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RESEARCH ARTICLE

SA-EXTENDING MODULES

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ABSTRACT

In this note we investigate modules with the Ads and the SIP-extending properties. Besides many results obtained, we show that the class of SA-extending modules is not closed under direct sums. Further we deal with direct summands of an SA-extending module. We give necessary and sufficient conditions for a module to be an SA-extending. It has been also shown that the concepts of UC, SA and SA-extending coincide when a module is distributive quasi continuous.

Keywords: Ads-module, SIP-extending module, UC module.

1. INTRODUCTION

Throughout this article, all rings are associative with unity and R signifies such a ring. All modules are unital right R-modules unless indicated otherwise. Let M be a module. Thus $N \leq M$, $N \leq_e M$, $End(M)$, and $r(X)$ will stand for N is a submodule of M, N is an essential submodule of M, the ring of endomorphism of M, and the right annihilator of a subset X in M, respectively. For any other terminology or unexplained notions, we refer to [1, 2].

It is well known that a module is said to be CS or *extending* or said to satisfy the C_1 *condition* if every submodule is essential in a direct summand. Extending modules and their several generalizations have been worked out extensively by many authors, see for example [3-6].

After Fuchs [7] defined the absolute direct summand property (briefly, Ads) for abelian groups in 1970, Burgess and Raphael [8] defined the absolute direct summand property for modules and investigated these modules, A module M is said to have the *absolute direct summand property* (briefly, Ads) if for every decomposition $M = A \oplus B$ of M and every complement C of A in M we have $M = A \oplus C$. A ring R is said to have the right Ads if as a right R-module R has the Ads.

In 1986, Wilson [9] defined the summand intersection property (briefly, SIP) for modules. A module is said to be SIP or said to have *summand intersection property (SIP)*, if the intersection of every pair of direct summands of M is a direct summand of M. Several generalizations of SIP modules have been studied to date (for example, [10, 11]). One of the generalizations of the SIP modules is SIP-extending module. A module M is said to be an *SIP-extending module* if the intersection of any two direct summands is essentially contained in a direct summand. A ring R is said to be an SIP-extending ring if as a right R-module R is an SIP-extending module.

Since the date which these concepts were defined, studies on these two module property have been increasingly continuing. In 2015, Takıl Mutlu showed that these two concepts do not require each other, and the author called the modules with these two property as SA-modules and examined this module class in detail (see, [6]). In 2018, the author examined the matrix rings with both the Ads

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and the SIP properties. In addition, some well-studied rings (such as semisimple rings, V-rings, regular rings) have been characterized in terms of SA modules in [12].

In this paper, SIP-extending modules with the Ads property (briefly, SA-extending module) which are a proper generalization of SA-modules, are examined. This module class has been studied specifically for direct sum and direct summand. A characterization of SA-extending modules is provided in Theorem 2.3. We give examples showing that the SA-extending condition is not carried over to direct sums. A necessary condition for the equivalence of SA and SA-extending conditions are given in Proposition 2.12 that if a module M is distributive quasi-continuous, then M is an SA-module if and only if M is an SA-extending module. We also proved that if A and B are both SA-extending R-modules with the property $r(A)+r(B)=R$, then $A\oplus B$ is an SA-extending module. Also, it is proved that for fully invariant submodules M_i of M, $M=\oplus_{i=1}^{n} M_i$ is SA-extending module if and only if each M_i is SA-extending and M_i is $\oplus_{i\neq j}^{n} M_j$-injective for all $i=1,2,\dots,n$.

2. RESULTS

We begin the following examples which show that SA-extending modules are proper generalization of SA-modules.

