J. Algebra Comb. Discrete Appl.
8(2) • 73–90

Received: 23 April 2020
Accepted: 24 November 2020

Journal of Algebra Combinatorics Discrete Structures and Applications

On optimal linear codes of dimension 4*

Research Article

Nanami Bono, Maya Fujii, Tatsuya Maruta

Abstract: In coding theory, the problem of finding the shortest linear codes for a fixed set of parameters is central. Given the dimension k, the minimum weight d, and the order q of the finite field $\mathbb{F}_q$ over which the code is defined, the function $n_q(k,d)$ specifies the smallest length n for which an $[n,k,d]_q$ code exists. The problem of determining the values of this function is known as the problem of optimal linear codes. Using the geometric methods through projective geometry, we determine $n_q(4,d)$ for some values of d by constructing new codes and by proving the nonexistence of linear codes with certain parameters.

2010 MSC: 94B05, 94B27, 51E20

Keywords: Optimal linear codes, Griesmer bound, Geometric method

1. Introduction

We denote by $\mathbb{F}_q$ the field of q elements. Let $\mathbb{F}_q^n$ be the vector space of n-tuples over $\mathbb{F}_q$. An $[n,k,d]_q$ code $\mathcal{C}$ is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum weight $d = \min\{wt(c) \mid c \in \mathcal{C}, c \neq (0,\ldots,0)\}$, where $wt(c)$ is the number of non-zero entries in the vector c. The weight distribution of $\mathcal{C}$ is the list of integers A_i where A_i is the number of codewords of weight i, $0 \le i \le n$. The weight distribution with $(A_0, A_d, \ldots) = (1, \alpha, \ldots)$ is also expressed as $0^1 d^\alpha \cdots$. A fundamental problem in coding theory is to find $n_q(k,d)$, the minimum length n for which an $[n,k,d]_q$ code exists [10, 11]. An $[n,k,d]_q$ code satisfies the inequality called the *Griesmer bound* [8, 10]:

$$n \ge g_q(k,d) = \sum_{i=0}^{k-1} \left\lceil d/q^i \right\rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The values of $n_q(k,d)$ are determined for all d only for some small values of q and k. For $k=3$, $n_q(3,d)$ is known for all d for $q \le 9$ [1]. See

* *This research was partially supported by JSPS KAKENHI Grant Number 20K03722.*
Nanami Bono, Maya Fujii, Tatsuya Maruta (Corresponding Author); Department of Mathematical Sciences, Osaka Prefecture University, Sakai, Osaka 599-8531, Japan (email: bononanami@gmail.com, ddddy.maya0802@gmail.com, maruta@mi.s.osakafu-u.ac.jp).

https://doi.org/10.13069/jacodesmath.935947 ISSN 2148-838X

[26] for the updated table of $n_q(k,d)$ for some small q and k. The following theorems give some known values of $n_q(4,d)$.

Theorem 1.1 ([21, 25]). *$n_q(4,d) = g_q(4,d)$ for $1 \le d \le q-2$, $q^2-2q+1 \le d \le q^2-q$, $q^3-2q^2+1 \le d \le q^3-2q^2+q$, $q^3-q^2-q+1 \le d \le q^3+q^2-q$, $2q^3-5q^2+1 \le d \le 2q^3-5q^2+3q$ and any $d \ge 2q^3-3q^2+1$ for all q.*

Theorem 1.2 ([18, 21, 25]). *$n_q(4,d) = g_q(4,d)+1$ for the following d and q:*

(a) $q^2-q+1 \le d \le q^2-1$ *with* $q \ge 3$,

(b) $q^3-2q^2-q+1 \le d \le q^3-2q^2-\lfloor (q+1)/2 \rfloor$ *with* $q \ge 7$,

(c) $2q^3-3q^2-q+1 \le d \le 2q^3-3q^2$ *with* $q \ge 4$,

(d) $2q^3-3q^2-2q+1 \le d \le 2q^3-3q^2-q$ *with* $q \ge 5$.

Our main results are the following theorems.

Theorem 1.3. *$n_q(4,d) = g_q(4,d)$ for $2q^3-4q^2+1 \le d \le 2q^3-4q^2+2q$ for all q.*

Theorem 1.4. *$n_q(4,d) = g_q(4,d)+1$ for the following d and q:*

(a) $2q^3-3q^2-3q+1 \le d \le 2q^3-3q^2-2q$ *with* $q \ge 7$,

(b) $2q^3-4q^2-3q+1 \le d \le 2q^3-4q^2$ *with* $q \ge 7$,

(c) $2q^3-5q^2-q+1 \le d \le 2q^3-5q^2$ *with* $q \ge 7$.

We also tackle the problem to determine $n_8(4,d)$ for all d as a continuation of [14, 16, 24]. The problem to determine $n_8(4,d)$ for all d has been still open for the 447 values of d, see [26]. We determine $n_8(4,d)$ for 32 values of d and give new lower or upper bounds of $n_8(4,d)$ for 12 values of d as follows.

Theorem 1.5. (a) *$n_8(4,d) = g_8(4,d)+1$ for $d =$ 381-384, 574, 633-638, 690-701, 745-749, 809-812.*

(b) *$n_8(4,d) \le g_8(4,d)+1$ for $d = 133, 134, 145, 194$.*

(c) *$g_8(4,d)+1 \le n_8(4,d) \le g_8(4,d)+2$ for $d =$ 173-176, 178, 179, 247, 248.*

Remark 1.6. (a) *From Theorem 1.4 (a), the problem to determine $n_q(4,d)$ for $d = 2q^3-3q^2-3q+1$ is still open only for $q=5$, see [26].*

(b) *The nonexistence of a $[g_q(4,d),4,d]_q$ code for $d = 2q^3-rq^2-q+1$ for $3 \le r \le q-q/p$, $q=p^h$ with p prime, is proved in [19]. We conjecture that a $[g_q(4,d),4,d]_q$ code for $d = 2q^3-rq^2-q+1$ with $r = q-q/p-1$ does not exist for non-prime $q \ge 8$, which is valid for $q=8,9$ by Theorem 1.5 and [17].*

(c) *We conjecture that $n_q(4,d) = g_q(4,d)+1$ for $q^3-2q^2-q+1 \le d \le q^3-2q^2$ for all $q \ge 3$. To prove this, we need to show the existence of a $[g_q(4,d)+1,4,d]_q$ code for $d = q^3-2q^2$ by Theorem 1.2 (b). This is already known for $q=3,4,5$ and is also valid for $q=8$ by Theorem 1.5.*

We recall geometric methods through projective geometry and preliminary results in Section 2. We prove Theorem 1.3 and some upper bounds on $n_q(4,d)$ in Theorems 1.4 and 1.5 in Section 3. The proofs of Theorems 1.4 and 1.5 are completed by the nonexistence of some Griesmer codes, which are given in Section 4.

2. Geometric methods

In this section, we give geometric methods to construct new codes or to prove the nonexistence of codes with certain parameters. We denote by $\mathrm{PG}(r, q)$ the projective geometry of dimension r over $\mathbb{F}_q$. A j-flat is a projective subspace of dimension j in $\mathrm{PG}(r, q)$. The 0-flats, 1-flats, 2-flats, $(r-2)$-flats and $(r-1)$-flats are called *points, lines, planes, secundums* and *hyperplanes*, respectively. We denote by θ_j the number of points in a j-flat, i.e., $\theta_j = (q^{j+1}-1)/(q-1)$.

Let $\mathcal{C}$ be an $[n, k, d]_q$ code having no coordinate which is identically zero. The columns of a generator matrix of $\mathcal{C}$ can be considered as a multiset of n points in $\Sigma = \mathrm{PG}(k-1, q)$ denoted by $\mathcal{M}_{\mathcal{C}}$. We see linear codes from this geometrical point of view. A point P in Σ is called an *i-point* if it has multiplicity $m_{\mathcal{C}}(P) = i$ in $\mathcal{M}_{\mathcal{C}}$. Denote by γ_0 the maximum multiplicity of a point from Σ in $\mathcal{M}_{\mathcal{C}}$ and let C_i be the set of i-points in Σ, $0 \le i \le \gamma_0$. We denote by $\Delta_1 + \cdots + \Delta_s$ the multiset consisting of the s sets $\Delta_1, ..., \Delta_s$ in Σ. We write $s\Delta$ for $\Delta_1 + \cdots + \Delta_s$ when $\Delta_1 = \cdots = \Delta_s$. Then, $\mathcal{M}_{\mathcal{C}} = \sum_{i=1}^{\gamma_0} iC_i$. For any subset S of Σ, we denote by $\mathcal{M}_{\mathcal{C}}(S)$ the multiset $\{m_{\mathcal{C}}(P)P \mid P \in S\}$. The *multiplicity of S with respect to $\mathcal{C}$*, denoted by $m_{\mathcal{C}}(S)$, is defined as the cardinality of $\mathcal{M}_{\mathcal{C}}(S)$, i.e.,

$$m_{\mathcal{C}}(S) = \sum_{P\in S} m_{\mathcal{C}}(P) = \sum_{i=1}^{\gamma_0} i\cdot|S\cap C_i|,$$

where $|T|$ denotes the number of elements in a set T. Then we obtain the partition $\Sigma = \bigcup_{i=0}^{\gamma_0} C_i$ such that $n = m_{\mathcal{C}}(\Sigma)$ and

$$n - d = \max\{m_{\mathcal{C}}(\pi) \mid \pi \in \mathcal{F}_{k-2}\},$$

where $\mathcal{F}_j$ denotes the set of j-flats in Σ. Such a partition of Σ is called an *$(n, n-d)$-arc* of Σ. Conversely an $(n, n-d)$-arc of Σ gives an $[n, k, d]_q$ code in the natural manner. A line l with $t = m_{\mathcal{C}}(l)$ is called a *t-line*. A *t-plane*, a *t-hyperplane* and so on are defined similarly. For an m-flat Π in Σ we define

$$\gamma_j(\Pi) = \max\{m_{\mathcal{C}}(\Delta) \mid \Delta \subset \Pi,\ \Delta \in \mathcal{F}_j\},\ 0 \le j \le m.$$

Let $\lambda_s(\Pi)$ be the number of s-points in Π. We denote simply by γ_j and by λ_s instead of $\gamma_j(\Sigma)$ and $\lambda_s(\Sigma)$, respectively. It holds that $\gamma_{k-2} = n-d,\ \gamma_{k-1} = n$. When $\mathcal{C}$ is Griesmer, the values $\gamma_0, \gamma_1, ..., \gamma_{k-3}$ are also uniquely determined ([22]) as follows:

$$\gamma_j = \sum_{u=0}^{j} \left\lceil \frac{d}{q^{k-1-u}} \right\rceil \quad \text{for } 0 \le j \le k-1. \tag{1}$$

