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# A simple proof for Kazmi et al.'s iterative scheme

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**Abstract**

In this paper, a simple proof for the existence iterative scheme using two Hilbert spaces due to Kazmi et al. [K.R. Kazmi, R. Ali, M. Furkan, Hybrid iterative method for split monotone variational inclusion problem and hierarchical fixed point problem for a finite family of nonexpansive mappings, Numer. Algor., 2017] is provided.

*Keywords:* Weak convergence; Strongly convergence; Hilbert space.
*2010 MSC:* 47H09, 47H10

## 1. Introduction

To see the definitions of maximal monotone operators and $\alpha$-inverse strongly monotone mappings one can refer to for example [1, 2, 3, 4, 6, 7, 8]. The following theorem have been proved in [5, Theorem 3.1 ].

**Theorem 1.1.** *[5, Theorem 3.1] Let $H_1$ and $H_2$ be two real Hilbert spaces and $C \subseteq H_1$, $Q \subseteq H_2$ two nonempty, closed and convex sets, $A : H_1 \to H_2$ a bounded linear operator with its adjoint operator $A^*$, $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ two multi-valued maximal monotone operators, $f : C \to H_1$ and $g : Q \to H_2$ two $\theta_1$- and $\theta_2$-inverse strongly monotone mappings, respectively, $S : C \to H_1$ a nonexpansive mapping, $\{T_i\}_{i=0}^{\mathbb{N}} : C \to C$ a finite family of nonexpansive mappings, and $W_n$ a $W$-mapping generated by $T_1, \cdots, T_N$ and $\lambda_{n,1}, \cdots, \lambda_{n,N}$ for every $n \in \mathbb{N} \cup \{0\}$. Assume that $\Gamma = \Omega \bigcap \Phi \neq \emptyset$. Suppose that the iterative*

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*sequences* $\{u_n\}$, $\{y_n\}$ *and* $\{x_n\}$ *are generated by the following hybrid iterative algorithm:*

$$\begin{aligned}
&x_0 \in C, \quad C_0 = C\\
&u_n = (1-\alpha_n)x_n + \alpha_n P_C(\sigma_n S x_n + (1-\sigma_n)W_n x_n);\\
&z_n = U(u_n); \quad w_n = V(Az_n); \quad y_n = z_n + \gamma A^*(w_n - Az_n);\\
&C_n = \{z \in C : \|y_n - z\|^2 \le (1-\alpha_n\sigma_n)\|x_n - z\|^2 + \alpha_n\sigma_n\|Sx_n - z\|^2\};\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\};\\
&x_{n+1} = P_{C_n \bigcap Q_n} x_0, n \ge 0.
\end{aligned} \tag{1.1}$$

*where* $U := J_\lambda^{M_1}(I-\lambda f)$, $V := J_\lambda^{M_2}(I-\lambda g)$, $A(Range(U)) \subseteq Q$, $\gamma \in (0, \frac{1}{\|A\|^2})$. *Let* $\{\lambda_{n,i}\}_{i=1}^N$ *be a sequence in* $[0,1]$ *such that* $\lambda_{n,i} \to \lambda_i$, $(i = 1, 2, \cdots, \mathbb{N})$, $\lambda \in (0, \alpha)$ *with* $\alpha = 2\min\{\theta_1, \theta_2\}$, *and* $\{\alpha_n\}$, $\{\sigma_n\}$ *two real sequences in* $(0,1)$ *satisfying the conditions:*

*(i)* $\lim_{n\to\infty} \sigma_n = 0$,

*(ii)* $\lim_{n\to\infty} \dfrac{\|x_n - u_n\|}{\alpha_n\sigma_n} = 0$.

*Then* $\{x_n\}$ *converges strongly to* $z \in \Gamma$, *where* $z = P_\Gamma x_0$.

In this paper, some simple proof is introduced for the existence of the above Theorem.

## 2. A simple proof for Kazmi et al.'s iterative scheme

The following relations between the relations (3.25) in [5], i.e,

$$\lim_{n\to\infty} \|fu_n - fp\| = 0$$

and (3.26) in [5], i.e,

$$\|y_n - z_n\|^2 \le L_1\|x_n - y_n\| + 2\gamma K_1\|w_n - Az_n\| + \alpha_n\sigma_n K,$$

have been proved to prove the relation (3.27) i.e.:

$$\lim_{n\to\infty} \|y_n - z_n\| = 0,$$

as follows:

