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A new closed form for Libby-Novick beta class of generalized distributions with applications to Weibull model

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Abstract: It is useful to derive new classes of distributions having a simple closed form instead of classes having no closed form to get better flexibility in deriving mathematical properties, generating random numbers and applying real data sets. In this paper, a new closed form class of generalized distributions, so-called the modified Libby-Novick (MLN) class, is derived from the implicit form of Libby-Novick beta class. Two important classes of distributions are nested by the MLN class. Some generalized mathematical properties are derived and the MLN class parameters estimation using maximum likelihood estimation (MLE) method is obtained. A simulation study using bootstrapping approach is applied to investigate the estimators behavior of the MLN-Weibull (MLN-W) distribution. A real data set is used to illustrate the potentiality of the MLN-W

Keywords: the new form Libby-Novick distribution, moments, order statistics, maximum likelihood estimation.

1. Introduction

Eugene *et al.* [1] presented for the first time the beta class which has the following *CDF* and *PDF*, respectively,

$$P(y;\alpha,\beta,W)=\frac{1}{B(\alpha,\beta)}\int_0^{G(y;W)} t^{\alpha-1}(1-t)^{\beta-1}dt \;;0<t<1;\alpha,\beta>0\;;-\infty<y<\infty,$$

and

$$p(y;\alpha,\beta,W)=\frac{1}{B(\alpha,\beta)}g(y;W)\,G(y;W)^{\alpha-1}\left\{1-G(y;W)\right\}^{\beta-1},$$

where *G(y;W)* is the *CDF* of the baseline distribution, *W* is the parameter vector of the baseline distribution. Wahed [2] presented a general method for constructing an extended class of generalized distributions by using the following *CDF*

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$$F(y;T,W)=\int_{0}^{G(y;W)} g(t;T)\,dt\,;0<t<1, \tag{1}$$

where $G(y;W)$ is the *CDF* of the baseline distribution and $g(t;T)$ is the *PDF* of the generator distribution, *T* is the parameter vector of the generator distribution and *W* is the parameter vector of the baseline distribution. Based on [2] many classes of generalized distributions are derived as the Kumaraswamy (*KW*) class, [3] and [4], the Kummer beta class [5], the McDonald class, [6] and [7], the Kumaraswamy- Kumaraswamy (*KW-KW*) class, [8] and [9], the Libby-Novick beta class, [10] and [11], which have the following *CDF* and *PDF*

$$F_{LNB}(y;a,b,c,W)=\frac{c^{a}}{B(a,b)}\int_{0}^{G(y;W)}\frac{t^{a-1}(1-t)^{b-1}}{\{1-(1-c)t\}^{a+b}}dt;\ 0<t<1;\ a,b,c>0,$$

and

$$f_{LNB}(y;a,b,c,W)=\frac{c^{a}g(x;W)G(x;W)^{a-1}\{1-G(x;W)\}^{b-1}}{B(a,b)\{1-(1-c)G(x;W)\}^{a+b}}.$$

The main goal of this study is to derive a new explicit form class of distributions depending on the new form Libby-Novick distribution, Ali Ahmed [12], as a simple generator.

The rest of this paper is organized as follows: In section 2, the proposed class is derived. In section 3, some mathematical properties are given. In section 4, the Hazard function is obtained. In section 5, the Rényi entropy is given. In section 6, order statistics are obtained. In Section 7, the *MLE* method is performed. In Section 8, a simulation study using bootstrapping approach is used. Finally, in Section 9, an application is used practically.

2. The New *MLN* Class of Generalized Distributions

Ali Ahmed [12] presented, for the first time, the new form Libby-Novick *(NLN)* distribution where he derived the *NLN* distribution using a transformation into the Libby- Novick beta distribution, the *NLN* distribution is much simpler to use than Libby-Novick beta distribution, in simulation studies or in mathematical properties, because of his simple closed form of both its quantile function and cumulative distribution function.

The *NLN* distribution has the following *CDF* and *PDF*, respectively,

$$G(t;\alpha,\beta,c)=1-\frac{(1-t^{\alpha})^{\beta}}{\left[1-(1-c)t^{\alpha}\right]^{\beta}};o<t<1;\alpha,\beta,c>0, \tag{2}$$

and

$$g(t;\alpha,\beta,c)=\frac{\alpha\,\beta\,c\,t^{\alpha-1}(1-t^{\alpha})^{\beta-1}}{\left[1-(1-c)t^{\alpha}\right]^{\beta+1}}, \tag{3}$$

substituting (3) into (1) gives

$$F(y;\alpha,\beta,c,W)=\alpha\,\beta c\int_{0}^{G(y;W)}\frac{t^{\alpha-1}(1-t^{\alpha})^{\beta-1}}{\left[1-(1-c)t^{\alpha}\right]^{\beta+1}}dt, \tag{4}$$

setting $x=t^{\alpha}$ gives

$$F\left(y;\alpha,\beta,c,W\right)=\beta c\int_{0}^{G^{\alpha}(y;W)}\frac{\left(1-x\right)^{\beta-1}}{\left[1-\left(1-c\right)x\right]^{\beta+1}}dx,$$

using the following method of integration, Prudnikov *et al*. [13],

$$\int_{r}^{s}\frac{\left(1+\delta z\right)^{\beta-1}}{\left(1+\zeta z\right)^{\beta+1}}dz=\frac{\left(\delta s+1\right)^{\beta}}{\delta\beta\left(\zeta s+1\right)^{\beta}-\beta\zeta\left(\zeta s+1\right)^{\beta}}-\frac{\left(\delta r+1\right)^{\beta}}{\delta\beta\left(\zeta r+1\right)^{\beta}-\beta\zeta\left(\zeta r+1\right)^{\beta}};\delta,\zeta\in R;\beta,z>0,$$

then,

$$F\left(y;\alpha,\beta,c,W\right)=\beta c\left\{\frac{1}{\beta c}-\frac{\left(1-x\right)^{\beta}}{\beta c\left(cx-x+1\right)^{\beta}}\right\}_{0}^{G^{\alpha}(y;W)},$$

hence,

$$F\left(y;\alpha,\beta,c,W\right)=1-\frac{\left[1-G^{\alpha}\left(y;W\right)\right]^{\beta}}{\left[1-\left(1-c\right)G^{\alpha}\left(y;W\right)\right]^{\beta}};-\infty<y<\infty;\ \alpha,\beta,c>0, \tag{5}$$

differentiating (4) gives, directly, the *PDF* of the *MLN* class, as follows,

$$f\left(y;\alpha,\beta,c,W\right)=\frac{\alpha\beta c\ g\left(y;W\right)G^{\alpha-1}\left(y;W\right)\left(1-G^{\alpha}\left(y;W\right)\right)^{\beta-1}}{\left[1-\left(1-c\right)G^{\alpha}\left(y;W\right)\right]^{\beta+1}}, \tag{6}$$

setting $c=1$ gives the *KW* class and setting $c=1$, $\beta=1$ gives the exponentiated (*EX*) class. Many special distributions can be derived by the *MLN* class as the *MLN*-Weibull (*MLN-W*) distribution, some shapes of the density function for the *MLN-W* distribution are illustrated in figure 1.