When $\gamma_0 = 2$, we obtain

$$\lambda_2 = \lambda_0 + n - \theta_{k-1} \tag{2}$$

from $\lambda_0 + \lambda_1 + \lambda_2 = \theta_{k-1}$ and $\lambda_1 + 2\lambda_2 = n$. Denote by a_i the number of i-hyperplanes in Σ. Note that

$$a_i = A_{n-i}/(q-1) \quad \text{for} \quad 0 \le i \le n-d. \tag{3}$$

The list of a_i's is called the *spectrum* of $\mathcal{C}$. We usually use τ_j's for the spectrum of a hyperplane of Σ to distinguish from the spectrum of $\mathcal{C}$. Simple counting arguments yield the following:

$$\sum_{i=0}^{\gamma_{k-2}} a_i = \theta_{k-1}, \tag{4}$$

$$\sum_{i=1}^{\gamma_{k-2}} ia_i = n\theta_{k-2}, \tag{5}$$

$$\sum_{i=2}^{\gamma_{k-2}} i(i-1)a_i = n(n-1)\theta_{k-3} + q^{k-2}\sum_{s=2}^{\gamma_0} s(s-1)\lambda_s. \tag{6}$$

When $\gamma_0 \le 2$, we get the following from (4)-(6):

$$\sum_{i=0}^{n-d-2} \binom{n-d-i}{2} a_i = \binom{n-d}{2}\theta_{k-1} - n(n-d-1)\theta_{k-2} + \binom{n}{2}\theta_{k-3} + q^{k-2}\lambda_2. \tag{7}$$

Lemma 2.1 ([17, 31]). *Put $\epsilon = (n-d)q - n$ and $t_0 = \lfloor (w+\epsilon)/q \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. Let Π be a w-hyperplane through a t-secundum δ. Then $t \le (w+\epsilon)/q$ and the following hold.*

(a) *$a_w = 0$ if an $[w, k-1, d_0]_q$ code with $d_0 \ge w - t_0$ does not exist.*

(b) *$\gamma_{k-3}(\Pi) = t_0$ if an $[w, k-1, d_1]_q$ code with $d_1 \ge w - t_0 + 1$ does not exist.*

(c) *Let c_j be the number of j-hyperplanes through δ other than Π. Then $\sum_j c_j = q$ and*

$$\sum_j (\gamma_{k-2} - j)c_j = w + \epsilon - qt. \tag{8}$$

(d) *A γ_{k-2}-hyperplane with spectrum $(\tau_0, \dots, \tau_{\gamma_{k-3}})$ satisfies $\tau_t > 0$ if $w + \epsilon - qt < q$.*

(e) *If any γ_{k-2}-hyperplane has no t_0-secundum, then $m_{\mathcal{C}}(\Pi) \le t_0 - 1$.*

An $[n, k, d]_q$ code is called *m-divisible* if all codewords have weights divisible by an integer $m > 1$.

Lemma 2.2 ([31]). *Let $\mathcal{C}$ be an m-divisible $[n, k, d]_q$ code with $q = p^h$, p prime, whose spectrum is*

$$(a_{n-d-(w-1)m}, a_{n-d-(w-2)m}, \dots, a_{n-d-m}, a_{n-d}) = (\alpha_{w-1}, \alpha_{w-2}, \dots, \alpha_1, \alpha_0),$$

where $m = p^r$ for some $1 \le r < h(k-2)$ satisfying $\lambda_0 > 0$ and

$$\bigcap_{H \in \mathcal{F}_{k-2},\ m_{\mathcal{C}}(H) < n-d} H = \emptyset. \tag{9}$$

Then there exists a t-divisible $[n^*, k, d^*]_q$ code $\mathcal{C}^*$ with $t = q^{k-2}/m$, $n^* = \sum_{j=0}^{w-1} j\alpha_j = ntq - \frac{d}{m}\theta_{k-1}$, $d^* = ((n-d)q - n)t$ whose spectrum is

$$(a_{n^*-d^*-\gamma_0 t}, a_{n^*-d^*-(\gamma_0-1)t}, \dots, a_{n^*-d^*-t}, a_{n^*-d^*}) = (\lambda_{\gamma_0}, \lambda_{\gamma_0-1}, \dots, \lambda_1, \lambda_0).$$

$\mathcal{C}^*$ is called a *projective dual* of $\mathcal{C}$, see also [4] and [11]. The condition (9) is needed to guarantee that $\mathcal{C}^*$ has dimension k although it was missing in Lemma 5.1 of [31]. Note that a generator matrix for $\mathcal{C}^*$ is given by considering $(n-d-jm)$-hyperplanes as j-points in the dual space Σ^* of Σ for $0 \le j \le w-1$ [31].

Example 2.3. *Let $\mathcal{C}$ be a $[16, 5, 9]_3$ code with generator matrix*

$$G = \begin{pmatrix} 1&0&0&0&0&1&1&1&1&1&1&1&1&0&0&0 \\ 0&1&0&0&0&1&1&1&2&2&2&0&0&1&1&0 \\ 0&0&1&0&0&2&0&0&1&2&2&2&0&1&2&1 \\ 0&0&0&1&0&2&2&0&0&1&0&2&2&2&1&1 \\ 0&0&0&0&1&2&0&2&2&1&0&1&1&0&2&1 \end{pmatrix}.$$

Then, the weight distribution of $\mathcal{C}$ is $0^1 9^{116} 12^{114} 15^{12}$, and $\mathcal{C}$ is 3-divisible. Hence, from (3), $\mathcal{C}$ has spectrum $(a_1, a_4, a_7) = (6, 57, 58)$. In this case, $\mathcal{M}_{\mathcal{C}}$ is not a multiset but a set of 16 points of $\Sigma = \mathrm{PG}(4,3)$ corresponding to the columns of G, for $\gamma_0 = 1$. Considering the $(7-3j)$-hyperplanes as j-points in the dual space Σ^* of Σ for $j = 0, 1, 2$, one can get a 9-divisible $[69, 5, 45]_3$ code $\mathcal{C}^*$. Actually, $[69, 5, 45]_3$ codes are unique up to equivalence, see [5] for the detail.

Lemma 2.4 ([27]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code and let $\cup_{i=0}^{\gamma_0} C_i$ be the partition of $\Sigma = \mathrm{PG}(k-1, q)$ obtained from $\mathcal{C}$. If $\cup_{i \geq 1} C_i$ contains a t-flat Δ and if $d > q^t$, then there exists an $[n - \theta_t, k, d']_q$ code $\mathcal{C}'$ with $d' \geq d - q^t$.*

The punctured code $\mathcal{C}'$ in Lemma 2.4 can be constructed from $\mathcal{C}$ by removing the t-flat Δ from the multiset $\mathcal{M}_{\mathcal{C}}$. We denote the resulting multiset by $\mathcal{M}_{\mathcal{C}} - \Delta$. The method to construct new codes from a given $[n, k, d]_q$ code by deleting the coordinates corresponding to some geometric object in $\mathrm{PG}(k-1, q)$ is called *geometric puncturing*, see [25].

Lemma 2.5 ([3]). *Let $\mathcal{C}_1$ be an $[n_1, k, d_1]_q$ code containing a codeword of weight $d_1 + m$ with $m > 0$ and let $\mathcal{C}_2$ be an $[n_2, k-1, d_2]_q$ code. Then, adding $\mathcal{M}_{\mathcal{C}_2}$ to an $(n_1 - d_1 - m)$-hyperplane for $\mathcal{C}_1$ gives an $[n_1 + n_2, k, d]_q$ code with $d = d_1 + m$ if $m < d_2$ and $d = d_1 + d_2$ if $m \geq d_2$.*

An $[n, k, d]_q$ code with generator matrix G is called *extendable* if there exists a vector $h \in \mathbb{F}_q^k$ such that the extended matrix $[G\ h^{\mathrm{T}}]$ generates an $[n+1, k, d+1]_q$ code. The following theorems will be applied to prove the extendability of codes with certain parameters in Sections 4 and 5.

Theorem 2.6 ([23],[32]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $q \geq 5$, $d \equiv -2 \pmod q$, $k \geq 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0, -1, -2 \pmod q$.*

Theorem 2.7 ([30]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $\gcd(d, q) = 1$. Then $\mathcal{C}$ is extendable if $\sum_{i \not\equiv n, n-d \pmod q} a_i < q^{k-2}$.*

Theorem 2.8 ([29]). *Let $\mathcal{C}$ be an $[n, k, d]_q$ code with $q = 2^h$, $h \geq 3$, d odd, $k \geq 3$. Then $\mathcal{C}$ is extendable if $A_i = 0$ for all $i \not\equiv 0, d \pmod{q/2}$.*

A set of s lines in $\mathrm{PG}(2, q)$ is called an *s-arc of lines* if no three of which are concurrent. An f-multiset $\mathcal{F}$ in $\mathrm{PG}(2, q)$ is an *(f, m)-minihyper* if every line meets $\mathcal{F}$ in at least m points and if some line meets $\mathcal{F}$ in exactly m points with multiplicity.

Lemma 2.9 ([20]). *For $x = \dfrac{q}{2} + 1$ with q even, every $(x(q+1), x)$-minihyper in $PG(2, q)$ is either the sum of x lines or the union of the lines forming a $(q+2)$-arc of lines.*

3. Construction results

In this section, we prove Theorems 1.3, 1.4(b) and a part of Theorem 1.5.

Lemma 3.1. *There exist $[n = 2q^3 - 2q^2 + 1 - t(q+1), 4, 2q^3 - 4q^2 + 2q - tq]_q$ codes for $0 \leq t \leq q-1$ for $q \geq 7$.*

Proof. For $q \geq 7$, let $\mathcal{H}$ be a hyperbolic quadric in $\mathrm{PG}(3, q)$, see [12] for hyperbolic quadric. Let l_1 and l_2 be two skew lines contained in $\mathcal{H}$. Take two skew lines l_3 and l_4 contained in $\mathcal{H}$ meeting l_1, l_2 and four points $P_1, \ldots, P_4$ of $\mathcal{H}$ so that $l_1 \cap l_3 = P_1$, $l_1 \cap l_4 = P_2$, $l_2 \cap l_3 = P_3$, $l_2 \cap l_4 = P_4$. Let $l_5 = \langle P_1, P_4 \rangle$, $l_6 = \langle P_2, P_3 \rangle$ and $\Delta_{ij} = \langle l_i, l_j \rangle$, where $\langle \chi_1, \chi_2, \cdots \rangle$ denotes the smallest flat containing $\chi_1, \chi_2, \cdots$. We set

$$C_0 = l_1 \cup l_2 \cup \cdots \cup l_6, \quad C_1 = (\Delta_{13} \cup \Delta_{14} \cup \Delta_{23} \cup \Delta_{24} \cup \mathcal{H}) \setminus C_0$$

and $C_2 = \mathrm{PG}(3,q) \setminus (C_0 \cup C_1)$. Then $\lambda_0 = 6q-2$, $\lambda_1 = 5q^2-10q+5$, $\lambda_2 = q^3-4q^2+5q-2$, where $\lambda_i = |C_i|$, and the multiset $C_1 + 2C_2$ gives a Griesmer $[2q^3-3q^2+1, 4, 2q^3-5q^2+3q]_q$ code, say $\mathcal{C}$. This construction is due to [16].

