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Non-local Fractional Differential Equation On The Half Line in Banach Space

Kheireddine Benia[a], El Hadi Ait Dads[b], Moustafa Beddani[a], Benaouda Hedia[c]

[a]*Department of Mathematics, Djillali Liabes University of Sidi Bel-Abbés, Sidi Bel-Abbés, Algeria.*
[b]*Department of Mathematics, LMDP, University Cadi Ayyad, Marrakech, Morocco.*
UMMISCO.UMI.209, Sorbonne University IRD, Bondy France.
[c]*Department of Mathematics, University of Tiaret Ibn Khaldoun, Tiaret, Algeria.*

Abstract

Our aim in this paper is to study the existence of solution sets and its topological structure for non-local fractional differential equations on the half-line in a Banach space using Riemann-Liouville definition. The main result is based on Meir-Keeler fixed point theorem for condensing operators combined with measure of non-compactness. An example is given to illustrate the feasibility of our main result.

Keywords: Nonlocal boundary value problem, measure of non-compactness, unbounded domain, Banach space, fixed point theorem, Riemann-Liouville fractional derivative, Meir-Keeler condensing operator, fractional differential equations.
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1. Introduction

Fractional calculus can be seen as a generalization of the ordinary differentiation and integration to arbitrary non integer order, and has been recognized as one of the most powerful tools to describe long memory processes in the last decades. For a long time, the theory of fractional Calculus developed only as a pure theoretical field of mathematics. However, in the last decades, it was found that fractional derivatives and integrals provide, in some situations, a better tool to understand some physical phenomena, especially when

Email addresses: `kheireddine.benia@univ-tiaret.dz` (Kheireddine Benia), `aitdads@uca.ac.ma` (El Hadi Ait Dads), `m.beddani@univ-chlef.dz` (Moustafa Beddani), `b_hedia@univ-tiaret.dz` (Benaouda Hedia)

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dealing with processes with memory[1]. Applications include modeling viscoelastic and viscoplastic materials [20], chemical processes [24], and a wide range of engineering problems. Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc., involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives, see [2, 3, 5, 13, 25]and [4, 6, 9], fractional differential equations with non local conditions have been discussed in [16] and references therein. Non-local conditions were initiated by Byszewaski [15] where he proved the existence and uniqueness of mild and classical solutions of non-local Cauchy problems. As remarked by Byszewaski [14] and [21] the non-local conditions can be more useful than the standard condition to describe some physical phenomena.

Very recently, many research papers have appeared concerning the fractional differential equations in Banach spaces, some of them investigated the existence results of solutions on finite intervals and unbounded domain by classical tools from functional analysis and measure of non compactness see, for example the following references: [8, 11, 12, 19].

In this paper we deal with the existence of solution sets and its topological structure for fractional differential equations on unbounded domain with the non-local conditions. We consider the following non-local boundary-value problem

$$^{\mathcal{RL}}\mathcal{D}^{\alpha}_{0+}y(t) = f(t, y(t)),\ t \in J = (0, +\infty), \tag{1}$$

$$\mathcal{I}^{2-\alpha}_{0^+}y(0^+) = \sum_{i=1}^{m} \lambda_i y(\tau_i), \tag{2}$$

$$^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+}y(\infty) = y_\infty, \tag{3}$$

where $^{\mathcal{RL}}\mathcal{D}^{\alpha}_{0+}$ denotes the Riemann-Liouville fractional derivative of order α, $1 < \alpha \leq 2$. The operator $\mathcal{I}^{2-\alpha}_{0^+}$ denotes the Riemann-Liouville fractional integral, the state $y(\cdot)$ takes values in a Banach space E, $f : (0, \infty) \times E \to E$ will be specified in section 3. τ_i, $i = 1, 2, \ldots, m$ are pre-fixed points satisfying $0 < \tau_1 \leq \cdots \leq \tau_m$, $\lambda_i \in \mathbb{R}^*_+$ and

$$\Gamma(\alpha - 1) \neq \sum_{i=1}^{m} \lambda_i \tau_i^{\alpha-2},\ \text{where}\ \Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}e^{-t}dt. \tag{4}$$

The condition (4) is used to define in section 3 a technical quantity. We can interpret this condition later in terms of non existence result or local blow-up once it is close to zero. Our starting point will be the property concerning Riemman-Liouville fractional derivative,

$$[\mathcal{I}^{\alpha} \circ D^{\alpha}_{0^+}y](t) = y(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \lim_{t\to 0^+} (\mathcal{I}^{1-\alpha}y)(t) - \frac{t^{\alpha-2}}{\Gamma(\alpha-1)} \lim_{t\to 0^+} (\mathcal{I}^{2-\alpha}y)(t).$$