Example 2.1. Let $\boldsymbol{R}=\begin{bmatrix}\mathbb{Z}_4 & 2\mathbb{Z}_4\\ \mathbf{0} & \mathbb{Z}_4\end{bmatrix}$. Then all idempotents of $\boldsymbol{R_R}$ are $\boldsymbol{e_1}=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{bmatrix}$, $\boldsymbol{e_2}=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}\end{bmatrix}$, $\boldsymbol{e_3}=\begin{bmatrix}\mathbf{0}&\mathbf{2}\\ \mathbf{0}&\mathbf{1}\end{bmatrix}$, $\boldsymbol{e_4}=\begin{bmatrix}\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{bmatrix}$, $\boldsymbol{e_5}=\begin{bmatrix}\mathbf{1}&\mathbf{2}\\ \mathbf{0}&\mathbf{0}\end{bmatrix}$, $\boldsymbol{e_6}=\begin{bmatrix}\mathbf{1}&\mathbf{0}\\ \mathbf{0}&\mathbf{1}\end{bmatrix}$ and hence all direct summands of $\boldsymbol{R_R}$ are $\boldsymbol{e_1R}=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}\end{bmatrix}$, $\boldsymbol{e_2R}=\begin{bmatrix}\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbb{Z}_4\end{bmatrix}$, $\boldsymbol{e_3R}=\left\{\begin{bmatrix}\mathbf{0}&\boldsymbol{2c}\\ \mathbf{0}&\boldsymbol{c}\end{bmatrix}:\boldsymbol{c}\in\mathbb{Z}_4\right\}$, $\boldsymbol{e_4R}=\begin{bmatrix}\mathbb{Z}_4&2\mathbb{Z}_4\\ \mathbf{0}&\mathbf{0}\end{bmatrix}$ and $\boldsymbol{R}$. Then we have $\boldsymbol{R_R}=\boldsymbol{e_2R}\oplus\boldsymbol{e_4R}=\boldsymbol{e_3R}\oplus\boldsymbol{e_4R}$. All complements of $\boldsymbol{e_2R}$ are $\boldsymbol{e_4R}$ and $\left\{\begin{bmatrix}\boldsymbol{2a}&\boldsymbol{2b}\\ \mathbf{0}&\boldsymbol{2b}\end{bmatrix}:\boldsymbol{a},\boldsymbol{b}\in\mathbb{Z}_4\right\}$, all complements of $\boldsymbol{e_3R}$ are $\boldsymbol{e_4R}$ and $\left\{\begin{bmatrix}\boldsymbol{2a}&\boldsymbol{2b}\\ \mathbf{0}&\boldsymbol{2b}\end{bmatrix}:\boldsymbol{a},\boldsymbol{b}\in\mathbb{Z}_4\right\}$ and all complements of $\boldsymbol{e_4R}$ are $\boldsymbol{e_2R},\boldsymbol{e_3R}$ and $\left\{\begin{bmatrix}\mathbf{0}&\boldsymbol{2b}\\ \mathbf{0}&\boldsymbol{2b}\end{bmatrix}:\boldsymbol{b}\in\mathbb{Z}_4\right\}$. Since $\boldsymbol{e_2R}\oplus\left\{\begin{bmatrix}\boldsymbol{2a}&\boldsymbol{2b}\\ \mathbf{0}&\boldsymbol{2b}\end{bmatrix}:\boldsymbol{a},\boldsymbol{b}\in\mathbb{Z}_4\right\}\neq\boldsymbol{R_R}$, then $\boldsymbol{R_R}$ is not an $\boldsymbol{Ads}$ module. However, since $\boldsymbol{e_2R}\cap\boldsymbol{e_3R}$ is not a direct summand of $\boldsymbol{R_R}$, $\boldsymbol{R_R}$ is not an $\boldsymbol{SIP}$ module. But $\boldsymbol{e_2R}\cap\boldsymbol{e_3R}\leq_e\boldsymbol{e_2R}$, then $\boldsymbol{R_R}$ is an $\boldsymbol{SIP}$-extending module.

The following example appears in [14, Example1.5].

Example 2.2. Let $\boldsymbol{T}$ the following matrix ring over a field $\boldsymbol{F}$

$$\boldsymbol{T}=\left\{\begin{bmatrix}\boldsymbol{a}&\boldsymbol{b}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\boldsymbol{c}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\boldsymbol{c}&\boldsymbol{d}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\boldsymbol{a}\end{bmatrix}:\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\in\boldsymbol{F}\right\}.$$

Then all idempotent elements of **T** are the followings:

$$0, I_{4\times4}, e=\begin{bmatrix}0&x&0&0\\0&1&0&0\\0&0&1&y\\0&0&0&0\end{bmatrix}\ and\ f=\begin{bmatrix}1&x&0&0\\0&0&0&0\\0&0&0&y\\0&0&0&1\end{bmatrix}$$

for some $\boldsymbol{x},\boldsymbol{y}\in\boldsymbol{F}$. All complements of the direct summand $\boldsymbol{eT}$ are $\boldsymbol{I}=\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{bmatrix}\boldsymbol{T}$ and

$J=\begin{bmatrix}\mathbf{1}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}\end{bmatrix}\boldsymbol{T}$. On the other hand, all complements of the direct summand $\boldsymbol{fT}$ are

$$\boldsymbol{K}=\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\boldsymbol{T}\text{ and }\boldsymbol{L}=\begin{bmatrix}\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\boldsymbol{T}.$$