2.1. An Expansion for the CDF and PDF

Using the binomial expansion for (4) gives

$$F\left(y;\alpha,\beta,c,W\right)=\alpha\beta c\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}t^{\alpha i}\int_{0}^{G(y;W)}\frac{t^{\alpha-1}}{\left[1-\left(1-c\right)t^{\alpha}\right]^{\beta+1}}dt,$$

then,

$$F\left(y;\alpha,\beta,c,W\right)=\alpha\beta c\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\int_{0}^{G(y;W)}\frac{t^{\alpha i+\alpha-1}}{\left[1-\left(1-c\right)t^{\alpha}\right]^{\beta+1}}dt,$$

substituting $x=t^{\alpha}$ into last equation yields

$$F\left(y;\alpha,\beta,c,W\right)=\beta c\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\int_{0}^{G^{\alpha}(y;W)}\frac{x^{i+1-1}}{\left[1-\left(1-c\right)x\right]^{\beta+1}}dx,$$

using the following method of integration, Gradshteyn and Ryzhik [14],

$$\int_{0}^{u}\frac{z^{\mu-1}}{\left[1+\lambda z\right]^{\nu}}dz=\frac{u^{\mu}}{\mu}\ {}_{2}F_{1}\left(\nu,\mu,1+\mu,-\lambda u\right), \tag{7}$$

where $u=G^{\alpha}\left(y;W\right)$, $\mu=i+1$, $\nu=\beta+1$, $\lambda=c-1$, gives

$$F\left(y;\alpha,\beta,c,W\right)=\beta c\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\frac{G^{\alpha(i+1)}\left(y;\mathrm{W}\right)}{(i+1)}\,{}_{2}F_{1}\left[(\beta+1),i+1,i+2,(1-c)G^{\alpha}\left(y;\mathrm{W}\right)\right],$$

using the hypergeometric expansion for the last equation yields

$$F\left(y;\alpha,\beta,c,W\right)=\beta c\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\frac{G^{\alpha(i+1)}\left(y;W\right)}{(i+1)}\sum_{j=0}^{\infty}\frac{(\beta+1)_{j}\,(i+1)_{j}}{(i+2)_{j}}\frac{\left[(1-c)G^{\alpha}\left(y;W\right)\right]^{j}}{j\,!},$$

where $(.)_j$ is the ascended factorial, then,

$$F\left(y;\alpha,\beta,c,W\right)=\beta c\sum_{i,j=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}\frac{(\beta+1)_{j}\,(1-c)^{j}\,G^{\alpha(i+1)+\alpha j}\left(y;W\right)}{j\,!},$$

hence,

$$F\left(y;\alpha,\beta,c,W\right)=\sum_{i,j=0}^{\infty}m_{i,j}\,G^{\alpha(i+1)+\alpha j}\left(y;W\right);$$

$$m_{i,j}=\beta c\,(-1)^{i}\binom{\beta-1}{i}\frac{(\beta+1)_{j}\,(1-c)^{j}}{j\,!}.$$

It can be seen that α , in the last equation, is a positive integer but one can generalizes the last equation after setting α a positive real, as follows, since

$$F\left(y;\alpha,\beta,c,W\right)=\sum_{i,j=0}^{\infty}m_{i,j}\left[1-\left(1-G\left(y;W\right)\right)\right]^{\alpha(i+1)+\alpha j},$$

using the binomial expansion twice for the last equation gives

$$F\left(y;\alpha,\beta,c,W\right)=\sum_{i,j=0}^{\infty}m_{i,j}\sum_{k=0}^{\infty}\left(1-G\left(y;W\right)\right)^{k}(-1)^{k}\binom{(i+1)+\alpha j}{k},$$

then,

$$F\left(y;\alpha,\beta,c,W\right)=\sum_{i,j=0}^{\infty}m_{i,j}\sum_{k=0}^{\infty}(-1)^{k}\binom{(i+1)+\alpha j}{k}\sum_{p=0}^{k}G^{p}\left(y;W\right)(-1)^{p}\binom{k}{p},$$

since,

$$\sum_{k=0}^{\infty}\sum_{p=0}^{k}=\sum_{p=0}^{\infty}\sum_{k=p}^{\infty},$$

hence,

$$F\left(y;\alpha,\beta,c,W\right)=\sum_{p=0}^{\infty}Q_{p}\,G^{p}\left(y;W\right);$$

$$Q_{p}=\sum_{i,j=0}^{\infty}\sum_{k=p}^{\infty}m_{i,j}\,(-1)^{k}\binom{(i+1)+\alpha j}{k}(-1)^{p}\binom{k}{p}, \tag{8}$$

differentiating (8) yields

$$f\left(y;\alpha,\beta,c,W\right)=\sum_{p=1}^{\infty}p\,Q_p\;g\left(y;W\right)G^{p-1}\left(y;W\right),$$

shifting the index p backward yields

$$f\left(y;\alpha,\beta,c,W\right)=\sum_{p=0}^{\infty}(p+1)\,Q_{p+1}\;g\left(y;W\right)G^{p}\left(y;W\right),$$

hence,

$$f\left(y;\alpha,\beta,c,W\right)=\sum_{p=0}^{\infty}Q_p^*\,g\left(y;W\right)G^{p}\left(y;W\right), \tag{9}$$

where,

$$Q_p^*=(p+1)\,Q_{p+1}.$$

2.2. The Condition for the PDF Expansion

since,

$$\sum_{p=0}^{\infty}Q_p^*\int_{-\infty}^{\infty}\text{g}\left(y;W\right)G^{p}\left(y;W\right)dy=1,$$

then,

$$\sum_{p=0}^{\infty}Q_p^*\left[\frac{G^{p+1}\left(y;W\right)}{\left(p+1\right)}\right]_{-\infty}^{\infty}=1,$$

hence,

$$\sum_{p=0}^{\infty}\frac{Q_p^*}{p+1}=1. \tag{10}$$

3. Some Properties of the *MLN* Class of Distributions

In this section some properties of the *MLN* class of distributions will be obtained as follows:

3.1. The r-th Moment

The random variable Y having the *PDF* of the *MLN* class of distributions has the following r-th moment, Johnson *et al.*[15],

