\right)dt = \sum_{m,n=1}^{+\infty} g_{m,n}\Psi_{m,n}(r,z). \tag{3.16}
$$

From (3.16), we deduce that

$$
g_{m^\dagger,n^\dagger} = \theta_1 u_{0,m^\dagger n^\dagger} \mathrm{E}_{\beta,1}(-\lambda_{mn}T^{\beta}) + \theta_1 f_{m^\dagger n^\dagger} \int_0^T (T-\tau)^{\beta-1} \mathrm{E}_{\beta,\beta}(-\lambda_{mn}(T-\tau)^{\beta})\Phi(\tau)d\tau
$$

$$
+ \theta_2 u_{0,m^\dagger n^\dagger} \int_0^T \mathrm{E}_{\beta,1}(-\lambda_{mn}t^{\beta})dt + \theta_2 f_{m^\dagger n^\dagger} \int_0^T \left(\int_0^t (t-\tau)^{\beta-1} \mathrm{E}_{\beta,\beta}(-\lambda_{mn}(t-\tau))\Phi(\tau)d\tau \right)dt. \tag{3.17}
$$

This implies that

$$g_{m^\dagger,n^\dagger} = \theta_1 u_{0,m^\dagger n^\dagger} \mathrm{E}_{\beta,1}(-\lambda_{mn}T^\beta) + \theta_2 u_{0,m^\dagger n^\dagger} \int_0^T \mathrm{E}_{\beta,1}(-\lambda_{mn}t^\beta)dt$$

$$+ \theta_1 f_{m^\dagger n^\dagger} \int_0^T (T-\tau)^{\beta-1}\mathrm{E}_{\beta,\beta}(-\lambda_{mn}(T-\tau)^\beta)\Phi(\tau)d\tau$$

$$+ \theta_2 f_{m^\dagger n^\dagger} \int_0^T \Big(\int_0^t (t-\tau)^{\beta-1}\mathrm{E}_{\beta,\beta}(-\lambda_{mn}(t-\tau)^\beta)\Phi(\tau)d\tau \Big)dt. \qquad (3.18)$$

For a shorter, by denoting $(t-\tau)^{\beta-1}\mathrm{E}_{\beta,\beta}(-\lambda_{mn}(t-\tau)^\beta)\Phi(\tau) = \mathcal{A}_{mn}(\beta,t-\tau,\Phi)$. From (3.17), the latter equality implies that

$$f_{m^\dagger n^\dagger} = \frac{g_{m^\dagger,n^\dagger} - \theta_1 u_{0,m^\dagger n^\dagger} \mathrm{E}_{\beta,1}(-\lambda_{mn}T^\beta) - \theta_2 u_{0,m^\dagger n^\dagger} \int_0^T \mathrm{E}_{\beta,1}(-\lambda_{mn}t^\beta)dt}{\theta_1 \int_0^T \mathcal{A}_{mn}(\beta,T-\tau,\Phi)d\tau + \theta_2 \int_0^T \Big(\int_0^t \mathcal{A}_{mn}(\beta,t-\tau,\Phi)d\tau \Big)dt}. \qquad (3.19)$$

The mild solution is given by

$$f(r,z) = \sum_{m,n=1}^{+\infty} \frac{g_{m^\dagger,n^\dagger} - \theta_1 u_{0,m^\dagger n^\dagger} \mathrm{E}_{\beta,1}(-\lambda_{mn}T^\beta) - \theta_2 u_{0,m^\dagger n^\dagger} \int_0^T \mathrm{E}_{\beta,1}(-\lambda_{mn}t^\beta)dt}{\theta_1 \int_0^T \mathcal{A}_{mn}(\beta,T-\tau,\Phi)d\tau + \theta_2 \int_0^T \Big(\int_0^t \mathcal{A}_{mn}(\beta,t-\tau,\Phi)d\tau \Big)dt} \Psi_{m,n}(r,z). \qquad (3.20)$$

$\square$

3.1. The ill-posedness and stability of problem (1.1)

Theorem 3.2. *The inverse source problem* (1.1) *is ill-posed.*

Proof. A linear operator $\mathcal{P} : L_r^2(\Omega) \to L_r^2(\Omega)$ as follows.