In this situation $\lim_{t\to 0^+} (\mathcal{I}^{1-\alpha}y)(t)$, $\lim_{t\to 0^+} (\mathcal{I}^{2-\alpha}y)(t)$ take the place of initial data values and are null once the state y is continuous on the domain J. Our thinking is focused on the continuity of the state y only on J' and the existence of the above values without nullity, hence this hardness combined with the unboundedness of the domain imposes us a choice of a special Banach space that will be specified later. We show that this constructed space is in a natural way, in the sense that, one recovers the characterization of the relatively compact subset in the space $C(J, E)$ when J is compact.

This paper is organized in the following way. In Section 2 we give some preliminaries and general results, in Section 3 we present the existence results for the problem (1)-(3), by using the fixed point theorem for Meir-Keeler condensing operators via measure of non-compactness. In the last section, we give an illustrative example that will be presented in Section 4.

2. Preliminary results

In this section, we introduce some notation and technical results which are used throughout this paper. Let $I \subset J = (0, \infty)$ be a compact interval and denote by $C(I, E)$ the Banach space of continuous functions $y : I \to E$ with the usual norm

$$\|y\|_\infty = \sup\{\|y(t)\|,\ t \in I\}.$$

$L^1(J, E)$ the space of E valued Bochner integrable functions on J with the norm

$$\|f\|_{L^1} = \int_0^{+\infty} \|f(t)\|dt.$$

We consider the following Banach space

$$C_\alpha([0,\infty), E) = \{y \in C((0,\infty), E) :\ \lim_{t\to 0^+} t^{2-\alpha} y(t) \text{ and } \lim_{t\to\infty} \frac{t^{2-\alpha}y(t)}{1+t^\alpha} \text{ exist and are finite}\}.$$

A norm in this space is given by

$$\|y\|_\alpha = \sup_{t\in J} \frac{t^{2-\alpha}\|y(t)\|}{1+t^\alpha}.$$

For $y \in C_\alpha([0,\infty), E)$, we define y_α by

$$y_\alpha(t) = \begin{cases} \dfrac{t^{2-\alpha}y(t)}{1+t^\alpha}, & t \in (0,\infty), \\ \lim\limits_{t\to 0} \dfrac{t^{2-\alpha}y(t)}{1+t^\alpha}, & t = 0. \end{cases}.$$

It is clear that $y_\alpha \in C([0,\infty), E)$.
We begin with some definitions from the theory of fractional calculus.

Definition 2.1 ([20]). *(i) Let Γ be the gamma function and α a non-negative real number. We recall that the fractional (arbitrary) integral of order α of a function $h \in L^1(J, E)$ is given by*

$$\mathcal{I}^\alpha_{a^+} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)ds.$$

(ii) Let $0 < \alpha < 1$. The Riemann-Liouville fractional derivative of order α of the function h is given by :

$$^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+} h(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_0^t (t-s)^{-\alpha} h(s)ds \right).$$

For the existence of solutions for the problem (1)-(3), we need the following auxiliary lemmas.

Lemma 2.2. *[20, 23] Let $\alpha > 0$ and $h \in C(J, E) \cap L^1(J, E)$. Then the differential equation*

$$^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+} h(t) = 0,$$

has as a unique solution given by

$$h(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R},\ i = 1 \ldots n$ and $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of the real number α.

Lemma 2.3. *[20, 23] Let $\alpha > 0$. Suppose that $h \in C(J, E) \cap L^1(J, E)$ with a fractional derivative of order α belonging to $C(J, E) \cap L^1(J, E)$. Then*

$$\mathcal{I}^\alpha_{0^+} {}^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+} h(t) = h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R},\ i = 0, \ldots, n,$ where $n = [\alpha] + 1$.

Remark 2.4. *For $\alpha > 0,\ k > -1$, we have*

$$\mathcal{I}_{0+}^{\alpha} t^{k} = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} t^{\alpha+k} \text{ and } {}^{\mathcal{RL}}\mathcal{D}_{0+}^{\alpha} t^{k} = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha},\ t > 0,$$

giving in particular ${}^{\mathcal{RL}}\mathcal{D}_{0+}^{\alpha} t^{\alpha-m} = 0,\ m = 1, \ldots, n$, where n is the smallest integer greater than or equal to α.