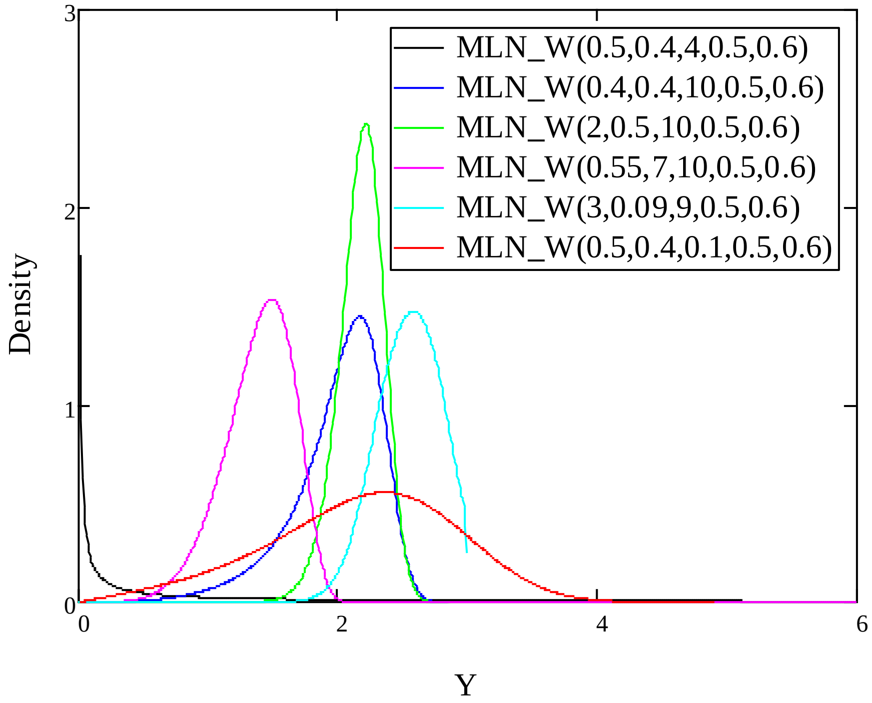

Figure 1: The *MLN-W* density functions

$$E\left(y^r\right)=\int_y y^r f\left(y\right)dy,$$

substituting (9) into last equation yields

$$E\left(y^r;\alpha,\beta,c,W\right)=\sum_{p=0}^{\infty}Q_p^*\int_{-\infty}^{\infty}y^r g\left(y;W\right)G^p\left(y;W\right)dy, \tag{11}$$

then,

$$E\left(y^r;\alpha,\beta,c,W\right)=\sum_{p=0}^{\infty}Q_p^*\,\tau_{r,p,0}\,.$$

where τ is the probability weighted moment (*PWM*), Greenwood *et al.*[16].
One easily can find that, setting $r=0$ and using (11) gives

$$E\left(y^0;\alpha,\beta,c,W\right)=\sum_{p=o}^{\infty}Q_p^*\int_{-\infty}^{\infty}g\left(y;W\right)G^p\left(y;W\right)dy,$$

then,

$$E\left(y^0;\alpha,\beta,c,W\right)=\sum_{p=o}^{\infty}Q_p^*\left[\frac{G^{p+1}\left(y;W\right)}{p+1}\right]_{-\infty}^{\infty},$$

substituting (10) into last equation yields

$$E\left(y^0;\alpha,\beta,c,W\right)=1.$$

Using the Parent Quantile Function

Setting $G(y;W)=u$, $y=q(u)$ and substituting into (11) gives

$$E\left(y^{r};\alpha,\beta,c,W\right)=\sum_{p=o}^{\infty}Q_{p}^{*}\int_{0}^{1}q^{r}(u)u^{p}du,$$

then,

$$E\left(y^{r};\alpha,\beta,c,W\right)=\sum_{p=o}^{\infty}Q_{p}^{*}\,\tau_{r,p,o}.$$

3.2. The PWM

The random variable *Y* having the *CDF* and *PDF* of the *MLN* class of distributions has the following *PWM*, Greenwood *et al*. [16],

$$\tau_{r,s,0}=\int_{y}y^{r}f(y)F^{s}(y)dy,$$

substituting (8) and (9) into last equation gives

$$\tau_{r,s,o}=\int_{-\infty}^{\infty}\left[\sum_{p=o}^{\infty}Q_{p}^{*}G^{p}(y;W)\right]\left[\sum_{p=o}^{\infty}Q_{p}G^{p}(y;W)\right]^{s}y^{r}g(y;W)dy,$$

since,

$$\tau_{r,s,o}=\int_{-\infty}^{\infty}\left[\sum_{p=o}^{\infty}Q_{p}^{*}G^{p}(y;W)\right]\left[\sum_{p=o}^{\infty}c_{p}G^{p}(y;W)\right]y^{r}g(y;W)dy,$$

where

$$c_{0}=Q_{0}^{s},\ c_{m}=\frac{1}{m\,Q_{0}}\sum_{p=1}^{m}(ps-m+p)Q_{p}\,c_{m-p};m\geq 1,$$

hence,

$$\tau_{r,s,o}=\sum_{p=o}^{\infty}d_{p}\int_{-\infty}^{\infty}y^{r}g(y;W)G^{p}(y;W)dy, \tag{12}$$

where

$$d_{p}=\sum_{p=o}^{n}Q_{p}^{*}\,c_{n-p},$$

then,

$$\tau_{r,s,o}=\sum_{p=o}^{\infty}d_{p}\tau_{r,p,o}.$$

Using the Parent Quantile Function

Setting $G(y;W) = u$, $y = q(u)$ and substituting into (12) yields

$$\tau_{r,s,o} = \sum_{p=o}^{\infty} d_p \int_0^1 q^r(u)\, u^p du,$$

then,

$$\tau_{r,s,o} = \sum_{p=o}^{\infty} d_p\, \tau_{r,p,o}.$$

3.3. The Moment Generating Function

The random variable *Y* having the *PDF* of the *MLN* class of distributions has the following moment generating function (*MGF*)

$$M_y(t) = E\left(e^{ty}\right) = \int_y e^{ty} f(y)dy, \tag{13}$$

using the exponential expansion for last equation gives

$$E\left(e^{ty}\right) = E\left(\sum_{r=0}^{\infty} \frac{t^r y^r}{r!}\right),$$

then,

$$E\left(e^{ty}\right) = \sum_{r=0}^{\infty} \frac{t^r E\left(y^r\right)}{r!}.$$

Using the Parent Quantile Function

Substituting (9) into (13) gives

$$E\left(e^{ty};\alpha,\beta,c,W\right) = \int_{-\infty}^{\infty} e^{ty} \sum_{p=o}^{\infty} Q_p^* g(y;W) G^p(y;W) dy,$$

then,

$$E\left(e^{ty};\alpha,\beta,c,W\right) = \sum_{p=o}^{\infty} Q_p^* \int_{-\infty}^{\infty} e^{ty} g(y;W) G^p(y;W) dy,$$

setting $G(y;W) = u$, $y = q(u)$ and substituting into last equation yields

$$E\left(e^{ty};\alpha,\beta,c,W\right) = \sum_{p=o}^{\infty} Q_p^* \int_0^1 e^{t\,q(u)} u^p du.$$

3.4. The Mean Deviation

The random variable *Y* having the *PDF* of the *MLN* class of distributions has the following mean deviation about the mean and about the median, respectively,

$$S_1(y)=\int_y |y-\mu| f(y)dy \quad \text{and} \quad S_2(y)=\int_y |y-M| f(y)dy,$$

easily, it can be given by, Ahmed [17], Ali Ahmed [12]

$$S_1(y)=2\mu F(\mu)-2t(\mu) \text{ and } S_2(y)=\mu-2t(M),$$

where $T(z)=\int_{-\infty}^{z} y f(y)dy$ is the linear incomplete moment.