"Since

$$\begin{aligned}
\|y_n - p\|^2 =& \|z_n + \gamma A^*(w_n - Az_n) - p\|^2\\
=& \langle z_n + \gamma A^*(w_n - Az_n) - p, y_n - p\rangle\\
=& \frac{1}{2}\Big[\|(z_n - p) + \gamma A^*(w_n - Az_n)\|^2 + \|y_n - p\|^2 + \|(z_n - y_n)\\
&+ \gamma A^*(w_n - Az_n)\|^2\Big]\\
&\vdots\\
&- Az_n\| - \|y_n - z_n\|^2 - \|\gamma A^*(w_n - Az_n)\|^2 - 2\gamma\langle y_n - z_n, A^*(w_n - Az_n)\rangle\Big],
\end{aligned}$$

which in turn yields

$$\begin{aligned}
\|y_n - p\|^2 \le& \|z_n - p\|^2 - \|y_n - z_n\|^2 + 2\gamma\|Az_n - Ap\|\|w_n - Az_n\|\\
&+ 2\gamma\|y_n - z_n\|\|A^*\|\|w_n - Az_n\|\\
\le& \|z_n - p\|^2 - \|y_n - z_n\|^2 + 2\gamma\|w_n - Az_n\|(\|Az_n - Ap\| + \|A^*\|\|y_n - z_n\|),
\end{aligned}$$

and this together with (3.3) and (3.5) implies that

$$\begin{aligned}\|y_n - z_n\|^2 \leq& \|z_n - p\|^2 - \|y_n - p\|^2 + 2\gamma\|Az_n - Ap\|\|w_n - Az_n\|\\ &\vdots\\ \leq& L_1\|x_n - y_n\| + 2\gamma K_1\|w_n - Az_n\| + \alpha_n\sigma_n K,\text{"}\end{aligned}$$

Now, a simple proof to prove the relation (3.27) i.e.,

$$\lim_{n\to\infty} \|y_n - z_n\| = 0,$$

instead of the above relations is proved in the following remark:

**Remark 1.** ***Simple proof:***
*Using the relation $y_n = z_n + \gamma A^*(w_n - Az_n)$ in the algorithm (3.1) in [5, Theorem 3.1], obviously, it is concluded that*

$$\|y_n - z_n\| \leq \gamma\|A^*\|\|w_n - Az_n\|, \tag{2.1}$$

*then from the relation (3.21) in [5] i.e., $\lim_{n\to\infty} \|w_n - Az_n\| = 0$, it is implied that $\lim\limits_{n\to\infty} \|y_n - z_n\| = 0$.*

Also the following relations between the relations (3.42) and (3.43) in [5] have been proved:

$$\begin{aligned}&\big|\big\langle Sx_n, x - \frac{u_n - x_n}{\alpha_n} - x_n\big\rangle - \langle Sx^*, x - x^*\rangle\big|\\ &=\big|\big\langle Sx_n, x - \frac{u_n - x_n}{\alpha_n} - x_n\big\rangle - \langle Sx^*, x - \frac{u_n - x_n}{\alpha_n} - x_n\rangle\\ &\quad + \langle Sx^*, x - \frac{u_n - x_n}{\alpha_n} - x_n\rangle - \langle Sx^*, x - x^*\big\rangle\big|\\ &\leq\big|\big\langle Sx_n - Sx^*, x - \frac{u_n - x_n}{\alpha_n} - x_n\big\rangle\big|\\ &\quad + \big|\big\langle Sx^*, x^* - \frac{u_n - x_n}{\alpha_n} - x_n\big\rangle\big|\\ &\leq\|Sx_n - Sx^*\|\big\|x - \frac{u_n - x_n}{\alpha_n} - x_n\big\| + \|Sx^*\|\big\|x^* - \frac{u_n - x_n}{\alpha_n} - x_n\big\|.\end{aligned} \tag{2.2}$$

**Remark 2.** *Note that the weak convergence*

$$\frac{\|u_n - x_n\|}{\alpha_n} + x_n \rightharpoonup x^*, \tag{2.3}$$

*have been claimed in [5](see page 15, line 8), but this is not valid since a real number can't be added with a member of a Hilbert space in general.*

**Remark 3.** *In [5, line 8 in page 15], $\frac{u_n - x_n}{\alpha_n} + x_n \rightharpoonup x^*$ must be replaced instead of the conclusion $\frac{\|u_n - x_n\|}{\alpha_n} + x_n \rightharpoonup x^*$. Indeed, from the fact that $x_n \rightharpoonup x^*$ ( [5, line 10 page 14]) and $\lim\limits_{n\to\infty} \frac{\|u_n - x_n\|}{\alpha_n} = 0$ ( [5, line 6 page 15]), it is implied that*

$$\begin{aligned}\lim_{n\to\infty} \langle\frac{u_n - x_n}{\alpha_n} + x_n - x^*, y\rangle &= \lim_{n\to\infty} \langle\frac{u_n - x_n}{\alpha_n}, y\rangle + \lim_{n\to\infty} \langle x_n - x^*, y\rangle\\ &\leq \lim_{n\to\infty} \frac{\|u_n - x_n\|}{\alpha_n}\|y\| + \lim_{n\to\infty} \langle x_n - x^*, y\rangle = 0\end{aligned}$$