$$\mathcal{P}f(r,z) = \int_0^{R_0} \int_0^{L_0} \ell(r,z,\xi)f(r,z)d\xi = \ell(r,z), \qquad (3.21)$$

where

$$\ell(r,z) = g_{m^\dagger,n^\dagger} - \theta_1 u_{0,m^\dagger n^\dagger} \mathrm{E}_{\beta,1}(-\lambda_{mn}T^\beta) - \theta_2 u_{0,m^\dagger n^\dagger} \int_0^T \mathrm{E}_{\beta,1}(-\lambda_{mn}t^\beta)dt, \qquad (3.22)$$

and

$$\Psi(r,z) = \sum_{m,n=1}^{+\infty} \Big[\theta_1 \int_0^T \mathcal{A}_{mn}(\beta,T-\tau,\Phi)d\tau + \theta_2 \int_0^T \Big(\int_0^t \mathcal{A}_{mn}(\beta,t-\tau,\Phi)d\tau \Big)dt \Big] \Psi_{m,n}(r,z).$$

Due to $\ell(r,z) = \ell(z,r)$, we know $\mathcal{P}$ is self-adjoint operator. Next, we are going to prove its compactness. Defining the finite rank operators $\mathcal{P}_{M,N}$ as follows

$$\mathcal{P}_{M,N}\, f(x) = \sum_{m,n=1}^{M,N} \left[\theta_1 \int_0^T \mathcal{A}_{mn}(\beta, T-\tau, \Phi)d\tau + \theta_2 \int_0^T \left(\int_0^t \mathcal{A}_{mn}(\beta, t-\tau, \Phi)d\tau \right)dt \right] \langle f, \Psi_{mn} \rangle \Psi_{mn}(r).$$

From $\mathcal{P}_{M,N}f$ and $\mathcal{P}f$, using the inequality $(a+b)^2 \le 2(a^2+b^2)$, $a, b \ge 0$, we have:

$$\begin{aligned}
\left\| \mathcal{P}_{M,N}f - \mathcal{P}f \right\|_{L_r^2(\Omega)}^2 &\le \sum_{m,n=M+1,N+1}^{+\infty} \left| \frac{\theta_1 \mathcal{C}_6}{\lambda_{mn}} + \frac{\theta_2 \mathcal{C}_6 T}{\lambda_{mn}} \right|^2 \left| \langle f, \Psi_{mn} \rangle \right|^2 \\
&\le \frac{1}{\lambda_{MN}^2} \left[2\mathcal{C}_6^2 \theta_1^2 + 2\mathcal{C}_6^2 (\theta_2 T)^2 \right] \sum_{m,n=M+1,N+1}^{+\infty} \left| \langle f, \Psi_{mn} \rangle \right|^2 \\
&\le \frac{1}{\lambda_{MN}^2} \left[2\mathcal{C}_6^2 \theta_1^2 + 2\mathcal{C}_6^2 (\theta_2 T)^2 \right] \left\| f \right\|_{L_r^2(\Omega)}^2.
\end{aligned} \tag{3.23}$$

Therefore, $\left\| \mathcal{P}_{M,N}f - \mathcal{P}f \right\|_{L_r^2(\Omega)}$ in the sense of operator norm in $L(L_r^2(\Omega); L_r^2(\Omega))$ as $M, N \to \infty$. Also, $\mathcal{P}$ is a compact operator. Next, the SVDs for the linear self-adjoint compact operator $\mathcal{P}$ are

$$\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi) = \theta_1 \int_0^T \mathcal{A}_{mn}(\beta, T-\tau, \Phi)d\tau + \theta_2 \int_0^T \left(\int_0^t \mathcal{A}_{mn}(\beta, t-\tau, \Phi)d\tau \right)dt. \tag{3.24}$$

and corresponding eigenvectors is $\Psi_{m,n}$ which is known as an orthonormal basis in $L_r^2(\Omega)$. Corresponding eigenvectors is Ψ_{mn} which is known as an orthonormal basis in $L_r^2(\Omega)$. From (3.21), the inverse source problem we introduced above can be formulated as an operator equation $\mathcal{P}f(r,z) = \Xi(r,z)$ and by Kirsch [30]. Assume that $u_{0,m^\dagger,n^\dagger} = 0$ and $g_{m^\dagger,n^\dagger}$ is noised data by and $g_{m^\dagger,n^\dagger}^\epsilon$ we have estimate