Next, take a line l contained in $\mathcal{H}$ such that l is skew to l_3 and l_4. Let $l \cap l_1 = Q_1$, $l \cap l_2 = Q_2$ and let $\delta_1, \dots, \delta_{q-1}$ be the planes through l other than $\langle l, l_1 \rangle$, $\langle l, l_2 \rangle$. Then each δ_i meets l_1 and l_2 in the points Q_1 and Q_2, respectively, and meets $l_3, \dots, l_6$ in some points out of l. Hence, we can take a line m_i in δ_i with $m_i \cap C_0 = \emptyset$ for $1 \le i \le q-1$ such that $m_1 \cap l, \cdots, m_{q-1} \cap l$ are distinct points. Now, take an elliptic quadric $\mathcal{E}$ and let $\mathcal{E}'$ be the projection of $\mathcal{E}$ from a point $R \in \mathcal{E} \setminus \Delta_{13}$ on to Δ_{13}. Since $m_{\mathcal{C}}(\Delta_{13}) = q^2-2q+1$, it follows from Lemma 2.5 that the multiset $\mathcal{M}' = \mathcal{M}_{\mathcal{C}} + \mathcal{E}'$ gives a $[2q^3-2q^2+1, 4, 2q^3-4q^2+2q]_q$ code, say $\mathcal{C}'$. Applying Lemma 2.4 by deleting t of the lines $m_1, \dots, m_{q-1}$, we get an $[n = 2q^3-2q^2+1-t\theta_1, 4, d = 2q^3-4q^2+2q-tq]_q$ code. □

The code constructed by Lemma 3.1 is Griesmer for $t = 0, 1$ and the length satisfies $n = g_q(4,d)+1$ for $2 \le t \le q-1$. Hence, Theorem 1.3 follows from the existence of Griesmer codes with $d = 2q^3-4q^2+2q, 2q^3-4q^2+3q$ by puncturing. We also have that $n_q(4,d) \le g_q(4,d)+1$ for $2q^3-5q^2+2q+1 \le d \le 2q^3-4q^2-2q$. Since Theorem 1.4 (2) is already known for $q \ge 9$ [19], it suffices to show the nonexistence of Griesmer codes for $d = 2q^3-4q^2-3q+1$ for $q = 7, 8$, which is given in Section 4, see Lemma 4.1.

Next, we give a method to construct good codes by some orbits of a given projectivity in $\mathrm{PG}(k-1,q)$. For a non-zero element $\alpha \in \mathbb{F}_q$, let $R = \mathbb{F}_q[x]/(x^N - \alpha)$ be the ring of polynomials over $\mathbb{F}_q$ modulo $x^N - \alpha$. We associate the vector $(a_0, a_1, ..., a_{N-1}) \in \mathbb{F}_q^N$ with the polynomial $a(x) = \sum_{i=0}^{N-1} a_i x^i \in R$. For $\mathbf{g} = (g_1(x), ..., g_m(x)) \in R^m$,

$$C_{\mathbf{g}} = \{(r(x)g_1(x), ..., r(x)g_m(x)) \mid r(x) \in R\}$$

is called the 1-*generator quasi-twisted* (QT) *code* with generator $\mathbf{g}$. $C_{\mathbf{g}}$ is usually called quasi-cyclic (QC) when $\alpha = 1$. When $m = 1$, $C_{\mathbf{g}}$ is called α-*cyclic* or *pseudo-cyclic* or *constacyclic*. All of these codes are generalizations of cyclic codes ($\alpha = 1$, $m = 1$). Take a monic polynomial $g(x) = x^k - \sum_{i=0}^{k-1} a_i x^i$ in $\mathbb{F}_q[x]$ dividing $x^N - \alpha$ with non-zero $\alpha \in \mathbb{F}_q$, and let T be the companion matrix of $g(x)$. Let τ be the projectivity of $\mathrm{PG}(k-1,q)$ defined by T. We denote by $[g^n]$ or by $[a_0 a_1 \cdots a_{k-1}^n]$ the $k \times n$ matrix $[P, TP, T^2P, ..., T^{n-1}P]$, where P is the column vector $(1,0,0,...,0)^{\mathrm{T}}$ (h^{T} stands for the transpose of a row vector h). Then $[g^N]$ generates an α^{-1}-cyclic code. Hence one can construct a cyclic or pseudo-cyclic code from an orbit of τ. For non-zero vectors $P_2^{\mathrm{T}}, ..., P_m^{\mathrm{T}} \in \mathbb{F}_q^k$, we denote the matrix

$$[P, TP, T^2P, ..., T^{n_1-1}P; P_2, TP_2, ..., T^{n_2-1}P_2; \cdots; P_m, TP_m, ..., T^{n_m-1}P_m]$$

by $[g^{n_1}] + P_2^{n_2} + \cdots + P_m^{n_m}$. Then, the matrix $[g^N] + P_2^N + \cdots + P_m^N$ defined from m orbits of τ of length N generates a QC or QT code, see [28]. It is shown in [28] that many good codes can be constructed from orbits of projectivities.

Example 3.2. *Take $g(x) = 1+x+x^2+x^4 \in \mathbb{F}_2[x]$ and a point $Q(1,0,0,1) \in \mathrm{PG}(3,2)$. Then, the matrix $[g^7]$ generates a cyclic Hamming $[7,4,3]_2$ code and the matrix*

$$[g^7] + Q^7 = \begin{pmatrix} 1&0&0&0&1&0&1&1&1&0&1&0&0&1 \\ 0&1&0&0&1&1&1&0&0&1&1&1&0&1 \\ 0&0&1&0&1&1&0&0&1&0&0&1&1&1 \\ 0&0&0&1&0&1&1&1&0&1&0&0&1&1 \end{pmatrix}.$$

generates a QC $[14,4,7]_2$ code with weight distribution $0^1 7^8 8^7$.

Let $\mathbb{F}_8 = \{0, 1, \alpha, \alpha^2, \dots, \alpha^6\}$, with $\alpha^3 = \alpha + 1$. For simplicity, we denote $\alpha, \dots, \alpha^6$ by $2, 3, \dots, 7$ so that $\mathbb{F}_8 = \{0, 1, 2, \dots, 7\}$. It sometimes happens that QC or QT codes are divisible or can be extended to divisible codes.

Lemma 3.3. *There exists a $[440, 4, 384]_8$ code.*

Proof. Let $\mathcal{C}$ be the QC $[40,4,32]_8$ code with generator matrix $[1111^5]+0121^5+0124^5+0141^5+0165^5+0171^5+1035^5+1053^5$. Then $\mathcal{C}$ is a 4-divisible code with weight distribution $0^1 32^{1155} 36^{2800} 40^{140}$. Applying Lemma 2.2, as the projective dual of $\mathcal{C}$, one can get a 16-divisible $[440,4,384]_8$ code $\mathcal{C}^*$ with weight distribution $0^1 384^{3815} 400^{280}$. □

Lemma 3.4. *There exist codes with parameters* $[156,4,134]_8$, $[169,4,145]_8$, $[208,4,179]_8$, $[225,4,194]_8$ *and* $[286,4,248]_8$.

Proof. The QC codes with generator matrices

$$[1464^{13}]+1004^{13}+1504^{13}+1524^{13}+1625^{13}+1145^{13}+1272^{13}+1643^{13}+1126^{13}$$
$$+1062^{13}+1144^{13}+1017^{13},$$
$$[1464^{13}]+1004^{13}+1504^{13}+1524^{13}+1523^{13}+1427^{13}+1471^{13}+1445^{13}+1643^{13}$$
$$+1126^{13}+1062^{13}+1510^{13}+1017^{13},$$
$$[1464^{13}]+1004^{13}+1504^{13}+1524^{13}+1523^{13}+1625^{13}+1471^{13}+1445^{13}+1232^{13}$$
$$+1126^{13}+1062^{13}+1401^{13}+1752^{13}+1731^{13}+1510^{13}+1017^{13},$$
$$[1001^{15}]+1004^{15}+1504^{15}+1523^{15}+1423^{15}+1133^{15}+1757^{15}+1277^{15}+1232^{15}$$
$$+1273^{15}+1036^{15}+1307^{15}+1707^{15}+1265^{15}+1144^{15},$$
$$[1464^{13}]+1004^{13}+1504^{13}+1524^{13}+1523^{13}+1423^{13}+1625^{13}+1427^{13}+1465^{13}$$
$$+1133^{13}+1232^{13}+1160^{13}+1231^{13}+1330^{13}+1062^{13}+1265^{13}+1144^{13}+1740^{13}$$
$$+1050^{13}+1274^{13}+1731^{13}+1017^{13}$$

give the desired codes with the following weight distributions

$$0^1 134^{1820} 136^{1183} 138^{364} 140^{364} 144^{182} 148^{182},$$
$$0^1 145^{1365} 146^{637} 147^{546} 148^{364} 149^{182} 150^{273} 152^{273} 154^{182} 156^{91} 157^{91} 160^{91},$$
$$0^1 179^{1092} 180^{637} 181^{728} 182^{546} 184^{728} 193^{364},$$
$$0^1 194^{1785} 196^{1050} 198^{420} 200^{210} 202^{105} 204^{210} 206^{210} 208^{105},$$
$$0^1 248^{3003} 256^{1001} 264^{91},$$

respectively. □

Since it is known that $g_8(4,d)+1 \le n_8(4,d) \le g_8(4,d)+2$ for $381 \le d \le 384$, Theorem 1.5 (a) for $381 \le d \le 384$, Theorem 1.5 (b) and (c) for $d = 178, 179, 247, 248$ follow from Lemmas 3.3 and 3.4.

4. Nonexistence of some Griesmer codes

Note that one can get an $[n-1,k,d-1]_q$ code from a given $[n,k,d]_q$ code by puncturing and that the nonexistence of an $[n-1,k,d-1]_q$ code implies the nonexistence of an $[n,k,d]_q$ code. Hence, to prove (a) and (b) of Theorem 1.4, it suffices to show the following.

Lemma 4.1. *There exists no* $[g_q(4,d),4,d]_q$ *code for* $d = 2q^3 - sq^2 - 3q + 1$ *with* $s = 3,4$ *for* $q \ge 7$.

Lemma 4.1 was proved for $q \ge 9$ in [19]. It follows from Theorem 1.5(a) that Lemma 4.1 is valid for $q = 8$, see Lemmas 4.14 and 4.17 in this section. We can also prove Lemma 4.1 for $q = 7$, but we omit the proof here because it is quite similar to the proof for $q = 8$, see [6] for the detail. The existence of a $[g_q(4,d)+1,4,d]_q$ code for $d = 2q^3 - 3q^2 - 2q$ is obtained from the result in [15]. It is also known that $n_q(4,d) = g_q(4,d)+1$ for $2q^3 - 5q^2 - q + 1 \le d \le 2q^3 - 5q^2$ with $q \ge 7$ except fpr $q = 8$ [18]. Hence, Theorem 1.4(c) follows from Theorem 1.5(a), see Lemma 4.12 below.