Substituting (9) into *T(.)* gives

$$T(z;\alpha,\beta,c,W)=\int_{-\infty}^{z} y \sum_{p=o}^{n} Q_p^{*} g(y;W) G^{p}(y;W)dy,$$

then,

$$T(z;\alpha,\beta,c,W)=\sum_{p=o}^{\infty} Q_p^{*} \int_{-\infty}^{z} y\, g(y;W) G^{p}(y,w)dy.$$

Using the Parent Quantile Function

Setting $G(y;W)=u$, $y=q(u)$ and substituting into last equation yields

$$T(z;\alpha,\beta,c,W)=\sum_{p=o}^{\infty} Q_p^{*} \int_{0}^{G(z)} q(u)u^{p}du.$$

4. The Hazard Function of the *MLN* Class of Distributions

The random variable *Y* having the *CDF* of the *MLN* class of distributions has the following survival function, Meeker and Escobar [18],

$$S(y)=1-F(y),$$

substituting (5) into last equation gives

$$S(y;\alpha,\beta,c,W)=\frac{\left[1-G^{\alpha}(y;W)\right]^{\beta}}{\left[1-(1-c)G^{\alpha}(y;W)\right]^{\beta}};\ -\infty<y<\infty;\ \alpha,\beta,c>0. \tag{14}$$

Simply, the Hazard function, Meeker and Escobar [18], can be given by

$$H(y)=\frac{f(y)}{S(y)},$$

substituting (6) and (14) into last equation yields

$$H(y;\alpha,\beta,c,W)=\frac{\alpha\,\beta\,c\,\,g(y;W)G(y;W)^{\alpha-1}}{\left[1-G^{\alpha}(y;W)\right]\left[1-(1-c)G(y;W)^{\alpha}\right]},$$

some shapes of the Hazard function, for example, for the *MLN-W* distribution are illustrated in figure 2.

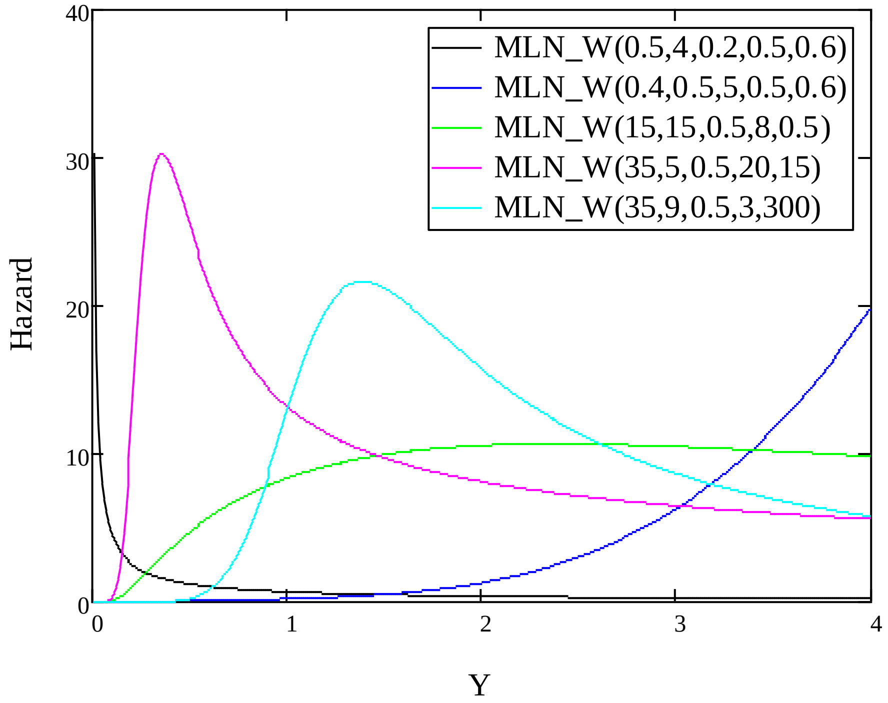

Figure 2: The *MLN-W* Hazard functions

One can see, in figure 2, three types of Hazard functions curves of the *MLN-W* distribution are described as follows: A decreasing then constant Hazard curve, an increasing then constant Hazard curve and an increasing then decreasing then constant Hazard curve.

5. The Rényi Entropy of the *MLN* Class of Distributions

The random variable Y having the PDF of the MLN class of distributions has the following Rényi entropy, Meeker and Escobar [18],

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\int_{-\infty}^{\infty}[f(y)]^{\rho}\,dy\right],$$

substituting (9) into last equation gives

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\int_{-\infty}^{\infty}\left(\sum_{p=0}^{\infty}Q_p^{*}\,g(y;W)G^{p}(y;W)\right)^{\rho}dy\right],$$

then,

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\int_{-\infty}^{\infty}g^{\rho}(y;W)\left(\sum_{p=0}^{\infty}Q_p^{*}G^{p}(y;W)\right)^{\rho}dy\right],$$

hence,

$$e_R(\rho)=\frac{1}{1-\rho}\log\left[\sum_{p=0}^{\infty}a_p\int_{-\infty}^{\infty}g^{\rho}(y;W)G^{p}(y;W)dy\right],$$

where,

$$a_0 = \left(Q_0^*\right)^{\rho},\ a_m = \frac{1}{m\,Q_0^*}\sum_{p=1}^{m}\left(p\rho - m + p\right)Q_p^*\,a_{m-p};m \geq 1.$$

6. Order Statistics of the *MLN* Class of Distributions

A density function $f\left(y_{u:v}\right)$ of the *u-th* order statistics for *u = 1,2,…, v* from *iid* random variables $Y_1, Y_2, \ldots, Y_v$ following any *MLN* generalized distribution, Arnold *et al.*[19], is given by

$$f\left(y_{u:v}\right) = \frac{f\left(y_u\right)}{B\left(u, v-u+1\right)}F^{u-1}\left(y_u\right)\left\{1-F\left(y_u\right)\right\}^{v-u},$$

using binomial expansion for last equation gives

$$f\left(y_{u:v}\right) = \sum_{w=0}^{v-u}\frac{(-1)^w\binom{v-u}{w}}{\beta\left(u, v-u+1\right)}f\left(y_u\right)F^{u+w-1}\left(y_u\right),$$

substituting (8) and (9) into last equation gives

$$f\left(y_{u:v};\alpha,\beta,c,W\right) = \sum_{w=0}^{v-u}\frac{(-1)^w\binom{v-u}{w}}{\beta\left(u, v-u+1\right)}\left[\sum_{p=0}^{\infty}Q_p^* g\left(y_u;W\right)G^P\left(y_u;W\right)\right]$$
$$\times\left[\sum_{p=0}^{\infty}Q_p G^P\left(y_u;W\right)\right]^{u+w-1},$$

then,

$$f\left(y_{u:v};\alpha,\beta,c,W\right) = \sum_{w=0}^{v-u}\frac{(-1)^w\binom{v-u}{w}}{\beta\left(u, v-u+1\right)}\left[\sum_{p=0}^{\infty}Q_p^* g\left(y_u;W\right)G^P\left(y_u;W\right)\right]\left[\sum_{p=0}^{\infty}c_p G^P\left(y_u;W\right)\right].$$

