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An Analytical Solution to Conformable Fractional Fokker-Planck Equation

Conformable Anlamında Kesirli Mertebeden Fokker-Planck Denkleminin Analitik Çözümü

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Abstract

It is the first time in this work that a newly defined conformable Laplace decomposition method (CLDM) is applied to nonlinear fractional Fokker Planck (FP) equation which has a major role in statistical physics. This new algorithm combines conformable Laplace transform and Adomian decomposition method. First, some basic theorems and definitions of fractional derivative are given in conformable sense. Then, the general algorithm of CLDM is presented. After that, the presented method is supported by numerical example by the aid of figures. As is seen from the numerical example, conformable Laplace decomposition method is strong, easy to use, reliable and applicable to a wide variety of fractional PDEs.

Keywords: Conformable fractional derivative, Fokker Planck equation, Laplace decomposition method

Öz

Bu çalışmada, istatistiksel fizikte önemli bir role sahip olan doğrusal olmayan kesirli Fokker Planck (FP) denklemi, yeni tanımlanmış olan conformable Laplace ayrıştırma metodu (CLAM) ile ilk kez çözülmektedir. Bu yeni algoritma, conformable Laplace dönüşümü ile Adomian ayrıştırma yöntemini birleştirmektedir. Çalışmamızda ilk olarak, kesirli türevin bazı temel tanımları ve teoremleri conformable anlamında verilmiştir. Ardından CLAM' ın genel algoritması anlatılmıştır. Bundan sonra, kullanılan metot grafikler yardımıyla sayısal örnekle verilerek desteklenmiştir. Sayısal örnekten görüldüğü üzere, conformable Laplace ayrıştırma yöntemi güçlü, güvenilir, kullanımı kolay ve kesirli mertebeden çok çeşitli kısmi türevli diferensiyel denklemlere uygulanabilir özelliklere sahiptir.

Anahtar Kelimeler: Conformable kesirli türev, Fokker Planck denklemi, Laplace ayrıştırma metodu

1. Introduction

In real life, many events can be modeled in a more accurate way in fractional analysis than classical analysis and also in fractional analysis it is possible to discuss various real world problems modeled. So fractional calculus started to get much attention lately (Podlubny 1998, Atangana et al. 2015). Some of the application areas of fractional calculus are; thermodynamics, entropy in dynamical systems, control theory of dynamical systems, mechanics, economics, motion, control problem, systems identification, signal processing, fluid flow, viscoelastic materials, polymers, diffusion problems, potential fields and many other areas of sciences (Atangana 2015, Miller and Ross 1993).

With the advances in technology, many physical and engineering problems gained an opportunity in modelling with the help of fractional differential equations. Until recently, Riemann-Liouville (RL) and Caputo were the most commonly used fractional derivative definitions for finding approximate solutions of differential equations. But some of the properties that must be in fractional derivative were not satisfied in these definitions. Some of them are as follows (Atangana 2015):

- Derivative of constant in RL does not equal to zero if the order of derivative is not a natural number.
- In order to get the Caputo derivative of a function, it is necessary to get the classical derivative of that function.
- Caputo and RL derivatives do not provide chain rule, product and quotient rules.

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- In Caputo and RL definitions, integer order derivatives do not give the same results as in classical derivatives.

Recently definition of conformable fractional derivative is presented by Khalil et al. (2014) which provides the fractional derivative properties that Caputo and RL definitions do not.

Now let us give some definitions and theorems of conformable fractional derivative that we are going to use.

Definition 1.1.

Consider a function g with $g:[0,\infty)\rightarrow\mathbb{R}$. When $t>0$, $\eta\in(0,1)$ in conformable sense,

- η-order fractional derivative belongs to g is defined as

$$D_t^{\eta}(g)(t)=\lim_{\lambda\rightarrow 0}\frac{g(t+\lambda t^{1-\eta})-g(t)}{\lambda}. \quad (1)$$

- $D_t^{\eta}(g)(0)=\lim_{t\rightarrow 0}D_t^{\eta}(g)(t)$ if g is η-differentiable in $(0,d)$ for some $d>0$ and if $\lim_{t\rightarrow 0}D_t^{\eta}(g)(t)$ exists (Khalil et al., 2014).