$$\left\| f - f^\epsilon \right\|_{L_r^2(\Omega)}^2 = \sum_{m,n=1}^{+\infty} \frac{\left| \ell_{m^\dagger,n^\dagger}^\epsilon - \ell_{m^\dagger,n^\dagger} \right|^2}{\left| \mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi) \right|^2} = \sum_{m,n=1}^{+\infty} \frac{\left| g_{m^\dagger,n^\dagger}^\epsilon - g_{m^\dagger,n^\dagger} \right|^2}{\left| \mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi) \right|^2}. \tag{3.25}$$

By the Lemma 2.3 and the Lemma 2.5, we know that

$$\frac{1}{\left| \mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi) \right|^2} \ge \frac{\lambda_{mn}^4}{\left[\theta_1 \mathcal{C}_6 + \theta_2 \mathcal{C}_6 T \right]^2}. \tag{3.26}$$

From (3.25) and (3.26), therefore in the computation of (3.25), the small data error can be amplified arbitrarily much by the factor $\left| \mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi) \right|^{-2}$ which increase without bound, so recovering the source $f(r,z)$ from a measured data $g^\epsilon(r,z)$ is ill-posed. Hence, regularization for this article needs to be considered. $\square$

3.2. *Conditional stability of source term f*

Theorem 3.3. *If $\left\| f \right\|_{H_r^{2j}(\Omega)} \le \mathcal{M}$ for $\mathcal{M}$ is the positive constant, then we get*

$$\left\| f \right\|_{L_r^2(\Omega)} \text{ is defined in } (3.29),$$

Proof. From (3.20), applying the Hölder inequality, we know

$$
\begin{aligned}
\|f\|_{L_r^2(\Omega)}^2 &= \sum_{m,n=1}^{+\infty} \frac{\left|\langle g, \Psi_{m,n}\rangle\right|^{\frac{2}{j+1}} \left|\langle g, \Psi_{m,n}\rangle\right|^{\frac{2j}{j+1}}}{\left|\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta,\Phi)\right|^2} \\
&\leq \left(\sum_{m,n=1}^{+\infty} \frac{\left|\langle g, \Psi_{m,n}\rangle\right|^2}{\left|\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta,\Phi)\right|^{2j+2}} \right)^{\frac{1}{j+1}} \left(\sum_{m,n=1}^{+\infty} \left|\langle g, \Psi_{m,n}\rangle\right|^2 \right)^{\frac{j}{j+1}} \\
&\leq \left(\sum_{m,n=1}^{+\infty} \frac{\left|\langle f, \Psi_{mn}\rangle\right|^2}{\left|\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta,\Phi)\right|^{2j}} \right)^{\frac{1}{j+1}} \|g\|_{L_r^2(\Omega)}^{\frac{2j}{j+1}},
\end{aligned}
\tag{3.27}
$$

and this inequality leads to

$$
\|f\|_{L_r^2(\Omega)}^2 \leq \left|\mathcal{Z}_{1,1}^{\theta_1,\theta_2}(\beta,T,\mathcal{C}_5)\right|^{-2j} \sum_{m,n=1}^{+\infty} \lambda_{mn}^{4j} \left|\langle f, \xi_j\rangle\right|^2 \leq \frac{\|f\|_{H_r^{2j}(\Omega)}^2}{\left|Z_{1,1}^{\theta_1,\theta_2}(\beta,T,\mathcal{C}_5)\right|^{2j}}.
\tag{3.28}
$$

Combining (3.27) and (3.28), we get

$$
\|f\|_{L_r^2(\Omega)}^2 \leq \frac{\mathcal{M}^{\frac{2}{j+1}}}{\left|\mathcal{Z}_{1,1}^{\theta_1,\theta_2}(\beta,T,\mathcal{C}_5)\right|^{\frac{2j}{j+1}}} \|g\|_{L_r^2(\Omega)}^{\frac{2j}{j+1}}.
\tag{3.29}
$$

whereby

$$
\mathcal{Z}_{1,1}^{\theta_1,\theta_2}(\beta,T,\mathcal{C}_5) = \left[\theta_1 \mathcal{C}_5 \big(1 - E_{\beta,1}(-\lambda_{11}T^\beta) \big) + \theta_2 \mathcal{C}_5 T \big(1 - E_{\beta,2}(-\lambda_{11}T^\beta) \big) \right].
\tag{3.30}
$$