In this section, we prove that there exists no $[g_8(4,d),4,d]_8$ code for $d = 173, 574, 633, 690, 697, 745, 809$, giving Theorem 1.5. $n_8(3,d)$ is already known for all d as follows, see [1, 7, 26].

Table 1. The spectra of some $[n, 3, d]_8$ codes.

parameters	possible spectra	reference
$[6, 3, 4]_8$	$(a_0, a_1, a_2) = (34, 24, 15)$	[14]
$[7, 3, 5]_8$	$(a_0, a_1, a_2) = (31, 21, 21)$	[14]
$[8, 3, 6]_8$	$(a_0, a_1, a_2) = (29, 16, 28)$	[14]
$[9, 3, 7]_8$	$(a_0, a_1, a_2) = (28, 9, 36)$	[14]
$[10, 3, 8]_8$	$(a_0, a_2) = (28, 45)$	[14]
$[26, 3, 22]_8$	$(a_0, a_2, a_3, a_4) = (10, 1, 16, 46)$	Lemma 4.3
$[33, 3, 28]_8$	$(a_0, a_3, a_5) = (9, 16, 48)$	[16]
	$(a_0, a_1, a_4, a_5) = (4, 5, 28, 36)$	
	$(a_0, a_3, a_4, a_5) = (6, 10, 18, 39)$	
$[42, 3, 36]_8$	$(a_0, a_4, a_5, a_6) = (4, 6, 24, 39)$	[2]
	$(a_0, a_3, a_5, a_6) = (3, 7, 21, 42)$	
	$(a_0, a_4, a_6) = (3, 21, 49)$	
	$(a_0, a_2, a_4, a_6) = (2, 3, 18, 50)$	
$[60, 3, 52]_8$	$(a_4, a_6, a_8) = (3, 16, 54)$	[16]
	$(a_0, a_4, a_7, a_8) = (1, 1, 32, 39)$	
	$(a_0, a_5, a_6, a_7, a_8) = (1, 1, 3, 27, 41)$	
	$(a_0, a_6, a_7, a_8) = (1, 1, 6, 24, 42)$	
$[61, 3, 53]_8$	$(a_0, a_5, a_7, a_8) = (1, 1, 24, 47)$	[16]
	$(a_0, a_6, a_7, a_8) = (1, 3, 21, 48)$	
$[62, 3, 54]_8$	$(a_0, a_6, a_7, a_8) = (1, 1, 16, 55)$	[16]
$[63, 3, 55]_8$	$(a_0, a_7, a_8) = (1, 9, 63)$	[9]
$[64, 3, 56]_8$	$(a_0, a_8) = (1, 72)$	[9]
$[69, 3, 60]_8$	$(a_5, a_8, a_9) = (1, 32, 40)$	[9]
	$(a_6, a_7, a_8, a_9) = (1, 3, 27, 42)$	
	$(a_7, a_8, a_9) = (6, 24, 43)$	
$[70, 3, 61]_8$	$(a_6, a_8, a_9) = (1, 24, 48)$	[9]
	$(a_7, a_8, a_9) = (3, 21, 49)$	
$[71, 3, 62]_8$	$(a_7, a_8, a_9) = (1, 16, 46)$	[9]
$[72, 3, 63]_8$	$(a_8, a_9) = (9, 64)$	[9]
$[73, 3, 64]_8$	$a_9 = 73$	[9]
$[92, 3, 80]_8$	$(a_0, a_8, a_{12}) = (1, 9, 63)$	[20]
	$(a_4, a_{12}) = (6, 67)$	
	$(a_4, a_8, a_{12}) = (1, 10, 62)$	
	$(a_8, a_{12}) = (12, 61)$	
$[101, 3, 88]_8$	$(a_5, a_{13}) = (5, 68)$	[20]
	$(a_9, a_{13}) = (10, 63)$	
$[108, 3, 94]_8$	$(a_4, a_6, a_{13}, a_{14}) = (1, 3, 16, 53)$	[16]
	$(a_5, a_6, a_{12}, a_{13}, a_{14}) = (2, 2, 1, 14, 54)$	
	$(a_5, a_6, a_{12}, a_{13}, a_{14}) = (1, 3, 1, 15, 53)$	
	$(a_6, a_{12}, a_{13}, a_{14}) = (4, 1, 16, 52)$	
	$(a_6, a_{12}, a_{14}) = (4, 9, 60)$	

Theorem 4.2. *$n_8(3,d) = g_8(3,d) + 1$ for $d = 13\text{-}16, 29\text{-}32, 37\text{-}40, 43\text{-}48$ and $n_8(3,d) = g_8(3,d)$ for any other d.*

Lemma 4.3. *Every $[26,3,22]_8$ code has spectrum $(a_0,a_2,a_3,a_4) = (10,1,16,46)$.*

Proof. Let $\mathcal{C}$ be a $[26,3,22]_8$ code. By (1), $\gamma_0 = 1$ and $\gamma_1 = 4$. Since $(\gamma_1 - \gamma_0)\theta_1 + \gamma_0 - 2 = 26$, any t-line though a fixed 1-point satisfies $t \geq \gamma_1 - 2 = 2$. Hence, there is no 1-line. From (4)-(6), we obtain $(a_0,a_2,a_3,a_4) = (s, 61-6s, 8s-64, 76-3s)$ with $8 \leq s \leq 10$. Let $l_1,\dots,l_8$ be 0-lines. Then, $\mathcal{L} = \{l_1,\dots,l_s\}$ forms an s-arc of lines, for $(\theta_1 - 3)\gamma_1 < 26$. Suppose $s = 8$. Then, one can find a line l so that $\mathcal{L} \cup \{l\}$ forms a 9-arc of lines since every 8-arc is contained in a 10-arc, see [13]. Since l meets $l_1,\dots,l_8$ in different points, l must be a 1-line, a contradiction. Similarly, we can rule out the case $s = 9$. Hence, our assertion follows. □

Lemma 4.4. *There exists no $[199,4,173]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[199,4,173]_8$ code. Then, $\gamma_0 = 1$, $\gamma_1 = 4$, $\gamma_2 = 26$ from (1). By Lemma 4.3, the spectrum of a γ_2-plane Δ_1 is $(\tau_0,\tau_2,\tau_3,\tau_4) = (10,1,16,46)$. An i-plane with a t-line satisfies

$$t \leq \frac{i+9}{8} \tag{10}$$

by Lemma 2.1. We have $a_1 = 0$ from Lemma 2.1(e) since Δ_1 has no 1-line. If a 14-plane δ exists, it follows from (10) that $\mathcal{M}(\delta)$ gives a $[14,3,12]_8$ code, which does not exist. In this way, using Theorem 4.2 and Lemma 2.1, one can get $a_i = 0$ for all $i \notin \{0,\ 7\text{-}10,\ 15,\ 23\text{-}26\}$. We refer to this procedure as the **first sieve** in the proofs of the nonexistence results. From (7), we get

$$\sum_{i \leq 24} \binom{26-i}{2} a_i = 4259. \tag{11}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (26-j)c_j = w + 9 - 8t. \tag{12}$$

Suppose $a_0 > 0$. Then, $a_0 = 1$ and $a_i > 0$ with $i > 0$ implies $i \geq 23$. Setting $w = t = 0$ in (12), the maximum possible contribution of c_j's to the LHS of (11) is $(c_{23}, c_{26}) = (3,5)$. Hence we get $4259 = (\text{LHS of (11)}) \leq 9 \times 73 + 325 = 982$, which contradicts (11). Hence $a_0 = 0$.

Now, setting $w = 26$ in (12), the maximum possible contribution of c_j's to the LHS of (11) are $(c_7, c_{10}, c_{26}) = (1,1,6)$ for $t = 0$; $(c_7, c_{26}) = (1,7)$ for $t = 2$; $(c_{15}, c_{26}) = (1,7)$ for $t = 3$; $(c_{23}, c_{26}) = (1,7)$ for $t = 4$. Hence we get

$$4259 = (\text{LHS of (11)}) \leq 291 \times 10 + 171 \times 1 + 55 \times 16 + 3 \times 46 = 4099,$$

a contradiction. This completes the proof. □

The following lemma is needed to prove the nonexistence of a $[657,4,574]_8$ code.

Lemma 4.5 ([24])**.** *There exists no $[658,4,575]_8$ code.*

Lemma 4.6 ([24])**.** *The spectrum of a $[83,3,72]_8$ code satisfies $a_i = 0$ for all i with $i \notin \{3,5,7,9,11\}$.*

Lemma 4.7. *There exists no $[657,4,574]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative $[657, 4, 574]_8$ code. Using Theorem 4.2 and Lemmas 2.1 and 4.6, one can get $a_i = 0$ for all $i \notin \{33, 49, 65\text{-}73, 81\text{-}83\}$ by the first sieve. From (7), we get

$$\sum_{i \le 81} \binom{83-i}{2} a_i = 64\lambda_2 - 2583. \tag{13}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (83-j)c_j = w + 7 - 8t. \tag{14}$$

Suppose $a_{72} > 0$. From Table 1, the spectrum of a 72-plane is $(\tau_8, \tau_9) = (9, 64)$. Setting $i = 72$, the maximum possible contributions of c_j's in (14) to the LHS of (13) are $(c_{68}, c_{83}) = (1, 7)$ for $t = 8$; $(c_{81}, c_{82}, c_{83}) = (3, 1, 4)$ for $t = 9$. Hence we get

$$64\lambda_2 - 2583 = (\text{LHS of (13)}) \le (105 \times 1 + 0 \times 7)9 + (1 \times 3 + 0 \times 1 + 0 \times 4)64 + 55 = 1192,$$

giving $\lambda_2 \le 58$. On the other hand, we have $\lambda_2 = \lambda_0 + 72 = 72$ from (2), a contradiction. Hence $a_{72} = 0$. Similarly, we can prove $a_{71} = a_{70} = a_{69} = a_{68} = 0$. Applying Theorem 2.6, $\mathcal{C}$ is extendable, which contradicts Lemma 4.5. This completes the proof. $\square$

As in the above proof, we often obtain a contradiction to rule out the existence of some i-plane by eliminating the value of λ_2 using (7), (8) and the possible spectra for a fixed w-plane. We refer to this proof technique as "(λ_2, w)-**ruling out method** ((λ_2, w)-**ROM**)" in what follows.