Theorem 1.1.

Assume s,h be η-differentiable functions at some point $t>0$ and $\eta\in(0,1]$. Then, (Khalil et al. 2014).

1. $D_t^{\eta}(hp+sk)=pD_t^{\eta}(h)+kD_t^{\eta}(s)$ for any real k,p constants.
2. $D_t^{\eta}(d)=0$ for any constant function $s(t)=d$.
3. $D_t^{\eta}(hs)=D_t^{\eta}(h)s+h\,D_t^{\eta}(s)$.
4. $D_t^{\eta}(\frac{h}{s})=\dfrac{D_t^{\eta}(h)s-hD_t^{\eta}(s)}{s^2}$.
5. $D_t^{\eta}(t^m)=mt^{m-\eta}$ for any real m.
6. $D_t^{\eta}(hog)=h'(s(t))D_t^{\eta}(s)(t)$ when h is differentiable at $s(t)$.

Definition 1.2.

Consider a function $w:[d,\infty)\rightarrow\mathbb{R}$ with $a\in\mathbb{R}$ and $0<\eta\leq 1$. Then η-order conformable for the function w, Abdeljawad (2015) defines Laplace transform as

$$\begin{aligned}\mathcal{L}_{\eta}^{d}[w(t)](s)&=\int_d^{\infty}e^{-s\frac{(t-d)^{\eta}}{\eta}}w(t)d_{\eta}(t,d)\\&=\int_d^{\infty}e^{-s\frac{(t-d)^{\eta}}{\eta}}w(t)(t-d)^{\eta-1}dt\end{aligned} \quad (2)$$

Theorem 1.2.

Consider a differentiable function $p:(r,\infty)\rightarrow\mathbb{R}$ with $r\in\mathbb{R}$ and $0<\eta\leq 1$. Then (Abdeljawad 2015)

$$\mathcal{L}_{\eta}^{r}[D_t^{\eta}(p(t))]=s\mathcal{L}_{\eta}^{r}[p(t)]-h(r). \quad (3)$$

2. Material and Methods

There are many studies in the literature to show that conformable fractional derivative definition is very useful, effective and powerful. Some of these st udies are as follows: Abdeljawad (2015) showed that conformable fractional derivative definition provides power series expansions, Laplace transform, integration by parts, chain rule. Hammad and Khalil (2014) presented the Wronskian form of linear conformable fractional differential equations. Tayyan and Sakka (2020) employed Lie analysis in searching properties of nonlinear conformable time-space fractiolanl PDEs. Eslami and Rezazadeh (2016) used first integral method in solving fractional conformable Wu-Zhang system. Awan et al. (2017) used preinvex functions on getting refinements of conformable fractional Hermite-Hadamard inequalities. Qi and Wang (2020) discussed the asymptotical stability of fractional system solution in conformable sense. Kurt et al. (2017) employed G'/G expansion and homotopy analysis method to find new solutions to fractional system of Nizhnik-Novikov-Veselov in conformable sense. Ayata and Ozkan (2020) found analytical solutions for conformable fractional Newell-Whitehead-Segel equation using conformable Laplace decomposition method. Zafar et al. (2019) explored conformable fractional mKdV equations by the help of exp(-$\phi(\tau)$) expansion method. Zhao and Luo (2017) generalized fractional derivative in conformable sense when describing real world problems. Ladrani and Cherif (2020) examined whether the damping conformable fractional equations have oscillating solutions or not. Chung (2015) studied harmonic, damped and forced oscillator problems in Newton mechanics with conformable fractional derivative. Ozkan and Kurt (2018a, 2018b) proved the existence of conformable Laplace transform and introduced conformable double Laplace transform. Özkan and Kurt (2019) also found new exact solutions for some conformable fractional PDE systems.

Due to the wide application areas of fractional analysis, many analytical and numerical methods such as homotopy perturbation method (Momani and Odibat 2007), Kudryashov method (Ray 2016), sub-equation method

(Zhang and Zhang 2011), first integral method (Eslami and Rezazadeh 2016), variational iteration method (Wu and Lee 2010) and Adomian decomposition method (Ray and Bera 2005) have been developed on the solutions of these fractional PDEs.