□

4. A Fractional Landweber Method and convergence rate

In the section, we show the fractional Landweber regularization solution for problem (1.1)

$$
f^{\gamma(\epsilon),b}(r,z) = \sum_{m,n=1}^{+\infty} \left[1 - \left(1 - \eta \left| \frac{\theta_1 + \theta_2 T}{\lambda_{mn}} \right|^2 \right)^{\gamma(\epsilon)} \right]^b \frac{\langle \ell, \Psi_{m,n}\rangle \Psi_{m,n}(r,z)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta,\Phi)}, \quad \frac{1}{2} < b \leq 1.
\tag{4.31}
$$

and measured data

$$
f_\epsilon^{\gamma(\epsilon),b}(r,z) = \sum_{m,n=1}^{+\infty} \left[1 - \left(1 - \eta \left| \frac{\theta_1 + \theta_2 T}{\lambda_{mn}} \right|^2 \right)^{\gamma(\epsilon)} \right]^b \frac{\langle \ell_\epsilon, \Psi_{m,n}\rangle \Psi_{m,n}(r,z)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta,\Phi_\epsilon)}, \quad \frac{1}{2} < b \leq 1.
\tag{4.32}
$$

where $b \in (\frac{1}{2}, 1]$ is called the fractional parameter, and $[\gamma(\epsilon)] \geq 1$ is a regularization parameter, and $\eta \in \left(0, \left(\frac{\lambda_{11}}{\theta_1 + \theta_2 T} \right)^2 \right)$. In the case $b = 1$, it is the classical Landweber method. In the proof section, we need the following lemmas:

Lemma 4.1. *For $0 < \lambda < 1, c > 0$, $n \in \mathbb{N}$, let $r_n(\lambda) := (1-\lambda)^n$, we get:*

$$
r_n(\lambda)\lambda^c \leq \theta_c(n+1)^{-c},
\tag{4.33}
$$

where $\theta_c = \begin{cases} 1, & 0 \leq c \leq 1, \\ c^c, & c > 1. \end{cases}$

Proof. This Lemma 4.1 can be found in [9]. □

Lemma 4.2. *For* $\dfrac{1}{2} < b < 1, \gamma \geq 1,$ *choosing* $\eta \in \left(0, \left(\dfrac{\lambda_{11}}{\theta_1 + \theta_2 T}\right)^2\right)$ *then* $0 < \eta\left(\dfrac{\theta_1 + \theta_2 T}{\lambda_{11}}\right)^2 < 1,$ *by denoting* $z = \eta\left(\dfrac{\theta_1 + \theta_2 T}{\lambda_{11}}\right)^2,$ *we have the following estimates*

$$a) \quad \left[1 - (1-z)^\gamma\right]^b \left(\frac{z}{\eta}\right)^{-\frac{1}{2}} \leq \eta^{\frac{1}{2}} \gamma^{\frac{1}{2}},$$

$$b) \quad (1-z)^\gamma \left(\frac{z}{\eta}\right)^{\frac{\varsigma}{2}} \leq \left(\frac{\varsigma}{2\eta}\right)^{\frac{\varsigma}{2}} \gamma^{-\frac{\varsigma}{2}}. \tag{4.34}$$

Proof. The proof can be found in [9]. □

Lemma 4.3. *Let ℓ be given by (3.22) depends on g and u_0 functions. Similarly, in a similar way we can find the function definition with the couple $(g_\epsilon, u_{0,\epsilon})$ are observed data by (g, u_0) as follows* $\langle \ell_\epsilon, \Psi_{m,n} \rangle = $

$$\langle g_\epsilon, \Psi_{m,n} \rangle - \langle u_{0,\epsilon,m,n}, \Psi_{m,n} \rangle \left(\theta_1 E_{\beta,1}(-\lambda_{mn} T^\gamma) + \theta_2 \int_0^T E_{\beta,1}(-\lambda_{mn} t^\beta) dt\right),$$

denoting $\mathcal{C}_7^2 = 2 + 2\left(\theta_1 \mathcal{C}_4 + \theta_2 \dfrac{\mathcal{C}_4 T}{\lambda_{11}}\right)^2$ *then*

