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A Robust Numerical Approach for Singularly Perturbed Problem with Integral Boundary Condition

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Highlights

- This study focuses on approximate solution of differential equation.
- A numerical approach is proposed for solving singularly perturbed problem with integral boundary value.
- The correctness and proficiency of the method were demonstrated.

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Abstract

In this research, the numerical integral method procedure on uniform mesh is used to solve the singularly perturbed problem which has integral boundary value. This method also includes the trapezoid method, the finite difference method, and the Thomas algorithm. The problem is converted to finite difference problem by using finite difference approximations and trapezoid method. Finally, the convergence of the presented method is analyzed through sample application. Thus, the correctness and sufficiency of the method are shown.

1. INTRODUCTION

The coefficient of the highest order derivative of the singularly perturbed equation is a parameter ε greater than zero but less than one. Due to this parameter ε, boundary layers (i.e. thin transition layers) are formed in the domain. While the behavior of the solution changes suddenly and rapidly in these parts, it changes slowly and regularly in other parts. These behaviors create boundless derivatives in solving singulary perturbed problems [1]. Some classical numerical methods applied cannot solve these problems. Therefore, suitable numerical methods that give stability and convergence with respect to ε as in this study should be preferred. For instance, finite difference method, perturbation methods and so on [2-10]. Numerical solution of singularly perturbed problems, existence-uniqueness and stability analysis have been extensively investigated by many researchers starting from the 1900s until now [11,12]. These problems occur in oceanography, electrical networks and various other practical engineering systems, medical science, aerodynamics, magnetic dynamics, control theory, spreading theory, reaction diffusion, fluid mechanics, electron plasma waves in plasma, communication lines, plasma dynamics, purified gas dynamics, mass motion, plastic, meteorology, electric current, and some physical models [13-17]. Thus, some researchers have focused on singularly perturbed problems with nonlocal boundary conditions [2-10, 14,16].

In some studies, the numerical integral method, after Taylor series expansion is made in the equation, the process is continued by taking the integral of the whole equation [18-22]. Unlike these, the equation given in [19] is integrated, then the difference derivatives are used respectively and it is continued with the trapezoidal method. Ultimately, an approximate solution is found with the Thomas algorithm. So the numerical integral method will contribute to the literature where it can be unhesitatingly and readily applied to many differential equations with nonlocal condition and delayed singular perturbation.

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In this research, we aimed to solve the following singular perturbated integral boundary value problems using the numerical integral method with the motivation given from the literature [2,19]. We can give this problem as

$$\varepsilon u'' + a(x)u' = g(x),\ x \in [0,1], \tag{1}$$

$$u(0) = A,\ \int_0^1 u(x)\,dx = B \tag{2}$$

where $0 < \varepsilon << 1$ is perturbation parameter, $a(x)$ and $g(x)$ are sufficiently smooth functions. A and B are real constants.

The contribution of this problem can be summarized as follows: Until now, singulary perturbed problems with integral boundary conditions have been solved by many methods [1,7,8,11]. Unlike these methods, it is shown in this study that singulary perturbed problems can be successfully solved by numerical integral method.

The study is planned as follows: In the introduction section, information about the singulary perturbed problem and the numerical integral method is given. In Section 2, the numerical integral method for singularly perturbed problem with right (left) boundary layer is expressed. In addition, the convergence and stability analysis of the proposed method is also presented. Two examples proposed to present the strength of method are examined and the results are shown in tables and figures.

2. NUMERICAL INTEGRAL METHOD AND ITS APPLICATIONS

In this section, we firstly give introduntion of numerical integral technique. The examples show us the left and right boundary layer problems, respectively.

2.1. Numerical Integral Method for the Left Boundary Layer Problem

Here the operation stages of the numerical integral method for the singularly perturbed problem whose solution is investigated are introduced according to the left boundary layer case. The method is similar for the problem with the right boundary layer [2].

If there is $a(x) > \alpha > 0,\ \alpha$ is constant in Equation (1), $x = 0$ is the left boundary layer. So the problem is a left boundary layer problem. For $i = 0,1,\dots,N-1$ in Equation (1), we integrate each term over the interval $[x_i, x_{i+1}]$ as

$$\varepsilon u'(x_{i+1}) - \varepsilon u'(x_i) + a_{i+1}u_{i+1} - a_i u_i + \int_{x_i}^{x_{i+1}} b(x)u(x)\,dx = \int_{x_i}^{x_{i+1}} g(x)\,dx.$$