Remark 2.5. *If h is a suitable function (see for instance [20, 22]), we have the composition relations*

$${}^{\mathcal{RL}}\mathcal{D}_{0+}^{\alpha} \mathcal{I}_{0+}^{\alpha} h(t) = h(t),\ \alpha > 0$$

and

$${}^{\mathcal{RL}}\mathcal{D}_{0+}^{\alpha} \mathcal{I}_{0+}^{k} h(t) = \mathcal{I}_{0+}^{k-\alpha} h(t),\ k > \alpha > 0,\ t > 0.$$

Let us now recall the definition of the measure of non-compactness in the sense of Kuratowski and its properties. For all $G \subseteq E$, we denote by $S_b(G)$ the set of all bounded subsets of G.

Definition 2.6. *[10, 18] Let $D \in S_b(E)$. The Kuratowski measure of non-compactness γ of the subset D is defined as follows:*

$$\gamma(D) = \inf \left\{ d > 0 : D \subset \bigcup_{i=1}^{n} D_i,\ diam\ D_i \leq d \right\}.$$

Lemma 2.7. *[10, 18] Let $A, B \in S_b(E)$. The following properties hold:*
($\mathbf{i}_1$) *$\gamma(A) = 0$ if and only if A is relatively compact,*
($\mathbf{i}_2$) *$\gamma(A) = \gamma(\overline{A})$, where $\overline{A}$ denotes the closure of A,*
($\mathbf{i}_3$) *$\gamma(A + B) \leq \gamma(A) + \gamma(B)$,*
($\mathbf{i}_4$) *$A \subset B$ implies $\gamma(A) \leq \gamma(B)$,*
($\mathbf{i}_5$) *$\gamma(a.A) = \|a\|.\gamma(A)$ for all $a \in E$,*
($\mathbf{i}_6$) *$\gamma(\{a\} \cup A) = \gamma(A)$ for all $a \in E$,*
($\mathbf{i}_7$) *$\gamma(A) = \gamma(Conv(A))$, where $Conv(A)$ is the smallest convex that contains A.*

Lemma 2.8. *[17] Let $D \in S_b(E)$ and $\varepsilon > 0$. Then, there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset D$, such that*

$$\gamma(D) \leq 2\gamma(\{u_n,\ n \in \mathbb{N}\}) + \varepsilon.$$

Lemma 2.9. *[18] If D is an equicontinuous and bounded subset of $\mathcal{C}([a, b], E)$, then $\gamma(D(.)) \in \mathcal{C}([a, b], \mathbb{R}^+)$*

$$\gamma_C(D) = \max_{t\in[a,b]} \gamma(D(t)),\ \gamma\left(\left\{ \int_a^b y(t)dt : t \in D \right\}\right) \leq \int_a^b \gamma(D(t))dt,$$

where $D(t) = \{y(t) : y \in D\}$ and γ_C is the non-compactness measure on the space $\mathcal{C}([a, b], E)$.

In 1969, Meir-Keeler introduced a notion of a contraction mapping in a metric space. Most recently in 2015, the author introduced the following definition and his fixed point theorem.

Definition 2.10. *[7] Let κ be an arbitrary measure of non-compactness on E and G be a non empty subset of E. Let Δ be an operator from G to G. Δ is said Meir-Keeler condensing operator if*

$$\forall\varepsilon > 0,\ \exists k(\varepsilon) > 0,\ \forall D \in S_b(G); \varepsilon \leq \kappa(D) < \varepsilon + k(\varepsilon) \Longrightarrow \kappa(\Delta D) < \varepsilon.$$

Theorem 2.11. *[7] Let κ be an arbitrary measure of non-compactness on E and G a closed, bounded and convex subset of E. Let Δ be an operator from G to G, assume that Δ is a Meir-Keeler condensing operator and continuous, then the set $\{w \in G : \Delta(w) = w\}$ is non empty and compact.*

3. Main result

In the sequel we denote

$$T = \frac{1}{\Gamma(\alpha-1) - \sum\limits_{i=1}^{m} \lambda_i \tau_i^{\alpha-2}}.$$

Let us list some assumptions to be used later.

(H₁) There exist nonnegative continuous functions a, b such that

$$\begin{cases} \|f(t,u)\| \le a(t) + t^{2-\alpha} b(t)\|u\| \,, & \text{for all } t \in J \text{ and } \ u \in E, \\ \int_0^\infty (1+t^\alpha) b(t)dt \le \frac{\Gamma(\alpha)}{3(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}, & \\ \int_0^\infty a(t)dt \le \infty. & \end{cases}$$

(H₂) $f : (0,\infty)\times E \to E$ is a continuous function and for all x, y and $(0,b] \subset (0,\infty)$:

$$\|f(t,x) - f(t,y)\| \le \alpha t^{2-\alpha}\|x-y\|, \text{ for all } t \in (0,b],$$

with $\alpha \in \mathbb{R}^+$.