where,

$$c_0 = Q_0^{u+w-1},\ c_m = \frac{1}{m\,Q_0}\sum_{p=1}^{m}\left[p\left(u+w-1\right)-m+p\right]Q_p\,c_{m-p};m \geq 1,$$

hence,

$$f\left(y_{u:v};\alpha,\beta,c,W\right) = \sum_{w=0}^{v-u}\frac{(-1)^w\binom{v-u}{w}}{\beta\left(u, v-u+1\right)}\sum_{p=0}^{\infty}d_p\,g\left(y_u;W\right)G^P\left(y_u;W\right)$$

where,

$$d_p = \sum_{p=o}^{n}Q_p^*\,c_{n-p},$$

moreover,

$$f\left(y_{u:v};\alpha,\beta,c,W\right)=\sum_{p=0}^{\infty}b_p\,g\left(y_u;W\right)G^{P}\left(y_u;W\right), \tag{15}$$

where,

$$b_p=\sum_{w=0}^{v-u}\frac{(-1)^w\binom{v-u}{w}}{\beta(u,v-u+1)}\sum_{p=o}^{n}Q_p^{*}\,c_{n-p}.$$

The *r-th* moment of order statistics of the *MLN* class of distributions is given by

$$E\left(y_{u:v}^{r}\right)=\int_{y_u}y_u^{r}\,f\left(y_{u:v}\right)dy_u,$$

substituting (15) into last equation yields

$$E\left(y_{u:v}^{r};\alpha,\beta,c,W\right)=\sum_{p=o}^{\infty}b_p\int_{-\infty}^{\infty}y_u^{r}\,g\left(y_u;W\right)G^{p}\left(y_u;W\right)dy_u,$$

so that,

$$E\left(y_{u:v}^{r};\alpha,\beta,c,W\right)=\sum_{p=o}^{\infty}b_p\,\tau_{r,\mathrm{p},o}.$$

7. Estimation for the *MLN* Class Parameters Using *MLE* Method

Let $Y_1,Y_2,\ldots,Y_n$ be *iid* random variables following any *MLN* generalized distribution $(y;\Lambda)$ then the likelihood function for the vector of parameter $\Lambda=\left(\alpha,\beta,c,W\right)$, Garthwait *et al*.[20], is obtained by

$$L\left(y;\alpha,\beta,c,W\right)=(\alpha\beta c)^n\prod_{i=1}^{n}g\left(y_i;W\right)\prod_{i=1}^{n}G^{\alpha-1}\left(y_i;W\right)\prod_{i=1}^{n}\left(1-G^{\alpha}\left(y_i;W\right)\right)^{\beta-1}$$
$$\times\prod_{i=1}^{n}\left[1-(1-c)G^{\alpha}\left(y_i;W\right)\right]^{-\beta-1},$$

the log likelihood function is given by

$$\ell\left(y;\alpha,\beta,c,W\right)=n\left[\log\alpha+\log\beta+\log c\right]+\sum_{i=1}^{n}\log g\left(y_i;W\right)+(\alpha-1)\sum_{i=1}^{n}\log G\left(y_i;W\right)$$
$$+(\beta-1)\sum_{i=1}^{n}\log\left(1-G^{\alpha}\left(y_i;W\right)\right)+(-\beta-1)\sum_{i=1}^{n}\log\left[1-(1-c)G^{\alpha}\left(y_i;W\right)\right],$$

the score functions for the parameters α, β, c and W are given by

$$\frac{\partial\ell\left(y;\alpha,\beta,c,W\right)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\log G\left(y_i;W\right)+(1-\beta)\sum_{i=1}^{n}\frac{G^{\alpha}\left(y_i;W\right)\log G\left(y_i;W\right)}{1-G^{\alpha}\left(y_i;W\right)}$$
$$+(1+\beta)\sum_{i=1}^{n}\frac{(1-c)G^{\alpha}\left(y_i;W\right)\log G\left(y_i;W\right)}{1-(1-c)G^{\alpha}\left(y_i;W\right)},$$

$$\frac{\partial \ell(y;\alpha,\beta,c,W)}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(1-G^{\alpha}(y_i;W)\right)-\sum_{i=1}^{n}\log\left[1-(1-c)G^{\alpha}(y_i;W)\right],$$

$$\frac{\partial \ell(y;\alpha,\beta,c,W)}{\partial c}=\frac{n}{c}-(\beta+1)\sum_{i=1}^{n}\frac{G^{\alpha}(y_i;W)}{\left[1-(1-c)G^{\alpha}(y_i;W)\right]},$$

and

$$\frac{\ell(y;\alpha,\beta,c,W)}{\partial w_j}=\sum_{i=1}^{n}\frac{1}{g(y_i;W)}\frac{\partial g(y_i;W)}{\partial w_j}+(\alpha-1)\sum_{i=1}^{n}\frac{1}{G(y_i;W)}\frac{\partial G(y_i;W)}{\partial w_j}$$
$$+(1-\beta)\sum_{i=1}^{n}\frac{\alpha G^{\alpha-1}(y_i;W)}{1-G^{\alpha}(y_i;W)}\frac{\partial G(y_i;W)}{\partial w_j}$$
$$+(1+\beta)\sum_{i=1}^{n}\frac{(1-c)\alpha G^{\alpha-1}(y_i;W)}{1-(1-c)G^{\alpha}(y_i;W)}\frac{\partial G(y_i;W)}{\partial w_j}.$$

8. A Simulation Study

This study aims to obtain *MLEs* of the *MLN-W* distribution parameters via random numbers to study the sample behavior of *MLEs* by the bootstrapping resample approach.

Obtaining parameters estimates algorithm is detailed in the following steps:

i. Generating a random sample $Y_1,Y_2,\ldots,Y_n$ having sizes *n=(5,15,30,50,100,300)* using the *MLN-W* distribution.

ii. Selecting different parameters set values as: set (1): *(α=0.1, β=0.2, θ=0.3, λ=4, c=3)*, set (2): *(α=0.1, β=0.2, θ=0.3, λ=6, c=3)* and set (3): *(α=0.1, β=0.2, θ=0.5, λ=4, c=3)*.

iii. Solving the *MLN-W* distribution normal equations by iteration to estimate distribution parameters.

iv. Replacing set (1), set (2) and set (3) with its estimators and repeating step (3) to calculate: Biases, *MLEs, RMSE* (the root of mean squared error) and the Pearson type of parameters estimators of the *MLN-W* distribution, Pearson (1895).

v. Repeating step (1) to step (4), 10000 times.

Random numbers samples are generated via Mathcad package v15 using the conjugate gradient iteration method. All outcomes are indicated in the appendix.

Clearly, from study results included in the appendix, Biases and *RMSEs* decrease as sample size increases. In all times, $\hat{\beta}$ and $\hat{\lambda}$ sampling distributions follow the Pearson type *IV* distribution, $\hat{\alpha}$, c and $\hat{\theta}$ sampling distributions differ according to sample size. As λ increases, for fixed values of α, β, c and θ, the biases and *MSEs* of $\hat{\beta}$ and $\hat{\theta}$ decrease.. Also, one can see that when sample size increases, the estimators can be consistent.