Instead of derivatives $u'(x_{i+1})$ and $u'(x_i)$ in the above equation, the following expressions are written

$$u'_{i+1} = \frac{u_{i+1} - u_i}{h}, \quad u'_i = \frac{u_i - u_{i-1}}{h}$$

$$\varepsilon\left(\frac{u_{i+1} - u_i}{h}\right) - \varepsilon\left(\frac{u_i - u_{i-1}}{h}\right) + a(x_{i+1})u_{i+1} - a_i u_i + \int_{x_{i-1}}^{x_i} b(x)u(x)\,dx = \int_{x_{i-1}}^{x_i} g(x)\,dx, \tag{3}$$

where we use the following trapezoidal method for integrals in Equation (3), $\mathrm{i}=1,2,...,N-1,$

$$\varepsilon\left(\frac{u_{i+1}-u_i}{h}\right)-\varepsilon\left(\frac{u_i-u_{i-1}}{h}\right)+a_{i+1}u_{i+1}-a_iu_i+\frac{h}{2}\left[b_{i+1}u_{i+1}+b_iu_i\right]=\frac{h}{2}\left[g_{i+1}+g_i\right],$$

here we get the difference equation after some arranging

$$u_{i-1}\left(\frac{\varepsilon}{h}\right)-u_i\left(\frac{2\varepsilon}{h}+a_i-\frac{hb_i}{2}\right)+u_{i+1}\left(\frac{\varepsilon}{h}+a_{i+1}+\frac{hb_{i+1}}{2}\right)=\frac{h}{2}\left[g_i+g_{i-1}\right]\ .$$

If we add the boundary conditions to difference equation, we get the following finite difference problem

$$u_{i-1}\left(\frac{\varepsilon}{h}\right)-u_i\left(\frac{2\varepsilon}{h}+a_i-\frac{hb_i}{2}\right)+u_{i+1}\left(\frac{\varepsilon}{h}+a_{i+1}+\frac{hb_{i+1}}{2}\right)=\frac{h}{2}\left[g_i+g_{i-1}\right],$$

$$u\left(x_0\right)=A,\quad u\left(x_N\right)=\sum_{i=0}^{m-2}c_iu_{N_i}. \tag{4}$$

We use Thomas algorithm to solve the finite difference problem (1) and find that

$$A_i=\frac{\varepsilon}{h},\quad B_i=\frac{\varepsilon}{h}+a_{i+1}+\frac{hb_{i+1}}{2},\quad C_i=\frac{2\varepsilon}{h}+a_i-\frac{hb_i}{2},\quad F_i=-\frac{h}{2}\left[f_i+f_{i-1}\right],$$

$$\alpha_{i+1}=\frac{B_i}{C_i-\alpha_iA_i},\quad \beta_{i+1}=\frac{F_i+\beta_iA_i}{C_i-\alpha_iA_i},\quad i=1,2,...,N-1,$$

$$u_i=\alpha_{i+1}u_{i+1}+\beta_{i+1},\quad i=N-1,...,2,1,$$

$$\alpha_1=A,\qquad \beta_1=0.$$

With all these steps, the numerical solution is obtained with the help of a mathematical program.

Example 1.

$$\varepsilon u''\left(x\right)+u'\left(x\right)=1,\ \ 0<x<1,$$

$$u\left(0\right)=\varepsilon(\varepsilon\text{-}1)(1\text{-}\mathrm{e}^{\frac{-1}{\varepsilon}}),\ \ \int_0^1 u\left(x\right)dx=\varepsilon-\frac{1}{2}, \tag{5}$$

the exact solution is given by

$$u\left(x\right)=x+\varepsilon(\varepsilon-1)(1-e^{\frac{-1}{\varepsilon}})+(\varepsilon-1)(1-e^{\frac{-x_i}{\varepsilon}}).$$

Let's solve this problem, which is the left boundary layer at the point, $x=0$ with the numerical integral method as follows.

Now, we integrate all the terms of the singular perturbate Equation (5) in the interval $\left[x_i, x_{i+1}\right]$, then we make the necessary arrangements

$$\varepsilon\int_{x_i}^{x_{i+1}} u''(x)dx+\int_{x_i}^{x_{i+1}} u'(x)dx=\int_{x_i}^{x_{i+1}} dx,$$

$$\varepsilon u'(x_{i+1})-\varepsilon u'(x_i)+u(x_{i+1})-u(x_i)=\int_{x_i}^{x_{i+1}} 1dx$$

where we use $u_i'(x_{i+1})=\dfrac{u_{i+1}-u_i}{h}$ and $u_{i-1}'(x_i)=\dfrac{u_i-u_{i-1}}{h}$ finite difference approximations and the trapezoidal method for integrals as follows:

$$\varepsilon\left(\frac{u_{i+1}-u_i}{h}\right)-\varepsilon\left(\frac{u_i-u_{i-1}}{h}\right)+u(x_{i+1})-u(x_i)=\frac{h}{2}(g_{i+1}+g_i)\quad .$$