Proof. Using the inequality $(a+b)^2 \leq 2a^2 + 2b^2, \forall a, b \geq 0,$ see the Lemma 2.1, it gives

$$\|\ell_\epsilon - \ell\|_{L^2_r(\Omega)}^2$$

$$= \sum_{m,n=1}^{+\infty} \left\| \langle g_\epsilon - g, \Psi_{m,n} \rangle - \langle u_{0,\epsilon} - u_0, \Psi_{m,n} \rangle \left(\theta_1 E_{\beta,1}(-\lambda_{mn} T^\gamma) + \theta_2 \int_0^T E_{\beta,1}(-\lambda_{mn} t^\gamma) d\tau\right) \right\|_{L^2_r(\Omega)}^2$$

$$\leq 2 \sum_{m,n=1}^{+\infty} \left| \langle g_\epsilon - g, \Psi_{m,n} \rangle \right|^2 + 2 \sum_{m,n=1}^{+\infty} \left\| \langle u_{0,\epsilon} - u_0, \Psi_{m,n} \rangle \left(\theta_1 \mathcal{C}_4 + \theta_2 \int_0^T E_{\beta,1}(-\lambda_{mn} t^\beta) dt\right) \right\|_{L^2_r(\Omega)}^2$$

$$\leq 2\|g_\epsilon - g\|_{L^2_r(\Omega)}^2 + 2\|u_{0,\epsilon} - u_0\|_{L^2_r(\Omega)}^2 \left(\theta_1 \mathcal{C}_4 + \theta_2 \dfrac{\mathcal{C}_4 T}{\lambda_{11}}\right)^2 \leq \epsilon^2 \mathcal{C}_7^2. \tag{4.35}$$

□

4.1. An a priori parameter choice rule

Theorem 4.1. *Suppose that f is given by (3.20). Let $f^{\gamma(\epsilon),b}$ is the its approximation, assume that conditions* $\|f\|_{H^{2j}_r(\Omega)} \leq \mathcal{M}$ *and (1.3) hold. By choosing* $\gamma(\epsilon) = \left(\dfrac{\mathcal{M}}{\epsilon}\right)^{\frac{2}{j+1}},$ *then*

$$\left\|f^{\gamma(\epsilon),b}_\epsilon - f\right\|_{L^2_r(\Omega)} \text{ is of order } \epsilon^{\frac{j}{j+1}}. \tag{4.36}$$

Proof. Using the triangle inequality, we get

$$\left\|f^{\gamma(\epsilon),b}_\epsilon - f\right\|_{L^2_r(\Omega)} \leq \left\|f^{\gamma(\epsilon),b}_\epsilon - f^{\gamma(\epsilon),b}\right\|_{L^2_r(\Omega)} + \left\|f^{\gamma(\epsilon),b} - f\right\|_{L^2_r(\Omega)}. \tag{4.37}$$

We divide the proof into two steps: We receive $\left\|f^{\gamma(\epsilon),b}_\epsilon - f^{\gamma(\epsilon),b}\right\|_{L^2_r(\Omega)}$ as follows:

$$f^{\gamma(\epsilon),b}_\epsilon(r,z) - f^{\gamma(\epsilon),b}(r,z)$$

$$= \sum_{m,n=1}^{+\infty} \left[1 - \left(1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right)^{\gamma(\epsilon)}\right]^b \left(\frac{\langle \ell_\epsilon, \Psi_{m,n} \rangle \Psi_{m,n}(\cdot, \cdot)}{\mathbf{V}^{\theta_1,\theta_2}_{m,n}(\beta, \Phi_\epsilon)} - \frac{\langle \ell, \Psi_{m,n} \rangle \Psi_{m,n}(\cdot, \cdot)}{\mathbf{V}^{\theta_1,\theta_2}_{m,n}(\beta, \Phi)}\right). \tag{4.38}$$