(H₃) There exists nonnegative function $\ell \in L^1(J,\mathbb{R}^+)$ such that for each non empty, bounded set $\Omega \subset C_\alpha(J,E)$

$$\begin{cases} \gamma(f(t,\Omega(t))) \le t^{2-\alpha}\ell(t)\gamma(\Omega(t)), & \text{for all } t \in J, \\ \int_0^\infty (1+t^\alpha)\ell(t)dt \le \frac{\Gamma(\alpha)}{4(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}. & \end{cases}$$

(H₄) There exists strictly positive real number R such that

$$R > \frac{\|y_\infty\| + 3\int_0^\infty a(t)dt}{\dfrac{\Gamma(\alpha)}{(1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}))} - 3\int_0^\infty (1+t^\alpha)b(t)dt}.$$

Definition 3.1. *A function $y \in \mathcal{C}_\alpha([0,+\infty))$ is said to be a solution of the problem (1)-(3) if y satisfies the equation ${}^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+}y(t) = f(t,y(t))$ and the conditions $(2)-(3)$.*

Lemma 3.2. *Let $1<\alpha<2$. A function y is a solution of the fractional integral equation*

$$\begin{aligned} y(t) = {} & \frac{y_\infty - \int_0^\infty f(s,y(s))ds}{\Gamma(\alpha)}\left[t^{\alpha-1} + T\left(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}\right)t^{\alpha-2}\right] \\ & + \frac{Tt^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - s)f(s,y(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds \end{aligned} \tag{5}$$

if and only if y is a solution of the problem

$${}^{\mathcal{RL}}\mathcal{D}^\alpha_{0^+}y(t) = f(t,y(t)), \quad t \in J = (0,+\infty), \tag{6}$$

$$\mathcal{I}^{2-\alpha}_{0^+}y(0^+) = \sum_{i=1}^{m}\lambda_i y(\tau_i), \tag{7}$$

$${}^{\mathcal{RL}}\mathcal{D}^{\alpha-1}_{0^+}y(\infty) = y_\infty. \tag{8}$$

Proof. Assume that y satisfies the problem (6)-(8). We may apply Lemma 2.3 to reduce equation (6) to an equivalent integral equation

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \mathcal{I}_{0^+}^{\alpha} f(t, y(t)), \tag{9}$$

for some $c_1, c_2 \in \mathbb{R}$. Applying $\mathcal{I}_{0^+}^{2-\alpha}$ to both sides of (9), we have

$$\mathcal{I}_{0^+}^{2-\alpha} y(t) = c_1 \mathcal{I}_{0^+}^{2-\alpha} t^{\alpha-1} + c_2 \mathcal{I}_{0^+}^{2-\alpha} t^{\alpha-2} + \mathcal{I}_{0^+}^{2-\alpha} \mathcal{I}_{0^+}^{\alpha} f(t, y(t)).$$

From Remark 2.4, we then get

$$\mathcal{I}_{0^+}^{2-\alpha} y(t) = \frac{c_1 \Gamma(\alpha)}{\Gamma(2)} t + c_2 \Gamma(\alpha - 1) + \frac{1}{\Gamma(2)} \int_0^t (t-s) f(s, y(s)) ds.$$

Taking $t \longrightarrow 0$, we obtain

$$c_2 = \frac{\mathcal{I}_{0^+}^{2-\alpha} y(0^+)}{\Gamma(\alpha-1)}.$$

Applying ${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}$ to both sides of (9), we obtain

$${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1} y(t) = c_1 {}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1} t^{\alpha-1} + c_2 {}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1} t^{\alpha-2} + {}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1} \mathcal{I}_{0^+}^{\alpha} f(t, y(t)).$$

From Remark 2.4 and Remark 2.5, we get

$${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1} y(t) = c_1 \Gamma(\alpha) + \frac{1}{\Gamma(1)} \int_0^t f(s, y(s)) ds.$$

Hence

$$c_1 = \frac{1}{\Gamma(\alpha)} \left[y_\infty - \int_0^\infty f(s, y(s))) ds \right].$$