We rearrange the above equation according to the coefficients x_{i-1}, x_i and x_{i+1}, and then we get the following equation

$$u_{i-1}\left(\frac{\varepsilon}{h}\right)-u_i\left(\frac{2\varepsilon}{h}+1\right)+u_{i+1}\left(\frac{\varepsilon}{h}+1\right)=h.$$

If we add boundary conditions (5) to this difference equation, the following finite difference problem occurs:

$$u_{i-1}\left(\frac{\varepsilon}{h}\right)-u_i\left(\frac{2\varepsilon}{h}+1\right)+u_{i+1}\left(\frac{\varepsilon}{h}+1\right)=h,$$

$$u_0=\varepsilon(\varepsilon\text{-}1)(1\text{-}e^{\frac{-1}{\varepsilon}}),\quad u_N=\left[\varepsilon-\frac{1}{2}-\sum_{i=1}^{N-1}hu_i\right]h^{-1}.$$

Approximate solutions of u_i, $i=1,2,\ldots,N\text{-}1$ are obtained by using the Thomas algorithm for the above problem

$$A_i=\frac{\varepsilon}{h},\;\; B_i=\frac{\varepsilon}{h}+1,\;\; C_i=\frac{2\varepsilon}{h}+1,\;\; F_i=-h,\;\; \alpha_1=0,\;\; \beta_1=\varepsilon(\varepsilon\text{-}1)(1\text{-}e^{\frac{-1}{\varepsilon}}),$$

$$\alpha_{i+1}=\frac{\left(\dfrac{\varepsilon}{h}+1\right)}{\left(\dfrac{2\varepsilon}{h}+1\right)-\left(\dfrac{\varepsilon}{h}\right)\alpha_i},\quad \beta_{i+1}=\frac{h+\left(\dfrac{\varepsilon}{h}\right)\beta_i}{\left(\dfrac{2\varepsilon}{h}+1\right)-\left(\dfrac{\varepsilon}{h}\right)\alpha_i},\quad i=1,\ldots N-1,$$

$$u_i=\alpha_{i+1}u_{i+1}+\beta_{i+1},\;\; i=N\text{-}1,\ldots,2,1.$$

Using a suitable mathematics program, numerical results are obtained with the help of the Thomas algorithm given above. Maximum errors are found for values of $N = 64, 28, 256, 5\,2, 1024$ ve $\varepsilon = 2^{-8}, ..., 2^{-40}$. Curves of exact solution, numerical solution and error curves for values of N ve ε are plotted. Thus, we show that the proposed method is very suitable for singulary perturbed with integral bounded problems. Maximum errors according to values of N and perturbation parameter of ε are given in Table 1 below.

Table 1. *Maximum errors calculated of Example 1 for different* ε, N

$\varepsilon \mid N$	64	128	256	512	1024
2^{-8}	0.1809746555	0.1972246194	0.1316044632	0.0599011211	0.0211164414
2^{-10}	0.0587659709	0.1106674686	0.0856909697	0.1815069339	0.1978046920
2^{-12}	0.0587659709	0.0302956326	0.0588090542	0.1107486032	0.1816400037
2^{-14}	0.0038908135	0.0077514651	0.0153836758	0.0303011814	0.0588198251
2^{-20}	0.0000610316	0.0001221176	0.0002439550	0.0004880419	0.0009761165
2^{-25}	0.0000019375	0.0000037523	0.0000077553	0.0000152585	0.0000300065
2^{-30}	0.0000000294	0.0000000119	0.0000003644	0.0000000477	0.0000004434
2^{-35}	0.0000000002	0.0000000586	0.0000001334	0.0000000015	0.0000000003
2^{-40}	0.0000000000	0.0000000000	0.0000000000	0.0000002544	0.0000000001