Also

$$\boldsymbol{T}=\boldsymbol{eT}\oplus\boldsymbol{I}=\boldsymbol{eT}\oplus\boldsymbol{J}=\boldsymbol{fT}\oplus\boldsymbol{K}=\boldsymbol{fT}\oplus\boldsymbol{L}.$$

Hence $\boldsymbol{T}$ is an $\boldsymbol{Ads}$ module. However, if we take

$$\boldsymbol{g}^{2}=\boldsymbol{g}=\begin{bmatrix}\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix}\text{ and }\boldsymbol{h}^{2}=\boldsymbol{h}=\begin{bmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{bmatrix},$$

then $\boldsymbol{gT}\cap\boldsymbol{hT}$ is not a direct summand of $\boldsymbol{T}$. So $\boldsymbol{T}$ is not an $\boldsymbol{SIP}$ module. But routine calculations show that $\boldsymbol{T}$ is an $\boldsymbol{SIP}$-extending module.

The following theorem is useful for checking whether a module is SA-extending module. The first part of this theorem was proved by Takıl Mutlu [6] and the second part of its was proved by Karabacak and Tercan [10].

Theorem 2.3. A module M is an SA-extending module if and only if the following statements are satisfied:
(i) for any decomposition $M = A\oplus B$, any complement C of A in M and the projection map $\pi: M\to B$, the restricted map $\pi_{|C}: C\to B$ is an isomorphism.
(ii) for any two direct summands M_1 and M_2, if $\pi: M\to M_1$ is the projection map, then the kernel of the restricted map $\pi_{|M_2}: M_2\to M_1$ is essential in a direct summand of M.

It is well known that both of the extending property and the SA property are inherited by direct summands (see [6, Lemma 2.7]). These results bring to mind the following natural question:

Is a direct summand of an SA-extending modules again an SA-extending module?

In [10], the authors gave a positive answer to this question under the condition that the direct summand is the unique closure for each of its essential submodules. So we can directly give the following proposition.

Proposition 2.4. Let M be an SA-extending module. If a direct summand K of M is the unique closure in M for each of its essential submodules, then K is also an SA-extending module.

The following example shows that the direct sum of two SA-extending modules is not an SA-extending module in general.

Example 2.5. (i) Let $M=\mathbb{Z}$ as a right $\mathbb{Z}$-module. Since M is indecomposable, it is an SA-extending module. But, M is not quasi-injective and hence $M\oplus M$ is not an SA-extending module.
(ii) Consider $M=\mathbb{Z}_2\oplus\mathbb{Z}_8$ as a right $\mathbb{Z}$-module. It is clear that $\mathbb{Z}_2$ and $\mathbb{Z}_8$ are uniform right $\mathbb{Z}$-modules and hence both of them are SA-extending modules. All direct decomposition of M are

$$\begin{aligned}M &= <(\bar{1},\bar{0})>\oplus<(\bar{0},\bar{1})>=<(\bar{1},\bar{0})>\oplus<(\bar{1},\bar{1})>\\ &= <(\bar{1},\bar{1})>\oplus<(\bar{1},\bar{4})>=<(\bar{0},\bar{1})>\oplus<(\bar{1},\bar{4})>.\end{aligned}$$

It can be shown that $<(\bar{1},\bar{2})>$ is one of the complement of $<(\bar{1},\bar{0})>$. But $<(\bar{1},\bar{2})>$ is not a direct summand of M. Hence M is not an SA-extending module.

(iii) Let K be a field and $R = \begin{bmatrix} K & K \\ 0 & K \end{bmatrix}$. Then $N = \begin{bmatrix} 0 & K \\ 0 & K \end{bmatrix}$ and $L = \begin{bmatrix} K & K \\ 0 & 0 \end{bmatrix}$ are right R-modules. Let M=R/L and $U = M \oplus N$. Although U is not Ads module from [6, Example 2.3], it is not an SIP-extending module from [14, Example 13]. Hence U is not an SA-extending module. On the other hand, since $M = R/L$ is a field and N is indecomposable, M and N are both SA-extending.

We give a condition under which the direct sum of SA-extending modules is an SA-extending module.

Theorem 2.6. Let M and N be two SA-extending modules such that $r(M) + r(N) = R$. Then $M \oplus N$ is an SA-extending module.