Lemma 4.8. *There exists no* $[725, 4, 633]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[725, 4, 633]_8$ code. Then, $\gamma_0 = 2$, $\gamma_1 = 12$, $\gamma_2 = 92$ by (1). From Table 1, the spectrum of a γ_2-plane Δ_1 is one of the following:

(A) $(\tau_0, \tau_8, \tau_{12}) = (1, 9, 63)$ with $\lambda'_0 = 9$,
(B) $(\tau_4, \tau_{12}) = (6, 67)$ with $\lambda'_0 = 15$,
(C) $(\tau_4, \tau_8, \tau_{12}) = (1, 10, 62)$ with $\lambda'_0 = 5$,
(D) $(\tau_8, \tau_{12}) = (12, 61)$ with $\lambda'_0 = 3$,

where $\lambda'_0 = \lambda_0(\Delta_1)$. Using Theorem 4.2 and Lemma 2.1, one can get $a_i = 0$ for all $i \notin \{0, 21\text{-}28, 53\text{-}64, 69\text{-}73, 85\text{-}92\}$ by the first sieve. From (7), we get

$$\sum_{i \le 90} \binom{92-i}{2} a_i = 64\lambda_2 - 5315. \tag{15}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (92-j)c_j = w + 11 - 8t. \tag{16}$$

We first prove $a_i = 0$ for $0 \le i \le 28$. Assume a t-plane δ_t with $0 \le t \le 28$ exists. Then, the multiset $\mathcal{M}_{\mathcal{C}} + \delta_t$ gives an $[N = 798, 4, D = 697]_8$ code $\mathcal{C}'$ since $m_{\mathcal{C}'}(\delta_t) = t + \theta_2 \le 28 + 73 \le 101$ and since $N = n + \theta_2 = 725 + 73 = 798$ and $N - D = n - d + \theta_1 = 92 + 9 = 101$. This contradicts that a $[798, 4, 697]_8$ code does not exist by Lemma 4.12. Hence, $a_i = 0$ for all $0 \le i \le 28$.

Since Δ_1 has no 9-line, we have $a_{73} = 0$. We can prove $a_i = 0$ for $i = 72, 71, 70, 69, 64, 63, 62, 61, 60$ by (λ_2, i)-ROM using the possible spectra of an i-plane in Table 1.

Next, we prove $a_i = 0$ for $53 \le i \le 59$. Suppose $a_{53} > 0$ and let δ_{53} be a 53-plane with spectrum $(\tau_0, \dots, \tau_8)$. Then, we have

$$\sum_{i \le 7} \binom{8-i}{2} \tau_i = 83. \tag{17}$$

Setting $w = 53$ in (16), the maximum possible contribution of c_j's to the left hand side of (15) are $(c_{53}, c_{85}, c_{88}, c_{92}) = (1, 3, 1, 3)$ for $t = 0$; $(c_{53}, c_{85}, c_{87}, c_{91}) = (1, 1, 1, 5)$ for $t = 1$; $(c_{53}, c_{89}, c_{91}) = (1, 1, 6)$ for $t = 2$; $(c_{59}, c_{91}) = (1, 7)$ for $t = 3$; $(c_{85}, c_{88}, c_{92}) = (4, 1, 3)$ for $t = 4$; $(c_{85}, c_{87}, c_{91}) = (2, 1, 5)$ for $t = 5$; $(c_{85}, c_{89}, c_{91}) = (1, 1, 6)$ for $t = 6$; $c_{91} = 8$ for $t = 7$; $c_{92} = 8$ for $t = 8$ since $c_{92} = 0$ for $t = 1, 2, 3, 5, 6, 7$. Hence we get

$$\begin{aligned} 64\lambda_2 - 5315 &= (\text{LHS of } (15)) \\ &\le 810\tau_0 + 772\tau_1 + 744\tau_2 + 528\tau_3 + 90\tau_4 + 52\tau_5 + 24\tau_6 \\ &< 53 \times (17) = 4399 \end{aligned}$$

giving $\lambda_2 \le 151$. On the other hand, we have $\lambda_2 = 140 + \lambda_0 \ge 140 + 73 - 53 \ge 160$, a contradiction. Hence $a_{53} = 0$. We can prove $a_{54} = a_{55} = a_{56} = a_{57} = a_{58} = a_{59} = 0$ similarly, see [6] for the detail.

Now, we have $a_i = 0$ for all $i < 85$. Setting $w = 92$, (16) has no solution for $t = 0, 4$. Hence every 92-plane has spectrum (D). Then, we get a contradiction by $(\lambda_2, 92)$-ROM. This completes the proof. □

Lemma 4.9. *Let $\mathcal{C}$ be a $[101, 3, 88]_8$ code and let $\Sigma = \mathrm{PG}(2, 8)$. Then,*
(A) $\mathcal{C}$ has spectrum $(a_5, a_{13}) = (5, 68)$ with $\lambda_0 = 10$ and $\mathcal{M}_\mathcal{C} = 2\Sigma - (l_1 + \cdots + l_5)$, where $\{l_1, \ldots, l_5\}$ is a 5-arc of lines; or
(B) $\mathcal{C}$ has spectrum $(a_9, a_{13}) = (10, 63)$ with $\lambda_0 = 0$ and $\mathcal{M}_\mathcal{C} = 2\Sigma - L$, where L is the union of a 10-arc of lines.

Proof. Let $\mathcal{C}$ be a $[101, 3, 88]_8$ code. Then $\gamma_0 = 2$ from (1) since $\mathcal{C}$ is Griesmer. Hence, our assertion follows from Lemma 2.9 since the multiset $2\Sigma - \mathcal{M}_\mathcal{C}$ is a $(45, 5)$-minihyper. □

Lemma 4.10. *Every $[100, 3, 87]_8$ code $\mathcal{C}$ is extendable and its spectrum is one of the following:*
(a) $(a_5, a_{12}, a_{13}) = (5, 9, 59)$,
(b) $(a_4, a_5, a_{12}, a_{13}) = (1, 4, 8, 60)$,
(c) $(a_8, a_9, a_{12}, a_{13}) = (2, 8, 7, 56)$,
(d) $(a_9, a_{12}, a_{13}) = (10, 9, 54)$.

Proof. Let $\mathcal{C}$ be a $[100, 3, 87]_8$ code. By Lemma 1, $\gamma_0 = 2$ and $\gamma_1 = 13$. Since $(\gamma_1 - \gamma_0)\theta_1 + \gamma_0 - 1 = n$, the lines though a fixed 2-point is one 12-line and eight 13-lines, and $a_{10} = a_{11} = 0$. Let l be a t-line containing a 1-point P. Considering the lines through P, we get $n \le (\gamma_1 - 1)8 + t$, so $4 \le t$. Hence $a_1 = a_2 = a_3 = 0$. Suppose a 0-line l_0 exists. Since there is no 9-line, for a point P on l_0, there are four 12-lines and four 13-lines through P. Hence, the spectrum is $(a_0, a_{12}, a_{13}) = (1, 36, 36)$, Then, from (6), we have $\lambda_2 = 648 - 4950/8$, a contradiction. Hence, there is no 0-line. Next, assume $a_6 > 0$ and let l_6 be a 6-line. For a 1-point P on l_6, there are exactly two 12-lines and six 13-lines through P. Hence $a_9 = 0$. For a 0-point Q on l_6, there are at most two lines whose multiplicities are less than 9. Hence we have $\sum_{i \equiv n, n-d} a_i \le (9 - 6)2 + 1 = 7$, and $\mathcal{C}$ is extendable by Theorem 2.7. One can prove this similarly when $a_7 > 0$. Finally, assume $a_6 = a_7 = 0$. Then, we have $a_i = 0$ for all $i \notin \{4, 5, 8, 9, 12, 13\}$, which implies that $A_i = 0$ for all $i \not\equiv 0, 87 \bmod 4$. Hence, $\mathcal{C}$ is extendable by Theorem 2.8. Assume that adding a point P to the multiset $\mathcal{M}_\mathcal{C}$ gives a 101-plane δ corresponding to a $[101, 3, 88]_8$ code. Then, δ satisfies (A) or (B) in the previous lemma. So, one can get the spectra (a)-(d) according to the cases (a) P is a 2-point on δ with case (A); (b) P is a 1-point from a 5-line on δ with case (A); (c) P is a 1-point from a 9-line on δ with case (B); (d) P is a 2-point on δ with case (B), respectively. □

Lemma 4.11. *There exists no $[790, 4, 690]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[790, 4, 690]_8$ code. Then, we have $\gamma_0 = 2$, $\gamma_1 = 13$, $\gamma_2 = 100$ from (1). Since $(\gamma_1 - \gamma_0)\theta_2 + \gamma_0 - 15 = 790$, an i-plane containing a 2-point satisfies $i \ge (\gamma_1 - \gamma_0)\theta_1 + \gamma_0 - 15 = 86$. From Table 1, the spectrum of a γ_2-plane Δ_1 is one of the following:

(A) $(\tau_5, \tau_{12}, \tau_{13}) = (5, 9, 59)$, (B) $(\tau_4, \tau_5, \tau_{12}, \tau_{13}) = (1, 4, 8, 60)$,
(C) $(\tau_8, \tau_9, \tau_{12}, \tau_{13}) = (2, 8, 7, 56)$, (D) $(\tau_9, \tau_{12}, \tau_{13}) = (10, 9, 54)$.

By the first sieve, one can get $a_i = 0$ for all $i \notin \{22\text{-}28, 30\text{-}33, 54\text{-}73, 86\text{-}92, 94\text{-}100\}$. From (7), we get

$$\sum_{i\le 98} \binom{100-i}{2} a_i = 64\lambda_2 - 8685. \tag{18}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (100-j)c_j = w + 10 - 8t. \tag{19}$$

We first prove $a_i = 0$ for $22 \le i \le 33$. Assume a t-plane δ_t with $22 \le t \le 33$ exists. Then, the multiset $\mathcal{M}_\mathcal{C} + \delta_t$ gives an $[N = 863, 4, D = 754]_8$ code $\mathcal{C}'$ since $m_{\mathcal{C}'}(\delta_t) = t + \theta_2 \le 33 + 73 \le 109$ and since $N = n + \theta_2 = 790 + 73 = 863$ and $N - D = n - d + \theta_1 = 790 - 690 + 9 = 109$. This contradicts that a $[863, 4, 754]_8$ code does not exist, see [26]. Hence, $a_i = 0$ for all $i \le 33$. We can prove $a_i = 0$ for $i = 73, 72, 71, 70, 64, 63, 62, 61, 60, 69$ in this order by (λ_2, i)-ROM using the possible spectra of each i-plane from Table 1.

Suppose $a_{68} > 0$ and let δ_{68} be a 68-plane. Since δ_{68} corresponds to a Griesmer $[68, 3, 59]_8$ code, $\mathcal{M}(\delta_{68})$ is obtained from δ_{68} by deleting five points, and the spectrum of δ_{68} is one of the following:

(a) $(\tau_4, \tau_8, \tau_9) = (1, 40, 32)$, (b) $(\tau_5, \tau_7, \tau_8, \tau_9) = (1, 4, 33, 35)$,
(c) $(\tau_6, \tau_7, \tau_8, \tau_9) = (1, 7, 28, 37)$, (d) $(\tau_6, \tau_7, \tau_8, \tau_9) = (2, 4, 31, 36)$,
(e) $(\tau_7, \tau_8, \tau_9) = (10, 25, 38)$.