9. Application

A real data set is used, practically, to investigate the flexibility of the new model using *MLE* method via the Mathematica package version 10, some distributions are used as: the *MLN-W* distribution, the gamma distribution, the Gumbel (Max) distribution, the Singh-Maddala distribution, Kumar [21], the Kumaraswamy-Weibull *(KW-W)* distribution, Cordeiro *et al.* [22], the exponentiated-Weibull (*EX-W*) distribution, Nassar *et al.* [23], and the Weibull (*W*) distribution. The lifetime (Hours) of classic lamps for 60 devices as follows, given from the *UK* National Physical Laboratory at http://www.npl.co.uk/

3.337, 0.988, 6.058, 0.347, 0.924, 2.484, 3.149, 0.478, 5.000, 5.273, 5.600, 0.348, 7.208, 3.087, 2.405, 2.123, 1.023, 2.154, 4.535, 1.164, 1.330, 0.494, 0.164, 4.766, 5.396, 3.338, 4.663, 3.124, 7.406, 4.494, 3.136, 8.305, 0.483, 3.623, 0.060, 1.761, 3.862, 2.451, 2.714, 4.937, 3.343, 2.225, 0.608, 5.151, 1.950, 5.504, 0.244, 3.860, 4.431, 0.434, 0.083, 2.456, 1.886, 1.396, 2.823, 0.430, 0.573, 4.499, 0.260, 0.634.

Some measures for goodness of fit and likelihood ratio tests are obtained and included in table (1) and table (2), respectively, the figure (3) shows *PDF*s for some distributions having skewness and kurtosis values similar to the *MLN-W* distribution (the gamma distribution, the Gumbel (Max) distribution, the Singh-Maddala distribution) and the figure (4) shows the empirical *CDF* compared to *CDFs* for some distributions (the gamma distribution, the Gumbel (Max) distribution, the Singh-Maddala distribution) and the figure (5) illustrates probability density functions for special cases from the *MLN-W* distribution (the *KW-W* distribution, the *EX*-W distribution and the *W* distribution).

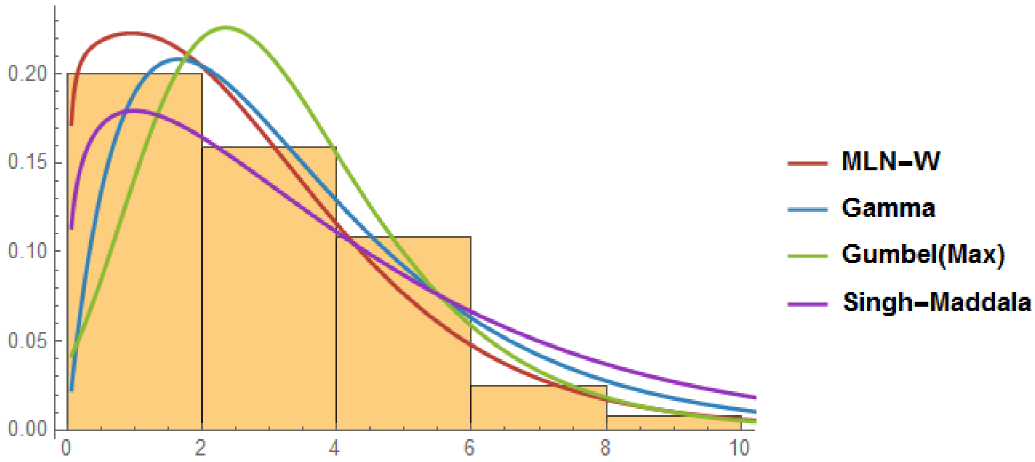

Figure 3: Probability density functions for some distributions having skewness and kurtosis values similar to the *MLN-W* distribution

Table 1: The *MLE* of the parameter(s) and the associated *AIC* and *BIC* values.

Distribution	MLE_parameters					Skewness	Kurtosis	*KS*	P-value	Log Likelihood	*AIC*	*BIC*	*CAIC*
	α	β	c	Λ	θ								
MLN_W	3.103	2.012	0.110	0.503	3.101	1.489	6.331	0.119	0.327	-119.502	249.003	259.475	250.114
	(0.142)	(0.107)	(0.012)	(0.121)	(0.725)								
Singh-Maddla	3.3	1.2088	5.1			1.505	4.159	0.174	0.044	-124.692	255.383	261.666	255.812
	(0.154)	(0.147)	(1.812)										
Gamma	1.8772	1.8662				1.459	5.196	0.190	0.021	-127.138	258.275	262.464	258.486

	(0.201)	(0.153)										
Gumbel (Max)	2.3438	1.628			1.140	6.400	0.194	0.018	-129.571	263.142	267.331	263.352
	(0.226)	(0.173)										

In table (1): *MLE*s of distributions parameters, test statistic of Kolmogorov-Smirnov (*KS*), the corresponding *RMSE* (given in parentheses), *AIC* (Akaike Information Criterion), *BIC* (Bayesian information criterion) and *CAIC* (the consistent Akaike Information Criterion), Merovcia and Puka [24], are computed for all distributions having skewness and kurtosis values similar to the *MLN-W* distribution as the gamma distribution, the Gumbel (Max) distribution and the Singh-Maddala distribution. The null hypothesis is the data follow the *MLN-W* distribution and it can be accepted at significance level α=0.05 where the *MLN-W* distribution has the smallest *KS*, *AIC*, *CAIC*, *BIC* , *SEs* and the largest p-value, so that, the *MLN*-W distribution can be the best fitted distribution to the data between other distributions.