From (4.38), we received

$$\left\|f_\epsilon^{\gamma(\epsilon),b} - f^{\gamma(\epsilon),b}\right\|_{L_r^2(\Omega)} = \sum_{m,n=1}^{+\infty} \left[1 - \left(1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right)^{\gamma(\epsilon)}\right]^b \left(\frac{\langle \ell_\epsilon - \ell, \Psi_{m,n}\rangle \Psi_{m,n}(\cdot,\cdot)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi_\epsilon)}\right)$$

$$+ \sum_{m,n=1}^{+\infty} \left[1 - \left(1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right)^{\gamma(\epsilon)}\right]^b \left(\frac{\langle \ell, \Psi_{m,n}\rangle \Psi_{m,n}(\cdot,\cdot)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi_\epsilon)} - \frac{\langle \ell, \Psi_{m,n}\rangle \Psi_{m,n}(\cdot,\cdot)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi)}\right). \quad (4.39)$$

From (4.39), to be able to use inequality in Lemma 4.2, we added quantities $\left|\dfrac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^{-1}$, one has

$$\left\|f_\epsilon^{\gamma(\epsilon),b} - f^{\gamma(\epsilon),b}\right\|_{L_r^2(\Omega)} \le \sum_{m,n=1}^{+\infty} \left|\frac{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi_\epsilon - \Phi)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi_\epsilon)} \frac{\langle \ell, \Psi_{m,n}\rangle \Psi_{m,n}(\cdot,\cdot)}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi)}\right|$$

$$+ \sum_{m,n=1}^{+\infty} \left[1 - \left(1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right)^{\gamma(\epsilon)}\right]^b \left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^{-1}$$

$$\times \left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right| \frac{\lambda_{m,n}|\langle \ell_\epsilon - \ell, \Psi_{m,n}\rangle \Psi_{m,n}(\cdot,\cdot)|}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)}, \quad (4.40)$$

whereby

$$\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T) = \theta_1\big(1 - E_{\beta,1}(-\lambda_{11}T^\beta)\big) + \theta_2 T\big(1 - E_{\beta,2}(-\lambda_{11}T^\beta)\big).$$

Using the Lemma 4.3, we can know that

$$\left\|f_\epsilon^{\gamma(\epsilon),b} - f^{\gamma(\epsilon),b}\right\|_{L_r^2(\Omega)} \le [\gamma(\epsilon)]^{\frac{1}{2}} \eta^{\frac{1}{2}} \frac{(\theta_1 + \theta_2 T)\|\ell_\epsilon - \ell\|_{L_r^2(\Omega)}}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)} + \frac{2\epsilon}{\mathcal{C}_5} \sum_{n=1}^{+\infty} \frac{|\langle \ell, e_n\rangle|}{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi)}$$

$$\le [\gamma(\epsilon)]^{\frac{1}{2}} \eta^{\frac{1}{2}} \frac{\mathcal{C}_7(\theta_1 + \theta_2 T)\epsilon}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)} + \left(\frac{2\epsilon}{\mathcal{C}_5}\right)\|f\|_{L_r^2(\Omega)}. \quad (4.41)$$

Next, we give

$$\left\|f^{\gamma(\epsilon),b} - f\right\|_{L_r^2(\Omega)}^2 = \sum_{m,n=1}^{+\infty} \left[1 - \left[1 - \left(1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right)^{\gamma(\epsilon)}\right]^b\right]^2 \frac{|\langle \ell, \Psi_{m,n}\rangle|^2}{|\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta, \Phi)|^2}$$

$$\le \sum_{m,n=1}^{+\infty} \left[1 - \left[1 - \left(1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right)^{\gamma(\epsilon)}\right]^b\right]^2 \lambda_{mn}^{-2j}\|f\|_{H_r^{2j}(\Omega)}^2$$

$$\le \sum_{m,n=1}^{+\infty} \left[1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right]^{2[\gamma(\epsilon)]} \lambda_{mn}^{-2j}\mathcal{M}^2. \quad (4.42)$$

Because of the Lemma 2.5, we know

$$\lambda_{mn}^{-1} \le \frac{\mathbf{V}_{m,n}^{\theta_1,\theta_2}(\beta)}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)} \text{ this leads to } \lambda_{mn}^{-2j} \le \left|(\theta_1 + \theta_2 T)\lambda_{mn}^{-1}\right|^{2j} \left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-2j}. \quad (4.43)$$