One can get a contradiction by the usual $(\lambda_2, 68)$-ROM for the possible spectra (b)-(e). Hence δ_{68} has spectrum (a). From (19), there is at most one i-plane with $i \le 68$ other than δ_{68}. We may assume that δ_{68} meets Δ_1 in a 9-line. Then Δ_1 has spectrum (C) or (D). Setting $w = 100$ in (19), the maximum possible contributions of c_j's to the LHS of (18) are $(c_{54}, c_{100}) = (1, 7)$ for $t = 8$; $(c_{86}, c_{96}, c_{100}) = (3, 1, 4)$ for $t = 8$ when $c_j = 0$ for $j < 86$; $(c_{65}, c_{97}, c_{100}) = (1, 1, 6)$ for $t = 9$; $(c_{86}, c_{90}, c_{100}) = (2, 1, 5)$ for $t = 9$ when $c_j = 0$ for $j < 86$; $(c_{86}, c_{100}) = (1, 7)$ for $t = 12$; $(c_{94}, c_{100}) = (1, 7)$ for $t = 13$. Hence, we get

$$64\lambda_2 - 8685 = (\text{LHS of } (18)) \le 1035 + 279(\tau_8 - 1) + 227\tau_9 + 91\tau_{12} + 15\tau_{13} = 4607$$

for the spectrum (C), giving $\lambda_2 \le 207$. On the other hand, we have $\lambda_2 = \lambda_0 + 205 \ge 205 + (73 - 69) = 209$, a contradiction. Similarly, we get a contradiction for spectrum (D). Hence $a_{68} = 0$. One can also prove $a_{67} = a_{66} = 0$ as well.

Suppose $a_{54} > 0$. Let δ_{54} be a 54-plane and l be 8-line in δ_{54}. Then, the other planes through l other than δ_{54} are 100-planes of spectrum (C), say $\Delta_1, \dots, \Delta_8$. Suppose that there is no plane with no 2-point meeting l in a 1-point. Then, one can get a contradiction by $(\lambda_2, 100)$-ROM using the spectrum (C) of a 100-plane. So, there is a plane δ with no 2-point meeting l in a 1-point P. Since δ meets each of $\Delta_1, \dots, \Delta_8$ in a 9-line, we have $m_\mathcal{C}(\delta) \ge (9 - 1)8 + 1 = 65$, whence δ is a 65-plane with spectrum $(\tau_1, \tau_8, \tau_9) = (1, 64, 8)$. Then, we get a a contradiction by $(\lambda_2, 65)$-ROM. Hence $a_{54} = 0$. Similarly, we can prove $a_{55} = a_{56} = a_{57} = a_{58} = a_{59} = 0$.

Suppose $a_{65} > 0$ and let δ_{65} be a 65-plane. Let l be a 9-line on δ_{65} and take a 100-plane Δ_1 through l. Since δ_{65} has no 2-point, there are eight 0-points in δ_{65}, and there are at most two lines on δ_{65} whose multiplicities are at most 5. Since any other 65-plane meets δ_{65} in some t-line with $t \le 5$ and since the spectrum of Δ_1 is (C) or (D), we have $a_{65} \le 3$ from (19) with $w = 100$. Setting $w = 100$ in (19), the maximum possible contributions of c_j's to the LHS of (18) are $(c_{65}, c_{89}, c_{100}) = (1, 1, 6)$ for $t = 8$; $(c_{86}, c_{96}, c_{100}) = (3, 1, 4)$ for $t = 8$ with $c_{65} = 0$; $(c_{65}, c_{97}, c_{100}) = (1, 1, 6)$ for $t = 9$; $(c_{86}, c_{90}, c_{100}) = (2, 1, 5)$ for $t = 9$ with $c_{65} = 0$; $(c_{86}, c_{100}) = (1, 7)$ for $t = 12$; $(c_{94}, c_{100}) = (1, 7)$ for $t = 13$. It follows from $\lambda_2 = \lambda_0 + 205 \ge 205 + (73 - 65) = 213$ that one can get a contradiction by $(\lambda_2, 100)$-ROM as

$$64\lambda_2 - 8685 = (\text{LHS of } (18)) \le 650\tau_8 + 227\tau_9 + 91\tau_{12} + 15\tau_{13} = 4593$$

when Δ_1 has spectrum (C) and

$$64\lambda_2 - 8685 = (\text{LHS of } (18)) \le 598 \times 2 + 227(\tau_9 - 2) + 91\tau_{12} + 15\tau_{13} = 4641$$

when Δ_1 has spectrum (D) since $a_{65} \le 3$, giving $\lambda_2 \le 208$. Hence, $a_{65} = 0$.

Now, we have $a_i = 0$ for all $i < 86$. One can get a contradiction by $(\lambda_2, 100)$-ROM using the possible spectra (A)-(D) as usual. This completes the proof. □

Lemma 4.12. *There exists no* $[798, 4, 697]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[798, 4, 697]_8$ code. By Lemma 4.9, the spectrum of a γ_2-plane Δ_1 is either (A) $(\tau_5, \tau_{13}) = (5, 68)$ or (B) $(\tau_9, \tau_{13}) = (10, 63)$. Using Theorem 4.2 and Lemma 2.1, one can get $a_i = 0$ for all $i \notin \{30\text{-}33,\ 62\text{-}73,\ 94\text{-}101\}$ by the first sieve. It follows from (7) that

$$\sum_{i \le 99} \binom{101-i}{2} a_i = 64\lambda_2 - 9123. \tag{20}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (101-j)c_j = w + 10 - 8t. \tag{21}$$

One can deduce that $a_i = 0$ by (λ_2, i)-ROM for $70 \le i \le 73$ using the possible spectra of the $[73 - j, 3, 64 - j]_8$ codes for $0 \le j \le 3$, see Table 1.

Suppose $a_{30} > 0$ and let δ_{30} be a 30-plane. It follows from (21) that $a_{30} > 0$ implies $a_{30} = 1$ and $a_j = 0$ for other $j < 94$. Since $\gamma_1(\delta_{30}) = 5$, one can find a 101-plane Δ of spectrum (A) meeting δ_{30} in a 5-line. Take another 5-line l_5 on Δ. Then, every plane through l_5 has multiplicity at least 94, which is impossible from (21) with $(w, t) = (101, 5)$. Hence $a_{30} = 0$. One can get $a_{31} = a_{32} = a_{33} = 0$, similarly. Then, using the possible spectra of the $[70 - j, 3, 61 - j]_8$ codes, we can also prove that $a_{70-j} = 0$ by $(\lambda_2, 70 - j)$-ROM for $1 \le j \le 3$.

Now, we have $a_i = 0$ for all $i \notin \{62\text{-}66,\ 94\text{-}101\}$. Note that a $(62 + e)$-plane with $0 \le e \le 3$ could have a 2-point because it corresponds to a $[62 + e, 3, 53 + e]_8$ code which is not Griesmer. Suppose a $(62+e)$-plane δ with $0 \le e \le 3$ has a 2-point. Then, one can find a 9-line l_9 through the 2-point on δ and a 101-plane through l_9 from (21) with $(w, t) = (62 + e, 9)$. This contradicts that a 9-line in a 101-plane with spectrum (B) has no 2-point by Lemma 4.9. Thus, a $(62 + e)$-plane with $0 \le e \le 4$ has no 2-point since a 66-plane corresponds to a Griesmer code.

Suppose $a_{62} > 0$ and let δ_{62} be a 62-plane and let l be a 9-line on δ_{62}. Then, the other planes through l are 101-planes, say $\Delta_1, \dots, \Delta_8$. For a fixed 1-point P on l, one can take a 9-line $l_j (\ne l)$ on Δ_j for $1 \le j \le 8$ from the geometric structure described in Lemma 4.9. Suppose that the plane $\delta = \langle l_1, l_2 \rangle$ is a $(62+e)$-plane with $0 \le e \le 3$ and let $\delta \cap \delta_{62}$ be an α-line. Since $\gamma_1(\delta) = 9$, δ contains all of $l_1, \dots, l_8$, and we have $m_{\mathcal{C}}(\delta) = 64 + \alpha$. One can rule out such cases by $(\lambda_2, 64 + \alpha)$-ROM. Hence, $a_{62} > 0$ implies that $a_{62} = 1$ and $a_j = 0$ for other $j < 94$. Setting $w = 101$, the maximum possible contributions of c_j's in (21) to the LHS of (20) are $(c_{62}, c_{101}) = (1, 7)$ for $t = 9$ with $c_{62} > 0$; $(c_{94}, c_{97}, c_{101}) = (5, 1, 2)$ for $t = 9$ with $c_{62} = 0$; $(c_{94}, c_{101}) = (1, 7)$ for $t = 13$. Using the spectrum of a 101-plane of spectrum (B), one can get a contradiction by $(\lambda_2, 101)$-ROM. Hence $a_{62} = 0$. One can similarly prove $a_{63} = 0$.

To rule out a 101-plane of spectrum (A), let Δ_1 be such a plane. From (21) with $(w, t) = (101, 5)$, there exists a $(64 + e)$-plane with $0 \le e \le 2$ through each of the 5-lines on Δ_1. One can rule out such a 66-plane by $(\lambda_2, 66)$-ROM using all possible spectra of a 66-plane with a 5-line. Hence $a_{66} = 0$. Note that $\lambda_0 \ge 8 - 4 + 10 = 14$ since a 101-plane of spectrum (A) has ten 0-points. Setting $w = 101$, the maximum possible contributions of c_j's in (21) to the LHS of (20) are $(c_{64}, c_{94}, c_{95}, c_{101}) = (1, 4, 1, 2)$ for $t = 5$; $(c_{94}, c_{101}) = (1, 7)$ for $t = 13$. Using the spectrum of a 101-plane of spectrum (A), one can get a contradiction by $(\lambda_2, 101)$-ROM. Hence every 101-plane has spectrum (B).

Suppose $a_{66} > 0$ and let δ_{66} be a 66-plane with spectrum $(\tau_2, \dots, \tau_9)$. Then, from the three equalities (4)-(6), we obtain $\tau_2 + \tau_3 + \tau_4 \le 2$ and $\tau_5 + \tau_6 + \tau_7 \le 21$. Setting $w = 66$, the maximum possible contributions of c_j's in (21) to the LHS of (20) are $(c_{64}, c_{94}, c_{95}, c_{99}) = (1, 1, 1, 5)$ for $t = 2$ since a 100-plane has no 2-line by Lemma 4.10; $(c_{94}, c_{96}, c_{100}) = (4, 1, 3)$ for $t = 5$ since $c_{101} = 0$; $(c_{96}, c_{100}) = (1, 7)$

for $t = 8$; $(c_{97}, c_{101}) = (1, 7)$ for $t = 9$. Using $(\tau_2, \tau_5, \tau_8, \tau_9) = (2, 21, 49, 1)$ instead of all possible spectra of a 66-plane, one can get a contradiction by $(\lambda_2, 66)$-ROM. Hence $a_{66} = 0$. we can prove $a_{65} = a_{64} = 0$ similarly.