Proof. $M \oplus N$ is an Ads module by [6, Theorem 2.10]. We show that $M \oplus N$ is an SIP-extending module. Let A and B two direct summands of $M \oplus N$. By [13, Proposition 3.9], $A = M_1 \oplus N_1$ and $B = M_2 \oplus N_2$, where M_1 and M_2 are submodules of M, N_1 and N_2 are submodules of N. Since A and B are direct summands of M, there exists two submodules, namely $A'$and B', such that $M \oplus N = A \oplus A' = B \oplus B'$. By the modular law

$$\begin{aligned} M = M \cap (M \oplus N) &= M_1 \oplus (M \cap (N_1 \oplus A')) \\ &= M_2 \oplus \big(M \cap (N_2 \oplus B')\big). \end{aligned}$$

Thus M_1 and M_2 are direct summands of M. Similarly, N_1 and N_2 are direct summands of N. By assumption, there exist a direct summand M' of M and a direct summand N' of N such that $M_1 \cap M_2 \leq_e M'$ and $N_1 \cap N_2 \leq_e N'$. Now

$$(M_1 \cap M_2) \oplus (N_1 \oplus N_2) = (M_1 \oplus N_1) \cap (M_2 \oplus N_2) = A \cap B.$$

Thus

$$A \cap B = (M_1 \cap M_2) \oplus (N_1 \cap N_2) \leq_e M' \oplus N'$$

and hence $\boldsymbol{M \oplus N}$ is an $\boldsymbol{SA}$-extending module.

Theorem 2.7. Let $M = \oplus_{i=1}^{n} M_i$ be a direct sum of fully invariant submodules M_i. Then M is an SA-extending module if and only if each M_i is an SA-extending and M_i is $\oplus_{i \neq j}^{n} M_j$-injective for all $i = 1,2,\ldots,n$.

Proof. By [15, Theorem 9] and [16, Proposition 2.16].

Recall that a ring R is called *abelian* if every idempotent of it is central. Now, we give the following result for SA-extending abelian rings.

Corollary 2.8. Let M_R be an SA-extending module with the abelian endomorphism ring $End(M_R)$. Then any direct summand of M is an SA-extending module.

Proof. Let M_R be an SA-extending module and N a direct summand of M_R. From the assumption that $End(M_R)$ is abelian, N is a fully invariant direct summand of M_R. By Theorem 2.7, N is an SA-extending module.

Proposition 2.9. An abelian ring R is an SA-extending module if and only if the polynomial ring over R, $R[x]$ is an SA-extending module.

Proof. Assume that R is an SA-extending module . Then every idempotent of $R[x]$ is in R by [17, Lemma 8]. Then $R[x]$ is an SA-extending module. The converse is clear.

Definition 2.10. A right R-module M is called a *multiplication module* if for each submodule N of M, $N = MI$ for some ideal I in R.

Corollary 2.11. Let M be an SA-extending module and N a direct summand of M. Then N is an SA-extending module if M satisfies one of the following conditions:

(i) $M_R = R_R$ and R is an abelian ring.

(ii) M is a multiplication module and R is a commutative ring.

Proof. (i) It is clear from Corollary 2.8.
(ii) Note that in a multiplication module, every submodule is fully invariant. Then, Theorem 2.7. yields the result.

Recall that a module M is called a UC*-module* if every submodule has a unique closure in M (see, [18]).

Proposition 2.12. Let M be a distributive quasi-continuous module. Then the following conditions are equivalent.
(i) M is a UC-module.
(ii) M is an SA-module.
(iii) M is an SA-extending module.

Proof. $(i) \Leftrightarrow (ii)$ This implication follows from [6, Theorem 2.20].
$(ii) \Rightarrow (iii)$ It is obvious.
$(iii) \Rightarrow (i)$ Let M be an SA-extending module and N a submodule of M. Suppose that $K \leq M$ and $L \leq M$ such that $N \leq_e K \leq_c M$ and $N \leq_e L \leq_c M$. Then there exist submodules $K'$and L' of M such that $M = K \oplus K' = L \oplus L'$. By assumption, there exist submodules T and $T'$of M such that $K \cap L \leq_e T$ and $M = T \oplus T'$. Since $N \leq_e K \cap L \leq_e T$, $N \cap T' \leq_e T \cap T' = 0$ and hence $N \cap T' = 0$. Therefore, $K \cap T' = 0$ and $L \cap T' = 0$. Thus

$$K = K \cap M = K \cap (T \oplus T') = K \cap T,$$
$$L = L \cap M = L \cap (T \oplus T') = L \cap T$$

and hence $K \leq T$ and $L \leq T$. Now, $N \leq_e K, N \leq_e L$, and $N \leq_e T$ imply $K \leq_e T$ and $L \leq_e T$. Since K and L are direct summands of M, $K = T = L$, as desired.

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CONFLICT OF INTEREST

There are no conflicts of interest regarding the publication of this article.

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