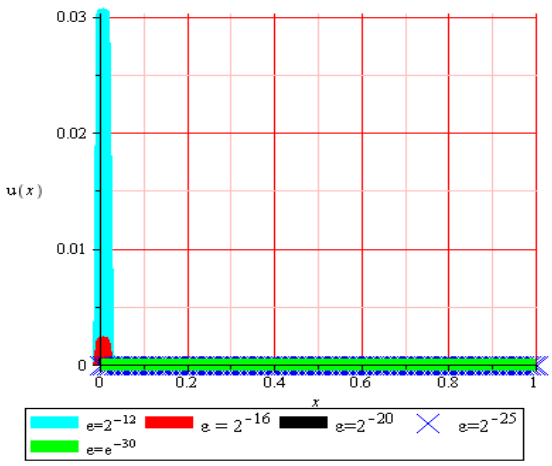

Figure 1. *Error curve of Example 1 for* $N = 128$

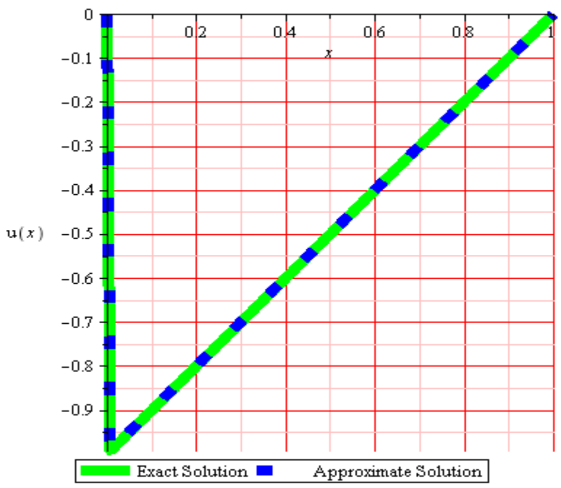

Figure 2. *Exact and approximate solution curves for* $N = 128$, $\varepsilon = 2^{-20}$

Example 2.

$$\varepsilon u''(x)-u'(x)=x+1,\quad 0<x<1,$$

$$u(0)=0,\quad \int_{0.5}^{1} u(x)dx=d, \tag{6}$$

$$d=\frac{-1}{e^{\frac{1}{\varepsilon}}-1}(0.02083333333(-102\varepsilon e^{\frac{1}{\varepsilon}}+48\varepsilon^2 e^{\frac{0.5}{\varepsilon}}+35+120\varepsilon e^{\frac{0.5}{\varepsilon}}+25e^{\frac{1}{\varepsilon}}+6\varepsilon-48\varepsilon^2 e^{\frac{1}{\varepsilon}})).$$

The problem has exact solution as

$$u(x)\ =\ \frac{-x^2}{2}+\frac{5+2\varepsilon}{2\left(e^{\frac{1}{\varepsilon}}-1\right)}\left(e^{\frac{x}{\varepsilon}}-1\right)-x-x\varepsilon.$$

Let's solve the singulary perturbed problem with integral boundary value with right boundary layer at point $x=1$ by numerical integral method.

First, we integrate the right boundary layer Equation (6) over the interval $[x_{i-1},x_i]$ and then we continue with the operations similar to the left boundary layer problem

$$\varepsilon\int_{x_{i-1}}^{x_i} u''(x)dx-\int_{x_{i-1}}^{x_i} u'(x)dx=\int_{x_{i-1}}^{x_i}(1+x)dx,$$

$$\varepsilon u'(x_i)-\varepsilon u'(x_{i-1})-u(x_i)+u(x_{i-1})=\int_{x_{i-1}}^{x_i}(1+x)dx$$

where $u_i'(x_i)=\dfrac{u_{i+1}-u_i}{h}$ and $u_{i-1}'(x_{i-1})=\dfrac{u_i-u_{i-1}}{h}$,

$$\varepsilon\left(\frac{u_{i+1}-u_i}{h}\right)-\varepsilon\left(\frac{u_i-u_{i-1}}{h}\right)-u(x_i)+u(x_{i-1})=\frac{h}{2}(g_i+g_{i-1}),\ \ \mathrm{g}_i=1+x_i.$$

If we arrange the above equation according to the coefficients x_{i-1}, x_i and x_{i+1}, we get the difference equation

$$u_{i-1}\left(\frac{\varepsilon}{h}+1\right)-u_i\left(\frac{2\varepsilon}{h}+1\right)+u_{i+1}\left(\frac{\varepsilon}{h}\right)=\frac{h}{2}(x_{i+1}+x_i+2).$$

We rewrite this difference equation with boundary conditions and construct the following finite difference problem

$$u_{i-1}\left(\frac{\varepsilon}{h}+1\right)-u_i\left(\frac{2\varepsilon}{h}+1\right)+u_{i+1}\left(\frac{\varepsilon}{h}\right)=h\left(x_{i+1}+x_i+2\right),$$

$$u\left(x_0\right)=0, \quad u_N=\left[d-\sum_{i=1}^{N-1}hu_i\right]h^{-1}. \tag{7}$$