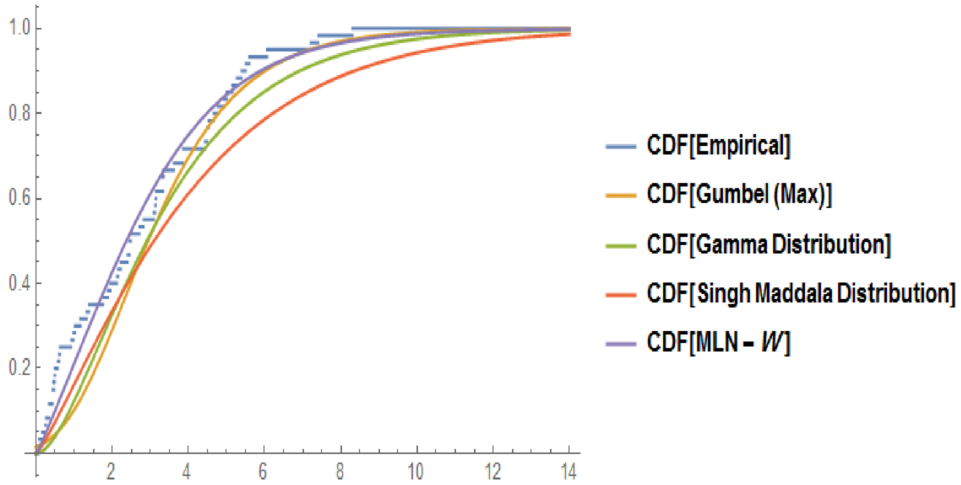

Figure 4: The empirical *CDF* compared to *CDF*s for some distributions

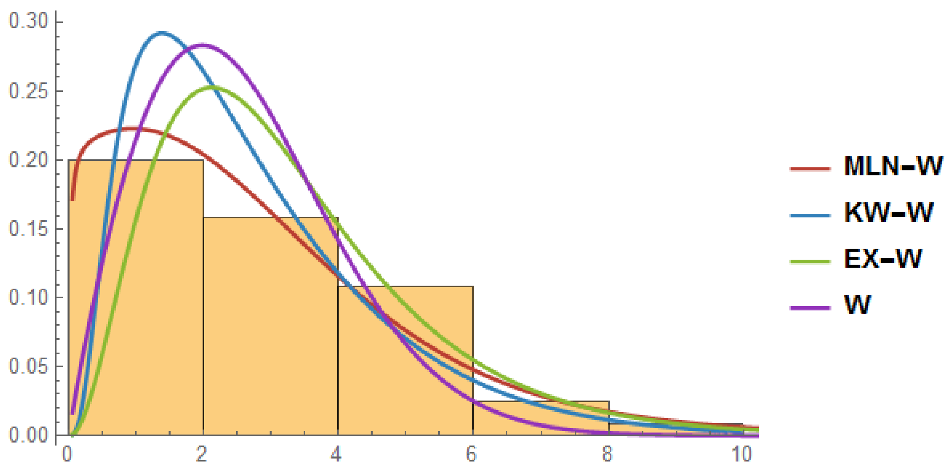

Figure 5: Probability density functions for special cases from the *MLN-W* distribution

Table 2: The log-likelihood function, the likelihood ratio tests statistic and p-values.

Distribution	Parameters					ℓ (log likelihood)	Λ (The likelihood ratio test statistic)	df degrees(of freedom)	p-value
	α	β	c	Θ	λ				
KW-W	2.985	0.151	–	1.298	1.632	-137.534	36.064	1	1.909×10^{-9}
	(0.149)	(0.118)		(0.134)	(0.709)				
EX-W	2.774	–	–	1.099	0.506	-143.153	47.302	2	5.352×10^{-11}
	(0.152)			(0.221)	(0.947)				
W	–	–	–	1.898	0.341	-133.532	28.06	3	3.528×10^{-6}
				(0.129)	(0.737)				

*Note that the log likelihood of the *MLN-W* distribution = -119.502

In table 2, upon the likelihood ratio test, the null hypothesis is the data follow the nested model and it can be rejected at the level of significance α = 0.05, so the *MLN-W* distribution can fit the data better than all nested distributions at significance level α=0.05, where the *KW-W* distribution, the *EX-W* distribution and the *W* distribution are nested by *MLN*-W distribution.

10. Conclusion

The modified Libby-Novick class of generalized distributions has several advantages as: it does not have any special function having implicit form, has flexible mathematical properties, simple quantile function and generalizes two important classes of distributions (the *KW* class and the *EX* class). The maximum likelihood estimation method is used easily to estimate the *MLN* class parameters, the *MLN-W* distribution works practically well when it be compared with other distributions. The author encourages researchers to do more researches and applications on the *MLN* class of generalized distributions.

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References

[1] Eugene, N., Lee, C. & Famoye, F. (2002). Beta-Normal Distribution and Its Applications. *Communications in Statistics, Theory and Methods, 31*, 497-512.

[2] Wahed, A.S. (2006). A General Method of Constructing Extended Families of Distribution from an Existing Continuous Class. *Journal of Probability and Statistical Science, 4*, 165-177.

[3] Kumaraswamy, P. (1980). Generalized Probability Density Function for Double-Bounded Random-Processes. *Journal of Hydrology, 46*, 79-88.

[4] Cordeiro, GM & de Castro, M. (2011). A New Family of Generalized Distributions. *Journal of Statistical Computation & Simulation, 81*, 883-898.

[5] Pescim, R. R., Cordeiro, G. M., Demétrio, C. G., Ortega, E. M. & Nadarajah, S. (2012). The new class of Kummer beta generalized distributions. *SORT-Statistics and Operations Research Transactions*, 153-180.

[6] McDonald, J.B. (1984). Some generalized functions for the size distribution of income. *Econometrica, 52*, 647-664.

[7] Alexander, C., Cordeiro, G.M., Ortega, E.M.M. & Sarabia, J.M. (2012). Generalized beta-generated distributions. *Comput Stat Data Anal, 56*, 1880-1897.

[8] El-Sherpieny, E.A. & Ahmed, MA. (2014). On The Kumaraswamy Kumaraswamy Distribution. *International Journal of Basic and Applied Sciences, 3*, 372-381

[9] Mahmoud, M.R., El-Sherpieny, E.A. & Ahmed, M.A. (2015). The New Kumaraswamy Kumaraswamy Family of Generalized Distributions with Application. *Pakistan Journal of Statistics and Operations Research, 11*, 159-180.

[10] Libby, D.L. & Novick, M.R. (1982). Multivariate generalized beta-distributions with applications to utility assessment. *Journal of Educational Statistics, 7*, 271-294.

[11] Cordeiro, G.M., de Santana, L.H. & Ortega, E.M., Pescim, R.R.(2014). A new family of distributions: Libby-Novick beta. *International Journal of Statistics and Probability, 3*, 63-80.

[12] Ali Ahmed, M. (2021). The new form Libby-Novick distribution. *Communications in Statistics-Theory and Methods, 50*, 540-559.

[13] Prudnikov, A.P., Brychkov, Y.A. & Marichev, O.I. (1986). *Integrals and Series*. Gordon and Breach Science Publishers, Amsterdam.

[14] Gradshteyn, I.S. & Ryzhik, I.M. (2000). *Tables of Integrals, Series, and Products*. Academic Press, San Diego, CA.

[15] Johnson, N.L., Kotz, S. & Balakrishnan, N.(1995). *Continuous Univariate Distributions*. John wiley and Sons, New York.

[16] Greenwood, J.A., Landwehr, J.M., Matalas, N.C. & Wallis, J.R. (1979). Probability Weighted Moments Definition and Relation to Parameters of Several Distributions. Expressable in Inverse Form. *Water Resources Research. 15*, 1049-1054.

[17] Ahmed, M.A. (2020). On the alpha power Kumaraswamy distribution: Properties, simulation and application. *Revista Colombiana de Estadística. 43*, 285-313.

[18] Meeker, W.Q. & Escobar, L.A. (1998). *Statistical Methods for Reliability Data*. John Wiley, New York.

[19] Arnold, C.B., Balakrishnan, N. & Nagaraja, H.N. (1992). *A first course in order statistics*. John Wiley and Sons, Inc. New York.

[20] Garthwait, P.H., Jolliffe, I.P. & jones, B. (1995). *Statistical Inference*. prentice Hall International (UK) Limited, London.

[21] Kumar, D. (2017). The Singh–Maddala distribution: properties and estimation. *International journal of system assurance engineering and management, 8*, 1297-1311.