Thus, we have

$$y(t) = \frac{1}{\Gamma(\alpha)} [y_\infty - \int_0^\infty f(s, y(s))) ds] t^{\alpha-1} + \frac{\mathcal{I}_{0^+}^{2-\alpha} y(0^+)}{\Gamma(\alpha-1)} t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds. \tag{10}$$

Next, we substitute t by τ_i into the above equation,

$$y(\tau_i) = \frac{1}{\Gamma(\alpha)} [y_\infty - \int_0^\infty f(s, y(s))) ds] \tau_i^{\alpha-1} + \frac{\mathcal{I}_{0^+}^{2-\alpha} y(0^+)}{\Gamma(\alpha-1)} \tau_i^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, y(s)) ds,$$

by multiplying both sides of the equality by λ_i, we obtain

$$\begin{aligned} \lambda_i y(\tau_i) = & \frac{1}{\Gamma(\alpha)} [y_\infty - \int_0^\infty f(s, y(s))) ds] \lambda_i \tau_i^{\alpha-1} + \frac{\mathcal{I}_{0^+}^{2-\alpha} y(0^+)}{\Gamma(\alpha-1)} \lambda_i \tau_i^{\alpha-2} \\ & + \frac{\lambda_i}{\Gamma(\alpha)} \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, y(s)) ds. \end{aligned}$$

From (2), we have

$$\begin{aligned} \mathcal{I}_{0^+}^{2-\alpha} y(0^+) = & \frac{1}{\Gamma(\alpha)} [y_\infty - \int_0^\infty f(s, y(s))) ds] \sum_{i=1}^m \lambda_i \tau_i^{\alpha-1} + \frac{\mathcal{I}_{0^+}^{2-\alpha} y(0^+)}{\Gamma(\alpha-1)} \sum_{i=1}^m \lambda_i \tau_i^{\alpha-2} \\ & + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, y(s)) ds, \end{aligned}$$

which implies

$$\mathcal{I}_{0^+}^{2-\alpha}y(0^+)=\frac{T}{\alpha-1}\left[(y_\infty-\int_0^\infty f(s,y(s)))ds)\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1}+\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\right]. \tag{11}$$

Substituting (11) into (10), we derive that (5).

Conversely, assume that y satisfies the integral equation (5). Applying $\mathcal{I}_{0^+}^{2-\alpha}$ to both sides of (5) and using Remark 2.4, we have

$$\begin{aligned}l\mathcal{I}_{0^+}^{2-\alpha}y(t)=&\left(y_\infty-\int_0^\infty f(s,y(s)))ds\right)\left(t+\frac{T}{\alpha-1}\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1}\right)\\&+\frac{T}{\alpha-1}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds+\mathcal{I}_{0^+}^{2}f(t,y(t))).\end{aligned}$$

As $t\longrightarrow 0$, we get

$$\begin{aligned}\mathcal{I}_{0^+}^{2-\alpha}y(0^+)=&\frac{T}{\alpha-1}(\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1})\left(y_\infty-\int_0^\infty f(s,y(s)))ds\right)\\&+\frac{T}{\alpha-1}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds,\end{aligned}$$

t by τ_i into (5), we have

$$\begin{aligned}y(\tau_i)=&\frac{y_\infty-\int_0^\infty f(s,y(s)))ds}{\Gamma(\alpha)}\left(\tau_i^{\alpha-1}+T(\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1})\tau_i^{\alpha-2}\right)\\&+\frac{T\tau_i^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds+\frac{1}{\Gamma(\alpha)}\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds.\end{aligned}$$

So, we derive

$$\begin{aligned}\sum_{i=1}^m\lambda_i y(\tau_i)=&\frac{y_\infty-\int_0^\infty f(s,y(s)))ds}{\Gamma(\alpha)}\left(\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1}+T(\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1})\sum_{i=1}^m\lambda_i\tau_i^{\alpha-2}\right)\\&+\frac{T\sum_{i=1}^m\lambda_i\tau_i^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\\&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\\=&\frac{(y_\infty-\int_0^\infty f(s,y(s)))ds)\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1}}{\Gamma(\alpha)}\left(1+T\sum_{i=1}^m\lambda_i\tau_i^{\alpha-2}\right)\\&+\frac{1}{\Gamma(\alpha)}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds\left(1+T\sum_{i=1}^m\lambda_i\tau_i^{\alpha-2}\right).\end{aligned}$$

Finally,

$$\begin{aligned}\sum_{i=1}^m\lambda_i y(\tau_i)=&\frac{T}{\alpha-1}(\sum_{i=1}^m\lambda_i\tau_i^{\alpha-1})\left(y_\infty-\int_0^\infty f(s,y(s)))ds\right)\\&+\frac{T}{\alpha-1}\sum_{i=1}^m\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}f(s,y(s))ds=\mathcal{I}_{0^+}^{2-\alpha}y(0^+)\end{aligned}$$