Hence, we have ruled out all possible i-planes with $i < 94$. Finally, using the spectrum (B) of a 101-plane, one can get a contradiction by $(\lambda_2, 101)$-ROM. This completes the proof. □

Lemma 4.13. *A* $[107, 3, 93]_8$ *code* $\mathcal{C}$ *satisfies* $\lambda_0 > 0$.

Proof. Suppose $\lambda_0 = 0$. It follows from Lemma 2.4 that the multiset $\mathcal{M}_\mathcal{C} - \mathrm{PG}(2, 8)$ gives a $[34, 3, 29]_8$ code, which does not exist by Theorem 4.2, a contradiction. □

Lemma 4.14. *There exists no* $[853, 4, 745]_8$ *code.*

Proof. Let $\mathcal{C}$ be a putative Griesmer $[853, 4, 745]_8$ code. From Table 1, the spectrum of a γ_2-plane Δ_1 is one of the following:

(A) $(\tau_4, \tau_6, \tau_{13}, \tau_{14}) = (1, 3, 16, 53)$, (B) $(\tau_5, \tau_6, \tau_{12}, \tau_{13}, \tau_{14}) = (2, 2, 1, 14, 54)$,
(C) $(\tau_5, \tau_6, \tau_{12}, \tau_{13}, \tau_{14}) = (1, 3, 1, 15, 53)$, (D) $(\tau_6, \tau_{12}, \tau_{13}, \tau_{14}) = (4, 1, 16, 52)$,
(E) $(\tau_6, \tau_{12}, \tau_{14}) = (4, 9, 60)$.

One can get $a_i = 0$ for all $i \notin \{$21-33, 37-42, 61-64, 69-73, 85-108$\}$ by the first sieve. From (7), we get

$$\sum_{i \le 106} \binom{108 - i}{2} a_i = 64\lambda_2 - 12251. \tag{22}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (108 - j) c_j = w + 9 - 8t. \tag{23}$$

We first prove $a_i = 0$ for $21 \le i \le 42$. Assume a t-plane δ_t with $21 \le t \le 42$ exists. Then, the multiset $\mathcal{M}_\mathcal{C} + \delta_t$ gives an $[N = 926, 4, D = 809]_8$ code $\mathcal{C}'$ since $m_{\mathcal{C}'}(\delta_t) = t + \theta_2 \le 42 + 73 \le 115$ and since $N = n + \theta_2 = 853 + 73 = 926$ and $N - D = n - d + \theta_1 = 853 - 745 + 9 = 117$. This contradicts that a $[926, 4, 809]_8$ code does not exist by Lemma 4.17. Hence, $a_i = 0$ for all $i \le 60$.

If $a_{73} > 0$, then any line on a 73-plane is a 9-line from Table 1, which contradicts that Δ_1 has no 9-line. Hence $a_{73} = 0$. Similarly, $a_{64} = a_{63} = a_{71} = a_{72} = 0$.

Suppose $a_{62} > 0$. The spectrum of a 62-plane is $(\tau_0, \tau_6, \tau_7, \tau_8) = (1, 1, 16, 55)$ and a 62-plane meets Δ_1 in a 6-line since the possible multiplicities of lines in Δ_1 are 4, 5, 6, 12, 13, 14. Setting $w = 62$ in (23), the maximum possible contributions of c_j's to the LHS of (22) are $(c_{106}, c_{107}) = (1, 7)$ for $t = 8$; $(c_{98}, c_{107}) = (1, 7)$ for $t = 7$; $(c_{85}, c_{106}, c_{108}) = (1, 1, 6)$ for $t = 6$; $(c_{42}, c_{107}) = (1, 7)$ for $t = 0$. Using the spectrum of a 62-plane, one can get a contradiction by $(\lambda_2, 62)$-ROM since $\lambda_2 = \lambda_0 + 268 \ge 268$. Hence $a_{62} = 0$. Similarly, we can prove $a_{61} = a_{69} = a_{70} = 0$ using the spectra from Table 1.

Now, we have $a_i = 0$ for all $i < 85$. Using the possible spectra (A)-(E) of a 108-plane, one can get a contradiction as follows.

Take a 14-line L on a 108-plane so that L has no 0-point. Setting $(w, t) = (108, 14)$ in (23), the solutions of c_j's are $(c_{101}, c_{108}) = (1, 7)$, $(c_{102}, c_{107}, c_{108}) = (1, 1, 6)$, $(c_{107}, c_{108}) = (7, 1)$ and so on. Counting the number of 0-points on the planes through L, we have $\lambda_0 \ge 6 + 6 + 7 = 19$ since a 108-plane has at least six 0-points and since a 107-plane has at least one 0-point by Lemma 4.13. Hence

$$\lambda_2 = \lambda_0 + 268 \ge 287. \tag{24}$$

Using the spectra (A)-(D) of a 108-plane, we get a contradiction by $(\lambda_2, 108)$-ROM. Hence every 108-plane has spectrum (E). Then, we have $64\lambda_2 - 12251 \le 6577$, giving

$$\lambda_2 \le 294. \tag{25}$$

Next, we rule out a possible 85-plane. Assume a 85-plane δ exists. Then, δ has a 12-line ℓ and the other planes through ℓ are 108-planes. Let s be the number of 0-points on ℓ. Since $s \le 3$ and since a 108-plane of spectrum (E) has seven 0-point, we obtain $\lambda_2 = \lambda_0 + 268 \ge 268 + (7-s)8 + s \ge 304$, which contradicts (25). Hence, $a_{85} = 0$.

Counting the number of 0-points on the planes through a fixed 14-line, the lower bound (24) can be improved to $\lambda_2 \ge 289$ since a 108-plane of spectrum (E) has seven 0-point.

On the other hand, since the maximum possible contributions of c_j's in (23) with $w = 108$ to the LHS of (22) are $(c_{86}, c_{103}, c_{108}) = (3, 1, 4)$ for $t = 6$ and $(c_{86}, c_{108}) = (1, 1, 6)$ for $t = 12$, the upper bound (25) can be also improved to $\lambda_2 \le 287$, a contradiction. This completes the proof. □

We recall that the multiset for a $[2q^2-q-1, 3, 2q^2-3q]_q$ code with $q \ge 5$ consists of two copies of PG$(2, q)$ with three non-concurrent lines deleted [16]. The following code is obtained from this code by deleting two (not necessarily distinct) points.

Lemma 4.15 ([16])**.** *A $[2q^2-q-3, 3, 2q^2-3q-2]_q$ code $\mathcal{C}'$ with $q \ge 7$ is extendable to a $[2q^2-q-1, 3, 2q^2-3q]_q$ code $\mathcal{C}$ and its spectrum is one of the following:*
(a) $(a_{q-3}, a_{q-1}, a_{2q-2}, a_{2q-1}) = (1, 2, 2q, q^2-q-2)$,
(b) $(a_{q-2}, a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (2, 1, 1, 2q-2, q^2-q-1)$,
(c) $(a_{q-2}, a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (1, 2, 1, 2q-1, q^2-q-2)$,
(d) $(a_{q-1}, a_{2q-3}, a_{2q-2}, a_{2q-1}) = (3, 1, 2q, q^2-q-3)$,
(e) $(a_{q-1}, a_{2q-3}, a_{2q-1}) = (3, q+1, q^2-3)$,
according to the cases (a) *P and Q are 1-points on the same $(q-1)$-line on δ;* (b) *P and Q are 1-points from different $(q-1)$-lines on δ;* (c) *P is a 1-point and Q is a 2-point on δ;* (d) *P and Q are distinct 2-points in on δ;* (e) *P and Q are the same 2-points in on δ, respectively, where P and Q are the points corresponding to the coordinates of $\mathcal{C}$ to be removed from the $(2q^2-q-1)$-plane δ stated in the previous lemma.*

One can get the following similarly to Lemma 4.13.

Lemma 4.16. *A $[116, 3, 101]_8$ code $\mathcal{C}$ satisfies $\lambda_0 > 0$.*

Lemma 4.17. *There exists no $[926, 4, 809]_8$ code.*

Proof. Let $\mathcal{C}$ be a putative $[926, 4, 809]_8$ code. From Lemma 4.15, the spectrum of a γ_2-plane Δ is one of the following:

(A) $(\tau_5, \tau_7, \tau_{14}, \tau_{15}) = (1, 2, 16, 54)$,
(B) $(\tau_6, \tau_7, \tau_{13}, \tau_{14}, \tau_{15}) = (2, 1, 1, 14, 55)$,
(C) $(\tau_6, \tau_7, \tau_{13}, \tau_{14}, \tau_{15}) = (1, 2, 1, 15, 54)$,
(D) $(\tau_7, \tau_{13}, \tau_{14}, \tau_{15}) = (3, 1, 16, 53)$,
(E) $(\tau_7, \tau_{13}, \tau_{15}) = (3, 9, 61)$.

Using Theorem 4.2 and Lemma 2.1, we obtain $a_i = 0$ for all $i \notin \{30\text{-}33, 38\text{-}42, 46\text{-}49, 62\text{-}64, 70\text{-}73, 94\text{-}117\}$ by the first sieve. From (7), we get

$$\sum_{i \le 115} \binom{117-i}{2} a_i = 64\lambda_2 - 17083. \tag{26}$$

Lemma 2.1(c) gives $\sum_j c_j = 8$ and

$$\sum_j (117-j)c_j = w + 10 - qt. \tag{27}$$

First, we prove $a_i = 0$ for $30 \le i \le 33$. Suppose $a_{30} > 0$ and let δ_{30} be a 30-plane. Then, it follows from (27) that $a_{30} = 1$ and any i-plane with $i > 30$ satisfies $i \ge 94$. From Lemma 2.1, δ_{30} meets Δ in a 5-line, say l, and Δ has the spectrum (A). Recall from Lemma 4.15 that there are two 7-lines in the 117-plane of spectrum (A) meeting the 5-line in 0-points. Since the other planes ($\neq \delta_{30}, \Delta$) through l are 117-planes of spectrum (A), say $\Delta_1, \ldots, \Delta_7$, and since there are four 0-points on l, one can take a 0-point Q on l which is on at least four 7-lines, say l_1, l_2, l_3, l_4. Without loss of generality, we may assume that l_j is on Δ_j for $1 \leq j \leq 4$. For the plane $\delta = \langle l_1, l_2 \rangle$, we have $m_{\mathcal{C}}(\delta) \leq 7+7+5+15\times 6 = 109$ since $m_{\mathcal{C}}(\delta_{30} \cap \delta) \leq 5$. Since a 109-plane has no 15-line, we have $m_{\mathcal{C}}(\delta \cap \Delta_j) = 7$ for $1 \leq j \leq 4$, and $m_{\mathcal{C}}(\delta) \leq 7\times 4 + 5 + 14 \times 4 = 89$, a contradiction. Hence $a_{30} = 0$. One can similarly prove $a_{31} = a_{32} = a_{33} = 0$.

If $a_{73} > 0$, then any line on a 73-plane is a 9-line from Table 1, which contradicts that Δ has no 9-line. Hence $a_{73} = 0$. Similarly, $a_{64} = a_{72} = 0$. Using the spectrum of a w-plane from Table 1, one can get a contradiction by (λ_2, w)-ROM for $w = 62, 63, 70, 71$. Hence, $a_{62} = a_{63} = a_{70} = a_{71} = 0$.