ADM is an effective method to solve linear-nonlinear PDE's (Rach 1984). For a better accuracy and convergence, there are so many modifications and hybrid forms of ADM. Some of which are Fourier Transform ADM (Nourazar et al. 2013), spectral ADM (Hosseini and Abbasbandy 2015), Legendre polynomials combined with ADM (Keyanpour and Mahmoudi 2012), combination of reproducing kernel method and ADM (Geng and Cui 2011). Here, CLDM we propose for conformable fractional systems is also one of those hybrid forms (Ayata and Özkan 2020). In CLDM, Laplace transform is combined with ADM in conformable sense.

Here we consider fractional FP equation in conformable sense. Andrian Daniel Fokker (1914) and Max Karl Planck (1917) are the first to use FP equation in describing the Brownian motion of particles (Risken and Frank 1989). This FP equation examines the change in the probability density function belongs to velocity and position of an article (Jordan et al. 1998). It has a central role in a large variety of areas some of which are; surface physics, neurosciences, laser physics, nonlinear hydrodynamics, psychology, plasma physics, pattern formation, etc (Frank 2004).

In this work we employ CLDM to nonlinear fractional PDEs. Let's give the CLDM algorithm in details.

Consider the following general fractional systems of PDE's in the operator form

$$D_t^{\eta} v + H(v) + P(v) = g(x,t) \quad 0 < \eta \leq 1, x > 0, t > 0 \quad (4)$$

with the initial condition

$$v(x,0) = h(x). \quad (5)$$

Here D_t^{η} represents the linear derivative operator of η -order with respect to t in conformable sense. H represents linear operator, P represents nonlinear operator and g represents the nonhomogeneous part.

When conformable Laplace transform $\mathcal{L}_{\eta}$ is implemented to both sides of Equation 4, it turns into

$$\mathcal{L}_{\eta}[D_t^{\eta} v] + \mathcal{L}_{\eta}[H(v)] + \mathcal{L}_{\eta}[P(v)] = \mathcal{L}_{\eta}[g] \quad (6)$$

Using differential property of conformable Laplace transform (Abdeljawad 2015), Equation 6 becomes

$$\mathcal{L}_{\eta}[v] = \tfrac{1}{s}(v(x,0) + \mathcal{L}_{\eta}[g]) - \tfrac{1}{s}(H(v)) - \tfrac{1}{s}(P(v)) \quad (7)$$

When inverse of the conformable Laplace transform is implemented to Equation 7, we obtain the solution we are looking for as,

$$v = \mathcal{L}_{\eta}^{-1}\left[\tfrac{1}{s}(v(x,0) + \mathcal{L}_{\eta}[g])\right] - \mathcal{L}_{\eta}^{-1}\left[\tfrac{1}{s}(H(v))\right] - \mathcal{L}_{\eta}^{-1}\left[\tfrac{1}{s}(P(v))\right] \quad (8)$$

According to the ADM, we decompose the $v(x,t)$ solution in the Equation 4, nonlinear term

$P(x,t)$ and Adomian Polynomials E_n as an infinite series given below (Rach 1984, Adomian 1990, Cherruault et al. 1992),

$$v(x,t) = \sum_{n=0}^{\infty} v_n, \quad (9)$$

$$P(v(x,t)) = \sum_{n=0}^{\infty} E_n, \quad (10)$$

$$E_n = \tfrac{1}{n!}\tfrac{d^n}{d\lambda^n}\left[P(v_0 + \sum_{i=1}^{n} \lambda^i v_i\right]\lambda = 0 \quad , \quad n = 0,1,2,... \quad (11)$$

If Equation 9., Equation 10. and Equation 11. are put in the Equation 8. and both sides are matched, the following iterative algorithm is obtained

$$v_0 = \mathcal{L}_{\eta}^{-1}\left[\tfrac{1}{s}(v(x,0) + \mathcal{L}_{\eta}[g])\right] \quad (12)$$

$$v_{n+1} = -\mathcal{L}_{\eta}^{-1}\left[\tfrac{1}{s}(H(v_n))\right] - \mathcal{L}_{\eta}^{-1}\left[\tfrac{1}{s}(P(v_n))\right. . \quad (13)$$

When the desired number of v_n components are calculated, approximate analytical solutions of $v(x,t)$ can be found from Equation 13.