Now by applying ${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}$ to both sides of (5) and using Remark 2.4, Remark 2.5, we have

$$ {}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}y(t) = y_\infty - \int_0^\infty f(s, y(s)))ds + \mathcal{I}_{0^+}^1 f(t, y(t)). $$

Let $t \longrightarrow \infty$, then we get

$$ {}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha-1}y(\infty) = y_\infty. $$

Next, by applying ${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha}$ to both sides of (5) and using Remark 2.4, Remark 2.5, we obtain ${}^{\mathcal{RL}}\mathcal{D}_{0^+}^{\alpha}y(t) = f(t, y(t))$. Which ends the proof. □

Now, we are in a position to give the main result of this work. Let

$$ B = \{y \in \mathcal{C}_\alpha([0,\infty), E) : \|y\|_\alpha \le R\}. $$

Remark 3.3. *We can write Equation (5) in the following form,*

$$ \begin{aligned} y(t) = & \frac{y_\infty}{\Gamma(\alpha)}[t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2}] + \frac{Tt^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^m \lambda_i \int_0^{\tau_i}(\tau_i - s)^{\alpha-1} f(s, y(s))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1} + Tt^{\alpha-2}\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1} - (t-s)^{\alpha-1}] f(s, y(s))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_t^\infty (t^{\alpha-1} + Tt^{\alpha-2}\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1}) f(s, y(s))ds. \end{aligned} $$

Theorem 3.4. *Assume that conditions* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_3})$ *and* $(\mathbf{H_4})$ *are satisfied. Then, the problem (1)-(3) has at least one solution.*

Proof. Let the operator $N : \mathcal{C}_\alpha([0,\infty), E) \to \mathcal{C}_\alpha([0,\infty), E)$ be defined as

$$ \begin{aligned} N(y)(t) = & \frac{y_\infty}{\Gamma(\alpha)}[t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2}] + \frac{Tt^{\alpha-2}}{\Gamma(\alpha)}\sum_{i=1}^m \lambda_i \int_0^{\tau_i}(\tau_i - s)^{\alpha-1} f(s, y(s))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_0^t [t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2} - (t-s)^{\alpha-1}] f(s, y(s))ds \\ & - \frac{1}{\Gamma(\alpha)}\int_t^\infty (t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2}) f(s, y(s))ds. \end{aligned} $$

From the definition of the operator N and Lemma 3.2, we see that the fixed points of N are solutions of problem (1)-(3). For this reason, it suffices to verify the axioms of Theorem 2.11, which is done in four steps.
Step1: We start to prove that N is bounded.

Let $y \in C_\alpha([0,\infty), E)$, from $(\mathbf{H_1})$ it is easy to deduce that $Ny \in C_\alpha(J, E)$. Using $(\mathbf{H_1})$, for all $y \in B$ and

$t \in (0, \infty)$, we get

$$\begin{aligned}
\frac{t^{2-\alpha}\|N(y)(t)\|}{1+t^\alpha} &\leq \frac{\|y_\infty\|\left(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}\right)}{\Gamma(\alpha)} + \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\|f(s,y(s))\|ds \\
&+ \frac{2+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}}{\Gamma(\alpha)}\int_0^t\|f(s,y(s))\|ds + \frac{1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}}{\Gamma(\alpha)}\int_t^\infty\|f(s,y(s))\|ds \\
&\leq \frac{\|y_\infty\|(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)} + \frac{3|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+3}{\Gamma(\alpha)}\int_0^\infty a(t)dt \\
&+ \frac{3|T|\|y\|_\alpha\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+3}{\Gamma(\alpha)}\int_0^\infty(1+t^\alpha)b(t)dt.
\end{aligned}$$