[22] Cordeiro, G.M., Ortega, E.M.M. & Nadarajah, S. (2010) The Kumaraswamy WeibullDistribution with Application to Failure Data. *Journal of The Franklin Institute, 347*, 1399–1429.
[23] Nassar, M.M. & Eissa, F.H. (2003). On the exponentiated Weibull distribution. *Communications in Statistics-Theory and Methods, 32*, 1317-1336.
[24] Merovcia, F. & Puka, L. (2014).Transmuted Pareto Distribution. *Prob Stat Forum, 7*, 1-11.

Appendix

Set(1):(α=0.1, β=0.2, θ =0.3, λ=4, c=3)								
Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	α=0.1	0.053	-0.047	2.127	0.065	5.443	-0.538	I
	β=0.2	0.304	0.104		0.235		0.501	IV
	θ =0.3	0.341	0.041		0.091		-0.012	I
	λ=4	4.935	0.935		4.577		0.81	IV
	c=3	1.093	-1.907		2.934		0.346	IV
20	α=0.1	0.067	-0.033	2.034	0.056	4.309	0.295	IV
	β=0.2	0.282	0.082		0.083		0.238	IV
	θ =0.3	0.331	0.031		0.049		0.134	IV
	λ=4	4.862	0.862		3.138		0.302	IV
	c=3	1.160	-1.84		2.952		0.391	IV
30	α=0.1	0.075	-0.025	1.505	0.056	3.862	-0.314	I
	β=0.2	0.247	0.047		0.037		0.097	IV
	θ =0.3	0.33	0.03		0.042		-0.131	I
	λ=4	4.730	0.730		2.482		0.78	IV
	c=3	1.685	-1.315		2.957		0.407	IV
50	α=0.1	0.086	-0.014	1.088	0.058	3.599	0.332	IV
	β=0.2	0.235	0.035		0.027		0.085	IV
	θ =0.3	0.321	0.021		0.035		0.578	IV
	λ=4	4.511	0.511		2.038		0.324	IV
	c=3	2.04	-0.96		2.965		0.527	IV
100	α=0.1	0.094	-0.006	0.339	0.045	2.318	0.005	IV
	β=0.2	0.210	0.010		0.072		0.004	IV
	θ =0.3	0.305	0.005		0.035		-0.076	I
	λ=4	4.269	0.269		1.165		0.155	IV
	c=3	2.794	-0.206		2.002		-0.0004	I
300	α=0.1	0.096	-0.004	0.025	0.025	1.133	0.003	IV
	β=0.2	0.202	0.002		0.037		0.001	IV
	θ =0.3	0.300	0.000		0.014		-0.047	I
	λ=4	4.013	0.013		0.561		0.148	IV
	c=3	2.979	-0.021		0.984		0.269	IV

Set(2):(α=0.1, β=0.2, θ =0.3, λ=6, c=3)								
Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	α=0.1	0.056	-0.044	2.407	0.127	8.031	-0.903	I
	β=0.2	0.300	0.100		0.195		0.588	IV
	θ =0.3	0.331	0.031		0.083		-0.105	I
	λ=6	7.591	1.591		7.122		0.943	IV
	c=3	1.197	-1.803		3.704		0.881	IV
20	α=0.1	0.074	-0.026	1.998	0.055	6.219	0.271	IV
	β=0.2	0.281	0.081		0.076		0.62	IV
	θ =0.3	0.321	0.021		0.041		-0.064	I
	λ=6	7.319	1.319		4.828		0.56	IV
	c=3	1.501	-1.499		3.92		1.035	VI
30	α=0.1	0.084	-0.016	1.463	0.057	4.927	0.278	IV
	β=0.2	0.236	0.036		0.031		0.101	IV
	θ =0.3	0.320	0.020		0.040		-0.345	I
	λ=6	7.002	1.002		3.939		0.238	IV
	c=3	1.934	-1.066		2.959		0.428	IV
50	α=0.1	0.093	-0.007	0.794	0.057	4.421	0.311	IV
	β=0.2	0.231	0.031		0.026		0.094	IV
	θ =0.3	0.314	0.014		0.032		0.128	IV
	λ=6	6.467	0.467		3.286		0.246	IV
	c=3	2.358	-0.642		2.958		0.409	IV
100	α=0.1	0.097	-0.003	0.474	0.057	3.745	-0.909	I
	β=0.2	0.205	0.005		0.021		0.408	IV
	θ =0.3	0.301	0.001		0.028		0.269	IV
	λ=6	6.273	0.273		2.598		0.47	IV
	c=3	2.612	-0.388		2.697		-0.486	I
300	α=0.1	0.099	-0.001	0.061	0.037	1.392	0.105	IV
	β=0.2	0.201	0.001		0.018		0.643	IV
	θ =0.3	0.300	0.000		0.010		-0.57	I
	λ=6	6.053	0.053		0.618		0.671	IV
	c=3	2.968	-0.032		1.247		0.347	IV

Set(3):(α=0.1, β=0.2, θ =0.5, λ=4, c=3)								
Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSE	Total RMSE	Pearson System Coefficients	Pearson Type
10	α=0.1	0.058	-0.042	1.968	0.056	5.754	0.386	IV
	β=0.2	0.391	0.191		0.319		0.79	IV

	θ =0.5	0.575	0.075		0.263		0.372	IV
	λ=4	4.846	0.846		4.86		0.407	IV
	c=3	1.235	-1.765		3.052		-0.562	I
20	α=0.1	0.078	-0.022	1.544	0.055	4.524	0.302	IV
	β=0.2	0.369	0.169		0.085		0.24	IV
	θ =0.5	0.565	0.065		0.083		-0.172	I
	λ=4	4.743	0.743		3.423		0.617	IV
	c=3	1.658	-1.342		2.955		0.399	IV
30	α=0.1	0.085	-0.015	1.125	0.056	4.02	0.244	IV
	β=0.2	0.316	0.116		0.038		0.875	IV
	θ =0.5	0.543	0.043		0.069		0.905	IV
	λ=4	4.576	0.576		2.712		0.715	IV
	c=3	2.041	-0.959		2.966		0.518	IV
50	α=0.1	0.093	-0.007	0.578	0.057	3.641	0.319	IV
	β=0.2	0.283	0.083		0.032		0.105	IV
	θ =0.5	0.528	0.028		0.062		-2.276	I
	λ=4	4.334	0.334		2.108		0.618	IV
	c=3	2.536	-0.464		2.967		-0.579	I
100	α=0.1	0.096	-0.004	0.180	0.061	3.502	-0.399	I
	β=0.2	0.247	0.047		0.031		0.149	IV
	θ =0.5	0.513	0.013		0.059		4.216	VI
	λ=4	4.119	0.119		1.872		0.346	IV
	c=3	2.874	-0.126		2.959		0.709	IV
300	α=0.1	0.097	-0.003	0.075	0.037	1.393	0.105	IV
	β=0.2	0.205	0.005		0.048		0.643	IV
	θ =0.5	0.504	0.004		0.024		-0.57	I
	λ=4	4.071	0.071		0.618		0.671	IV
	c=3	2.976	-0.024		1.247		0.347	IV