3. Results and Discussion

Let us give a numerical example of fractional PDE systems to conformable Laplace decomposition method.

Consider the given non-linear conformable fractional Fokker Planck PDE below (Eslami and Taleghani 2019)

$$\tfrac{\partial^{\eta} v}{\partial t^{\eta}} = \left[-\tfrac{\partial}{\partial x}(3v - \tfrac{x}{2}) + \tfrac{\partial^2}{\partial x^2}(xv)\right]v \quad , \quad x,t > 0 \ , \ 0 < \eta \leq 1 \quad (14)$$

with the initial condition

$$v(x,0) = x. \quad (15)$$

To solve this problem above using CLDM presented and initial values given in Equation 14. and Equation 15. we obtain

$$
\begin{aligned}
v_0 &= x, \\
v_1 &= x\frac{t^{\eta}}{\eta}, \\
v_2 &= x\frac{t^{2\eta}}{\eta^2 2!}, \\
v_3 &= x\frac{t^{3\eta}}{\eta^3 3!}, \\
v_4 &= x\frac{t^{4\eta}}{\eta^4 4!}, \\
&\;\;\vdots
\end{aligned} \tag{16}
$$

Hence the solution $v(x,t)$ is obtained as follows

$$
\begin{aligned}
v(x,t) &= v_0 + v_1 + v_2 + v_3 + \cdots \\
&= x\left(1 + \tfrac{t^{\eta}}{\eta} + \tfrac{t^{2\eta}}{\eta^2 2!} + \tfrac{t^{3\eta}}{\eta^3 3!} + \tfrac{t^{4\eta}}{\eta^4 4!} + \cdots\right)
\end{aligned} \tag{17}
$$

with the closed form of

$$v(x,t) = xe^{\frac{t^{\eta}}{\eta}}. \tag{18}$$

For $\eta = 1$, the solution becomes

$$v(x,t) = xe^{t}. \tag{19}$$

that overlaps with the exact solution in (Eslami and Taleghani 2019).

In Figure 1, it is seen how CLDM solution behave for $\eta = 1$ when $0 \leq t \leq 1$ and $0 \leq x \leq 5$. In Figure 2, CLDM solution is given for different η values at $x = 1$ and $0 \leq t \leq 3$. Here it is aimed to show how the change in derivative order affects the solution. In Figure 3, the error between 10-step CLDM solution and the exact solution for $\eta = 1$ is given when $0 \leq t \leq 1$ and $0 \leq x \leq 5$. Here it is seen that CLDM gives us a very close result to the exact solution even in 10-step.

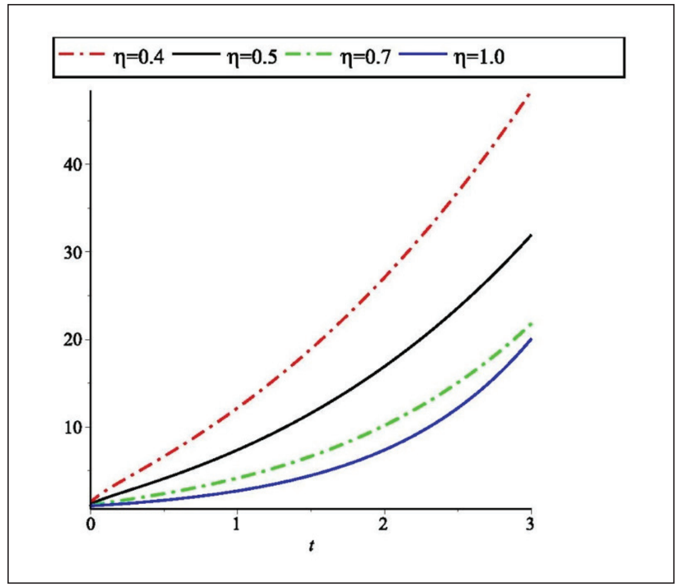

Figure 2. CLDM solution of $v(x,t)$ for different η values when $\mathbf{x = 1}$ and $\mathbf{0 \leq t \leq 3}$.