*for each $y \in H$ ($H$ is a real Hilbert space). Now, line 3 in page 16 in [5] in the above of the equation (3.43) should be changed by:*

$$\leq \|Sx_n - Sx^*\|\|x - \frac{u_n - x_n}{\alpha_n} - x_n\| + |\langle Sx^*, x^* - \frac{u_n - x_n}{\alpha_n} - x_n\rangle|.$$

**Remark 4.** *Note that the authors have taken limit on n on both side of the equation* (2.2) *to get the equation (3.43) in [5] as follows:*

$$\lim_{n\to\infty}\big\langle Sx_n, x-\frac{u_n-x_n}{\alpha_n}-x_n\big\rangle=\langle Sx^*, x-x^*\rangle, \tag{2.4}$$

*this means that the strong convergence* $x^*-\frac{u_n-x_n}{\alpha_n}-x_n$ *have been used instead of the weak convergence. But if* $x^*-\frac{u_n-x_n}{\alpha_n}-x_n$ *converges strongly to* 0*, since moreover,* $\lim_{n\to\infty}\frac{\|u_n-x_n\|}{\alpha_n}=0$ *[5, line 6 in page 15], it is implied that*

$$\lim_{n\to\infty}\|x^*-x_n\|\le\lim_{n\to\infty}\|x^*-\frac{u_n-x_n}{\alpha_n}-x_n\|+\lim_{n\to\infty}\frac{\|u_n-x_n\|}{\alpha_n}=0,$$

*then* $\{x_n\}$ *converges strongly to* $x^*$ *while the aim of Theorem 3.1 is to prove that* $\{x_n\}$ *converges strongly to* $x^*$ *in the step VI. Hence this is a scientific error in the article.*

Now, the following proof instead of (2.2) is given in the following remark:

**Remark 5.**

$$\begin{aligned}
&\big|\big\langle Sx_n, x-\frac{u_n-x_n}{\alpha_n}-x_n\big\rangle-\langle Sx^*, x-x^*\rangle\big|\\
=&\big|\big\langle Sx_n, x-\frac{u_n-x_n}{\alpha_n}-x_n\big\rangle-\langle Sx^*, x-\frac{u_n-x_n}{\alpha_n}-x_n\rangle\\
&+\langle Sx^*, x-\frac{u_n-x_n}{\alpha_n}-x_n\rangle-\langle Sx^*, x-x^*\rangle\big|\\
\le&\big|\big\langle Sx_n-Sx^*, x-\frac{u_n-x_n}{\alpha_n}-x_n\big\rangle\big|\\
&+\big|\big\langle Sx^*, x^*-\frac{u_n-x_n}{\alpha_n}-x_n\big\rangle\big|\\
\le&\|Sx_n-Sx^*\|\|x-\frac{u_n-x_n}{\alpha_n}-x_n\|+|\langle Sx^*, x^*-\frac{u_n-x_n}{\alpha_n}-x_n\rangle|.
\end{aligned}\tag{2.5}$$

*then from the weak convergence of* $x^*-\frac{u_n-x_n}{\alpha_n}-x_n$*, we conclude the relation (3.43) in [5].*

**Remark 6.** *Another problem in theorem 3.1 in [5] is that the authors have used from the continuity of* $S$ *from the weak topology to the norm topology in [5](see page 16 line 4) while this can not be valid in general. For example, let* $H=L^2(\mathbb{R})$ *equipped with the standard inner product. In* $L^2(\mathbb{R})$*, the strong convergence and the weak convergence is not equivalent. Indeed, define a sequence* $\{f_n\}$ *by* $f_n(x)=\chi_{(n,n+1)}(x)$ *where* $\chi$ *is the characteristic function. Then one can check that* $\{f_n\}$ *converges weakly, but not strongly, to zero in* $L^2(\mathbb{R})$*. Suppose* $S:L^2(\mathbb{R})\to L^2(\mathbb{R})$ *be the identity mapping that is nonexpansive, too. But, if* $S$ *is continuous from the weak topology to the norm topology , then the strong convergence and the weak convergence in* $L^2(\mathbb{R})$ *are equivalent which is a contradiction. Then we should to consider* $S$ *as a continuous mapping from the weak topology to the norm topology in theorem 3.1.*

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