From (4.42) and (4.43), we have

$$\left\|f^{\gamma(\epsilon),b} - f\right\|_{L_r^2(\Omega)}^2 \le \left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-2j} \mathcal{M}^2 \sum_{m,n=1}^{+\infty} \left[1 - \eta\left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^2\right]^{2[\gamma(\epsilon)]} \left|\frac{\theta_1 + \theta_2 T}{\lambda_{mn}}\right|^{2j}$$

$$\le \left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-2j} \mathcal{M}^2 \left(\frac{j}{2\eta}\right)^j [\gamma(\epsilon)]^{-j}. \quad (4.44)$$

Hence, it gives

$$\left\|f^{\gamma(\epsilon),b} - f\right\|_{L_r^2(\Omega)} \le \left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-j} \mathcal{M}\left(\frac{j}{2\eta}\right)^{\frac{j}{2}} [\gamma(\epsilon)]^{-\frac{j}{2}}. \tag{4.45}$$

Combining (4.41) to (4.45), it can be seen

$$\left\|f_\epsilon^{\gamma(\epsilon),b} - f\right\|_{L_r^2(\Omega)} \le \overbrace{[\gamma(\epsilon)]^{\frac{1}{2}} \eta^{\frac{1}{2}} \frac{\mathcal{C}_7(\theta_1 + \theta_2 T)\epsilon}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)} + \left(\frac{2\epsilon}{\mathcal{C}_5}\right) \|f\|_{L_r^2(\Omega)}}^{\mathcal{Z}_1}$$
$$+ \underbrace{\left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-j} \mathcal{M}\left(\frac{j}{2\eta}\right)^{\frac{j}{2}} [\gamma(\epsilon)]^{-\frac{j}{2}}}_{\mathcal{Z}_2}. \tag{4.46}$$

By choosing $[\gamma(\epsilon)]$ by

$$\gamma(\epsilon) = \left[\left(\frac{\mathcal{M}}{\epsilon}\right)^{\frac{2}{j+1}}\right], \tag{4.47}$$

We receive $\mathcal{Z}_1$ can be bounded as follows:

$$\mathcal{Z}_1 \le \epsilon^{\frac{j}{j+1}} \mathcal{M}^{\frac{1}{j+1}} \left(\frac{\mathcal{C}_7(\theta_1 + \theta_2 T)\eta^{\frac{1}{2}}}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)} + \frac{2\epsilon^{\frac{1}{j+1}}}{\mathcal{C}_5} \left|\mathcal{Z}_{1,1}^{\theta_1,\theta_2}(\beta, T, \mathcal{C}_5)\right|^{-\frac{j}{j+1}} \|g\|_{L_r^2(\Omega)}^{\frac{j}{j+1}}\right). \tag{4.48}$$

Similarly, from (4.45) and (4.47), $\mathcal{I}_2$ can be bounded as follows:

$$\mathcal{Z}_2 \le \epsilon^{\frac{j}{j+1}} \mathcal{M}^{\frac{1}{j+1}} \left(\left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-j} \left(\frac{j}{2\eta}\right)^{\frac{j}{2}}\right). \tag{4.49}$$

Finally, combining (4.48) to (4.49), the convergent rate can be established as follow

$$\left\|f_\epsilon^{\gamma(\epsilon),b} - f\right\|_{L_r^2(\Omega)} \le \epsilon^{\frac{j}{j+1}} \mathcal{M}^{\frac{1}{j+1}} \left(\left|\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)\right|^{-j} \left(\frac{j}{2\eta}\right)^{\frac{j}{2}}\right.$$
$$\left. + \frac{\mathcal{C}_7(\theta_1 + \theta_2 T)\eta^{\frac{1}{2}}}{\mathcal{L}(\theta_1, \theta_2, \lambda_{11}, \beta, T)} + \frac{2\epsilon^{\frac{1}{j+1}}}{\mathcal{C}_5} \left|\mathcal{Z}_{1,1}^{\theta_1,\theta_2}(\beta, T, \mathcal{C}_5)\right|^{-\frac{j}{j+1}} \|g\|_{L_r^2(\Omega)}^{\frac{j}{j+1}}\right). \tag{4.50}$$

$$\square$$

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