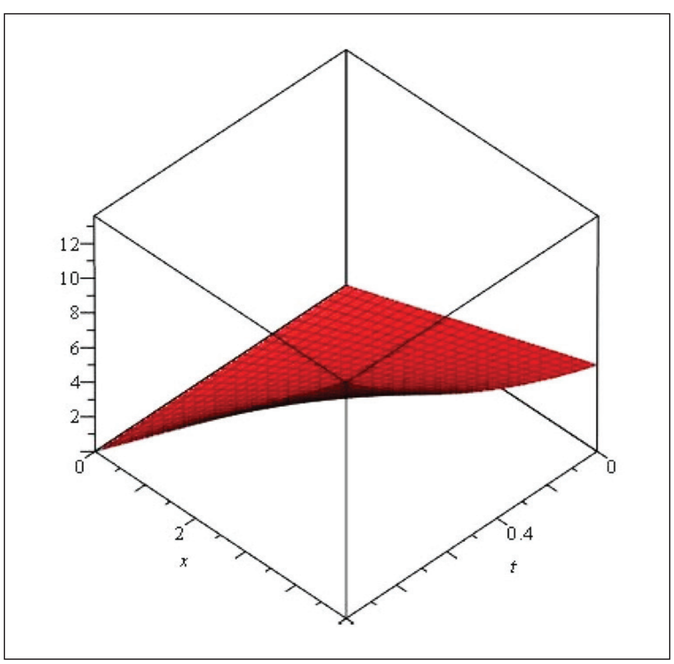

Figure 1. CLDM (exact) solution of $v(x,t)$ for $\boldsymbol{\eta = 1}$ when $\mathbf{0 \leq t \leq 1}$ and $\mathbf{0 \leq x \leq 5}$.

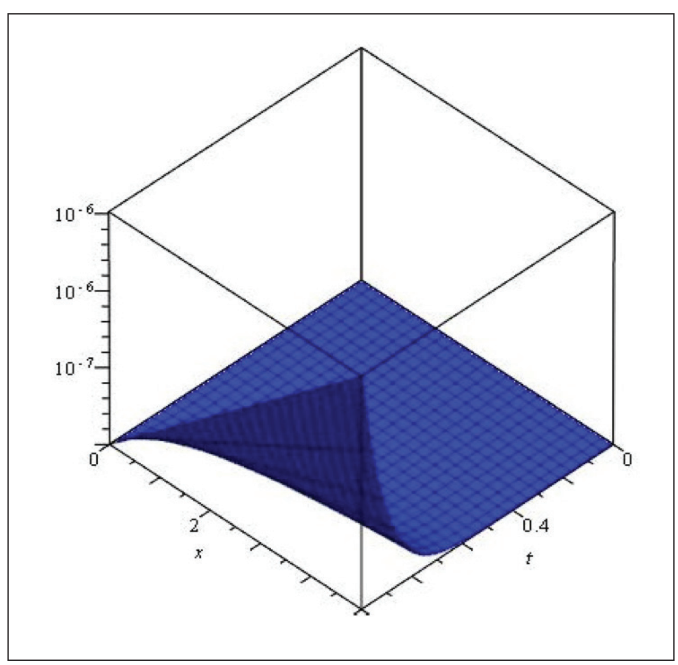

Figure 3. Error between 10-step CLDM solution and exact solution of $v(x,t)$ for $\boldsymbol{\eta = 1}$ when $\mathbf{0 \leq t \leq 1}$ and $\mathbf{0 \leq x \leq 5}$.

In this study the applicability of the new CLDM to nonlinear fractional PDEs is aimed to be demonstrated. In this new CLDM, there is no need of any restrictive assumptions and discretizations. Since the method does not need any discretization, it does not contain round of errors in calculations. As is seen from the numerical example, the solution obtained can be expressed as an infnite power series that can be expressed in closed form. As a result it can be said that CLDM is a powerful mathematical tool in solving nonlinear fractional PDEs and also it is a guide for researchers in that it offers analytical and numerical solutions to a large class of fractional PDEs.

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