We give the Thomas algorithm used to solve this difference problem (7) as follows:

$$A_i=\frac{\varepsilon}{h}+1,\ B_i=\frac{\varepsilon}{h},\ C_i=\frac{2\varepsilon}{h}+1,\ F_i=-h\left(x_{i+1}+x_i+2\right),\ \alpha_1=0,\ \beta_1=0,$$

$$\alpha_{i+1}=\frac{\left(\frac{\varepsilon}{h}\right)}{\left(\frac{2\varepsilon}{h}+1\right)-\left(\frac{\varepsilon}{h}+1\right)\alpha_i},\quad \beta_{i+1}=\frac{h\left(x_i+x_{i+1}+2\right)+\left(\frac{\varepsilon}{h}+1\right)\beta_i}{\left(\frac{2\varepsilon}{h}+1\right)-\left(\frac{\varepsilon}{h}+1\right)\alpha_i},\quad i=1,\ldots N-1,$$

$$u_i=\alpha_{i+1}u_{i+1}+\beta_{i+1},\ i=N-1,\ldots,2,1.$$

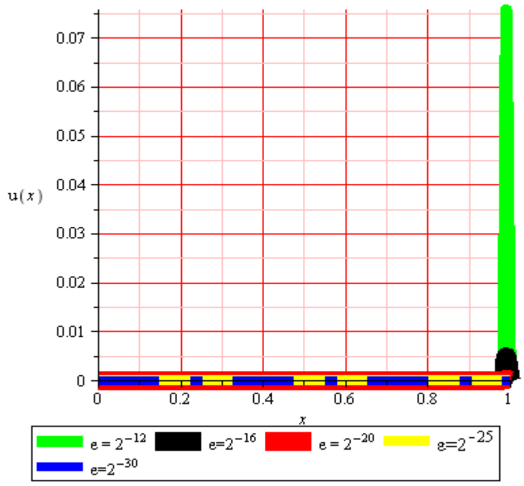

Figure 3. *Error curve of Example 2 for* $N=128$

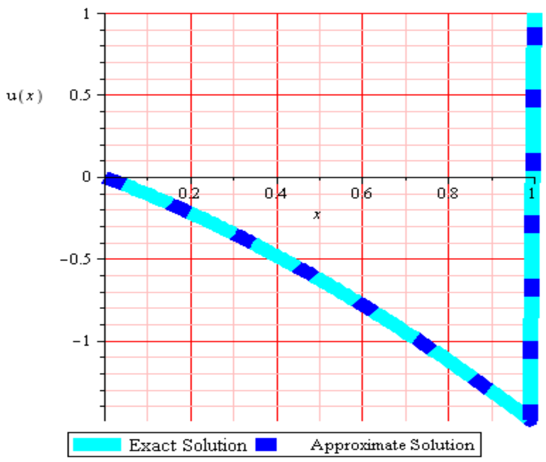

Figure 4. *Comparison of exact and approximate solution curves for* $N=128,\ \varepsilon=2^{-20}$

In Table 2, we give maximum errors for increasing values of N and decreasing values of ε. It can be seen from here that the errors approached zero.

Table 2. *Maximum errors calculated of Example 2 for different* ε, N

$\varepsilon \vert N$	64	128	256	512	1024
2^{-8}	0.4549206060	0.4957685556	0.3308174919	0.1917116380	0.1044645275
2^{-10}	0.1471159870	0.2770473000	0.4543883250	0.4951884830	0.3304304284
2^{-12}	0.0384652940	0.0757649740	0.1470729010	0.2769661670	0.4542552529
2^{-14}	0.0097278640	0.0193803180	0.0384624750	0.0757594230	0.0001525780
2^{-20}	0.0097278640	0.0003052240	0.0006100330	0.0012201070	0.0024397059
2^{-25}	0.0000048100	0.0000094560	0.0000192430	0.0000381470	0.0000756040
2^{-30}	0.0000000107	0.0000000298	0.0000000763	0.0000001192	0.0000001705

3. CONCLUSION

We have numerically solved the singularly perturbed integral boundary value problems involving right and left boundary layers by numerical integral method. The difference equation was obtained from the differential equation given by the numerical integral method. The trapezoidal method was used for the integrals in the equation. The difference equation is solved with the Thomas algorithm. We see that both the error values given in Tables 1 and 2 and the exact and approximate solution curves in Figure 2 and Figure 4 show that the approximate and exact solutions are quite close to each other. As seen in Figures 1 and 3, the error curves for Example 1 and Example 2 in the boundary layer regions are maximum due to the abrupt and quick change of the solution. In other words our results showed that the applied method worked very well.

As a contribution to the literature, the numerical integral method can also be applied to delayed differential equations and nonlocal problems with delayed singularly perturbed boundary values.

CONFLICTS OF INTEREST

No conflict of interest was declared by the author.

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