Suppose $a_{38} > 0$. Then, $a_{38} = 1$ and $a_j > 0$ with $j \neq 38$ implies $j \geq 94$. Let δ_0 be the 38-plane. Then, δ_0 contains a 6-line, say L, and the other planes through L other than δ_0 are 117-planes of spectrum (B) or (C), say $\Delta_1, \ldots, \Delta_8$. Recall from Lemma 4.15 that there are two 7-lines (resp. one 6-line and one 7-line) in the 117-plane of spectrum (C) (resp. (B)) meeting the 6-line L in 0-points. Let l_j and m_j be the 6- or 7-lines in Δ_j other than L. Since there are three 0-points on L, one can take a 0-point Q on l which is on at least six 6- or 7-lines. Without loss of generality, we may assume that l_2 and l_3 meet L in $Q = l_1 \cap L$ and that two of other l_j, m_j with $j \geq 2$ meet L in $Q' = m_1 \cap L$. Note that there is no s-line with $7 < s < 14$ in Δ_1 through Q or Q' by Lemma 4.15. Let δ be a t-plane through l_1 other than Δ_1. If $t < 102$, then δ meets Δ_2 and Δ_3 in l_2 and l_3, respectively, since a t-plane contains no 14- nor 15-line, whence $m_{\mathcal{C}}(\delta) \leq 7+7+7+|\delta \cap \delta_0| + 5 \times 14 \leq 97$. Then, from Lemma 2.1, δ contains no 14-plane, and we have $m_{\mathcal{C}}(\delta) \leq 7+7+7+6+5\times 13 = 92$, a contradiction. Hence any t-plane through l_1 or m_1 satisfies $t \geq 102$. Take Δ_1 as Π in Lemma 2.1, the maximum possible contribution of c_j's in (27) with $w = 117$ to the LHS of (26) are $(c_{38}, c_{117}) = (1, 7)$ for $t = 6$ with $c_{38} > 0$; $(c_{102}, c_{113}, c_{117}) = (5, 1, 2)$ for $t = 6$ with $c_{38} = 0$; $(c_{102}, c_{106}, c_{117}) = (4, 1, 3)$ for $t = 7$; $(c_{94}, c_{117}) = (1, 7)$ for $t = 13$; $(c_{102}, c_{117}) = (1, 7)$ for $t = 14$; $(c_{110}, c_{117}) = (1, 7)$ for $t = 15$. Note that $\lambda_2 = \lambda_0 + 341 \geq 341 + (73 - 38) + 8 = 384$ since each of $\Delta_1, \ldots, \Delta_8$ contains a 0-point out of L. Using the spectrum of a 117-plane of spectrum (B) or (C), one can get a contradiction by $(\lambda_2, 117)$-ROM. Hence $a_{38} = 0$. One can prove $a_{39} = a_{40} = a_{41} = a_{42} = 0$ similarly. (When δ_0 is a 42-plane, there are four 117-planes of spectrum (B) or (C), say $\Delta_1, \ldots, \Delta_4$, through a fixed 6-line L in δ_0, which contain 6- or 7-lines l_i, m_j as above. It could happen that l_2, l_3, l_4 meet L in $Q = l_1 \cap L$, m_2 meets L in $Q' = m_1 \cap L$ and that m_3 and m_4 meet L in the remaining 0-point of L other than Q, Q'. In this case, any t-plane ($\neq \langle m_1, m_2 \rangle$) through m_1 satisfies $t \geq 102$. Considering this situation, one can get a contradiction by $(\lambda_2, 117)$-ROM as above.) The above investigation for the case $a_{38} > 0$ is also valid to rule out possible i-planes for $46 \leq i \leq 49$, see [6] for the detail.

Now, we have $a_i = 0$ for all i without $94 \leq i \leq 117$. Using the spectrum (A)-(E) of a 117-plane, one can get a contradiction as follows. Take a 15-line with no 0-point on a 117-plane. Since the possible contributions of c_j's with $w = 117$ in (27) to the LHS of (26) are $(c_{110}, c_{117}) = (1, 7)$, $(c_{116}, c_{117}) = (7, 1)$ and so on for $t = 15$ and since a 116-plane has at least one 0-point by Lemma 4.16, we have $\lambda_2 = \lambda_0 + 341 \geq 341 + 3 + 3 \times 1 + 1 \times 7 \geq 354$. Hence, we can get a contradiction by $(\lambda_2, 117)$-ROM when the spectrum of the w-plane is one of (A)-(D). Now, we may assume that any 117-plane has spectrum (E). We first rule out a possible 94-plane. Assume a 94-plane δ exists. Then, δ has a 13-line ℓ and the other planes through ℓ are 117-planes. Since ℓ has at most two 0-points and since ℓ is on a 117-plane of spectrum (E) which has four 0-points, we get $\lambda_2 = \lambda_0 + 341 \geq 341 + (4-2)8 + 2 \geq 359$. On the other hand, we obtain $\lambda_2 \leq 358$ by $(\lambda_2, 117)$-ROM using the spectrum (E), a contradiction. Hence there is no 94-plane. Take a 15-line with no 0-point on a 117-plane. Counting the number of 0-points on the planes through the 15-line, we get a lower bound on λ_2 as $\lambda_2 = \lambda_0 + 341 \geq 341 + 4 + 4 \times 1 + 1 \times 7 \geq 356$ since a 117-plane of spectrum (E) has four 0-points. Then, we get a contradiction by $(\lambda_2, 117)$-ROM. This completes the proof. □

Now, Theorem 1.5 follows from Lemmas 4.4-4.17.

Acknowledgment: The authors wish to express their thanks to the anonymous reviewers for their careful reading and valuable comments that improved the presentation and the content of the paper.

References

[1] S. Ball, Table of bounds on three dimensional linear codes or (n, r)-arcs in PG$(2, q)$, available at https://web.mat.upc.edu/people/simeon.michael.ball/codebounds.html.

[2] A. Betten, E. J. Cheon, S. J. Kim, T. Maruta, The classification of $(42, 6)_8$ arcs, Adv. Math. Commun. 5 (2011) 209–223.

[3] I. Bouyukliev, Y. Kageyama, T. Maruta, On the minimum length of linear codes over $\mathbb{F}_5$, Discrete Math. 338 (2015) 938–953.

[4] A. E. Brouwer, M. van Eupen, The correspondence between projective codes and 2-weight codes, Des. Codes Cryptogr. 11 (1997) 261–266.

[5] M. van Eupen, R. Hill, An optimal ternary $[69, 5, 45]_3$ codes and related codes, Des. Codes Cryptogr. 4 (1994) 271–282.

[6] M. Fujii, Nonexistence of some Griesmer codes of dimension 4, Master Thesis, Osaka Prefecture University (2019).

[7] M. Grassl, "Bounds on the minimum distance of linear codes and quantum codes." Online available at http://www.codetables.de.

[8] J. H. Griesmer, A bound for error-correcting codes, IBM J. Res. Develop. 4 (1960) 532–542.

[9] N. Hamada, A characterization of some $[n, k, d; q]$-codes meeting the Griesmer bound using a minihyper in a finite projective geometry, Discrete Math., 116 (1993) 229–268.

[10] R. Hill, Optimal linear codes, In: C. Mitchell(Ed.), Cryptography and Coding II, Oxford Univ. Press, Oxford (1992) 75–104.

[11] R. Hill, E. Kolev, A survey of recent results on optimal linear codes, In: Combinatorial Designs and their Applications, F.C. Holroyd et al. Ed., Chapman and Hall/CRC Press Research Notes in Mathematics, CRC Press. Boca Raton (1999) 127–152.

[12] J. W. P. Hirschfeld, Finite Projective Spaces of Three Dimensions, Clarendon Press, Oxford (1985).

[13] J. W. P. Hirschfeld, Projective Geometries over Finite Fields, Clarendon Press, Oxford, second edition (1998).

[14] C. Jones, A. Matney, H. Ward, Optimal four-dimensional codes over GF(8), Electron. J. Combin. 13 (2006) #R43, .

[15] Y. Kageyama, T. Maruta, On the geometric constructions of optimal linear codes, Des. Codes Cryptogr., 81 (2016) 469–480.

[16] R. Kanazawa, T. Maruta, On optimal linear codes over $\mathbb{F}_8$, Electronic J. Combin., 18(1) (2011) #P34

[17] K. Kumegawa, T. Okazaki, T. Maruta, On the minimum length of linear codes over the field of 9 elements, Electron. J. Combin. 24(1) (2017) #P1.50.

[18] K. Kumegawa, T. Maruta, Nonexistence of some Griesmer codes over $\mathbb{F}_q$, Discrete Math. 339 (2016) 515–521.

[19] K. Kumegawa, T. Maruta, Non-existence of some 4-dimensional Griesmer codes over finite fields, J. Algebra Comb. Discrete Struct. Appl. 5 (2018) 101–116.

[20] I. Landjev, L. Storme, A study of $(x(q + 1), x; 2, q)$-minihypers, Des. Codes Cryptogr. 54 (2010) 135–147.

[21] T. Maruta, On the minimum length of q-ary linear codes of dimension four, Discrete Math., 208/209 (1999) 427–435.

[22] T. Maruta, On the nonexistence of q-ary linear codes of dimension five, Des. Codes Cryptogr. 22 (2001) 165–177.

[23] T. Maruta, A new extension theorem for linear codes, Finite Fields Appl. 10 (2004) 674–685.

[24] T. Maruta, Optimal 4-dimensional linear codes over $\mathbb{F}_8$, Proceedings of 13th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT 2012), Pomorie, Bulgaria (2012) 257–262.

[25] T. Maruta, Construction of optimal linear codes by geometric puncturing, Serdica J. Computing 7 (2013) 73–80.

[26] T. Maruta, Griesmer bound for linear codes over finite fields, available at http://mars39.lomo.jp/opu/griesmer.htm.

[27] T. Maruta, Y. Oya, On optimal ternary linear codes of dimension 6, Adv. Math. Commun. 5 (2011) 505–520.

[28] T. Maruta, M. Shinohara, M. Takenaka, Constructing linear codes from some orbits of projectivities,

Discrete Math. 308 (2008) 832–841.
[29] T. Maruta, T. Tanaka, H. Kanda, Some generalizations of extension theorems for linear codes over finite fields, Australas. J. Combin. 60 (2014) 150–157.
[30] T. Maruta, Y. Yoshida, A generalized extension theorem for linear codes, Des. Codes Cryptogr. 62 (2012) 121–130.
[31] M. Takenaka, K. Okamoto, T. Maruta, On optimal non-projective ternary linear codes, Discrete Math. 308 (2008) 842–854.
[32] Y. Yoshida, T. Maruta, An extension theorem for $[n, k, d]_q$ codes with $\gcd(d, q) = 2$, Australas. J. Combin. 48 (2010) 117–131.