Hence, $N : C_\alpha(J, E) \to C_\alpha(J, E)$ is bounded.
Step2: We will show that N is continuous.
Let $\{y_n\}_{n=1}^\infty \subset C_\alpha(J, E)$ and $y \in C_\alpha(J, E)$ such that $y_n \to y$ as $n \to \infty$. Then, $\{y_n\}_{n=1}^\infty$ is a bounded set of $C_\alpha(J, E)$, i.e. there exists $M > 0$ such that $\|y_n\|_\alpha \leq M$, for $n > 1$. We also have by taking the limit that $\|y\|_\alpha \leq M$. In view of condition $(\mathbf{H_1})$, for all $\varepsilon > 0$, there exists $L > \tau_m$ such that

$$\int_L^\infty a(t)dt < \frac{\Gamma(\alpha)\varepsilon}{3[4|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+4]},\ \int_L^\infty(1+t^\alpha)b(t)dt < \frac{\Gamma(\alpha)\varepsilon}{3[4|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}+4]M}, \tag{12}$$

and from $(\mathbf{H_2})$, there exists $\widetilde{N} \in \mathbb{N}$ such that, for all $n \geq \widetilde{N}$ and $t \in (0, L]$, we have

$$\|f(t, y_n(t)) - (t, y(t))\| < \frac{\Gamma(\alpha)}{3[2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})+3]L}\varepsilon. \tag{13}$$

Therefore, for all $t \in J$ and $n > \widetilde{N}$, we have

$$\begin{aligned}
\frac{t^{2-\alpha}}{1+t^\alpha}\|N(y_n)(t) - N(y)(t)\| &\leq \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\|f(s,y_n(s)) - (s,y(s))\|ds \\
&+ \frac{2+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^t\|f(s,y_n(s)) - f(s,y(s))\|ds \\
&+ \frac{1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_t^\infty\|f(s,y_n(s)) - f(s,y(s))\|ds.
\end{aligned}$$

If $t \leq L$ and $n > \widetilde{N}$, we have

$$\begin{aligned}\frac{t^{2-\alpha}}{1+t^{\alpha}}\|N(y_n)(t)-N(y)(t)\| \leq & \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}\int_0^L \|f(s,y_n(s))-f(s,y(s))\|ds \\ & + \frac{3+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^L \|f(s,y_n(s))-f(s,y(s))\|ds \\ & + \frac{2+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_L^{\infty} a(s)ds \\ & + \frac{(2+2|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}))M}{\Gamma(\alpha)}\int_L^{\infty}(1+s^{\alpha})b(s)ds.\end{aligned}$$

From (12) and (13), we obtain,

$$\frac{t^{2-\alpha}}{1+t^{\alpha}}\|N(y_n)(t)-N(y)(t)\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon.$$

The case when $t > L$ and $n > \widetilde{N}$ is treated similarly. Thus we conclude that,

$$\|y_n - y\|_{\alpha} \to 0 \text{ as } n \to \infty.$$

So, N is continuous.
Step 3: We prove the following results :
(i) $NB_{\alpha} = \{(Ny)_{\alpha} : y \in B_{\alpha}\}$ is equicontinuous on any compact $[0,d]$ of $[0,\infty)$.
(ii) For given $\varepsilon > 0$, there exists a constant $n_1 > 0$ such that

$$\|\frac{N(y)_{\alpha}(t_1)}{1+t_1^{\alpha}} - \frac{N(y)_{\alpha}(t_2)}{1+t_2^{\alpha}}\| < \varepsilon,$$

for any $t_1, t_2 \geq n_1$ and $y(.) \in B_{\alpha}$. We have, from $(\mathbf{H_1})$ and the boundedness of B, there exists $M > 0$ such that

$$\int_0^{\infty}\|f(t,y(t))\|dt \leq M \text{ for any } y \in B. \tag{14}$$

Let us show the equicontinuity of NB_{α} on any compact $[0,d]$. Indeed, let $y \in B$ and $t_1,\ t_2 \in [0,d]$, where

$t_2 > t_1$.Then

$$
\begin{aligned}
&\|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\| \\
&\le \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left(|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}| + |\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}|\right) \\
&+ \left[\frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i - s)^{\alpha-1}\|f(s,y(s))\|ds\right]\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds - \int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds\right\| \\
&\le \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| + \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|\|f(s,y(s))\|ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,y(s))\|ds \\
&\le \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| + \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|a(s)ds + \frac{R}{\Gamma(\alpha)}\int_0^{t_1}|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|(1+s^\alpha)b(s)ds \\
&+ \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}a(s)ds + \frac{R}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}(1+s^\alpha)b(s)ds \\
&\le \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| + \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
&+ \frac{a^*+b^*R}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}ds\right) + \frac{a^*+b^*R}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds \\
&+ \frac{2b^*R}{\Gamma(\alpha)}\left(\int_0^{t_2}(t_2-s)^{\alpha-1}s^\alpha ds - \int_0^{t_1}(t_1-s)^{\alpha-1}s^\alpha ds\right) \\
&\le \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| + \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
&+ \frac{a^*+b^*R}{\Gamma(1+\alpha)}(t_2^\alpha - t_1^\alpha - (t_2-t_1)^\alpha) \\
&+ \frac{a^*+b^*R}{\Gamma(1+\alpha)}(t_2-t_1)^\alpha + \frac{2b^*R\mathcal{B}(\alpha,\alpha+1)}{\Gamma(\alpha)}(t_2^{2\alpha} - t_1^{2\alpha}),
\end{aligned}
$$

where $a^* = \max\limits_{t\in[a,b]} a(t)$ and $b^* = \max\limits_{t\in[a,b]} b(t)$. As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero. Then $\dfrac{t^{2-\alpha}N(B)(t)}{1+t^\alpha}$ is equicontinuous on $[0,d]$.

Next, let us show the equiconvergence of NB_α. In fact, let $\varepsilon > 0$, we have

$$\begin{aligned}
&\|\frac{t_1^{2-\alpha}N(y)(t_1)}{1+t_1^\alpha} - \frac{t_2^{2-\alpha}N(y)(t_2)}{1+t_2^\alpha}\| \\
&\leq \frac{\|y_\infty\|+M}{\Gamma(\alpha)}\left|\frac{t_1}{1+t_1^\alpha} - \frac{t_2}{1+t_2^\alpha}\right| \\
&+ \frac{\|y_\infty\| + |T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})M + M}{\Gamma(\alpha)}\left|\frac{1}{1+t_1^\alpha} - \frac{1}{1+t_2^\alpha}\right| \\
&+ \frac{1}{\Gamma(\alpha)}\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds - \int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\|.
\end{aligned}$$

It suffices to show that

$$\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds - \int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\| \leq \varepsilon.$$

Relation (14) yields that there exits $N_0 > 0$ such that

$$\int_{N_0}^{\infty}\|f(t,y(t))\|dt \leq \frac{\varepsilon}{3} \text{ for any } y \in B. \tag{15}$$

On the other hand, since $\lim\limits_{t\to\infty}\dfrac{t^{2-\alpha}(t-N_0)^{\alpha-1}}{1+t^\alpha} = 0$, there exists $N_1 > N_0$ such that, for any $t_1, t_2 \geq N_1$ and $s \in [0, N_0]$, we have

$$\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right| < \frac{\varepsilon}{3M}. \tag{16}$$

Now taking $t_1, t_2 \geq N_1$, from (15), (16), we can arrive at

$$\begin{aligned}
&\left\|\int_0^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}f(s,y(s))ds - \int_0^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}f(s,y(s))ds\right\| \\
&\leq \int_0^{N_1}\left|\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha} - \frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\right|\|f(s,y(s))\|ds \\
&+ \int_{N_1}^{t_1}\frac{t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{1+t_1^\alpha}\|f(s,y(s))\|ds + \int_{N_1}^{t_2}\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{1+t_2^\alpha}\|f(s,y(s))\|ds \\
&< \frac{\varepsilon}{3M}\int_0^\infty\|f(s,y(s))\|ds + 2\int_{N_1}^\infty\|f(s,y(s))\|ds < \varepsilon.
\end{aligned}$$

Thus, NB_α is equiconvergent.

Step 4: Now, let us show that N satisfies the assumptions of Theorem. 2.11

First, we now show that N is defined from B to B, Indeed, for any $y \in B$, by above conditions $(\mathbf{H_1}), (\mathbf{H_4})$ and according to a little calculation, we have

$$\begin{aligned}
\|\frac{t^{2-\alpha}N(y)(t)}{1+t^\alpha}\| &\leq \frac{\|y_\infty\|}{\Gamma(\alpha)}(1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})) + \frac{|T|}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\int_0^{\tau_i}(\tau_i-s)^{\alpha-1}\|f(s,y(s))\|ds \\
&+ \frac{2+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^t\|f(t,y(t))\|dt + \frac{1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}{\Gamma(\alpha)}\int_0^\infty\|f(t,y(t))\|dt \\
&\leq \frac{(1+|T|(\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1}))}{\Gamma(\alpha)}\left(\|y_\infty\| + 3\int_0^\infty a(t)dt + 3R\int_0^\infty(1+t^\alpha))b(t)dt\right) < R.
\end{aligned}$$

Hence, $\|Ny\|_\alpha \leq R$, we conclude that $N : B \to B$.
We put $D = \overline{conv}(NB)$, it is clear that D is a closed, bounded and convex subset of B. As we know that $ND \subset NB \subset D$, then N remains defined from D to D. We denote by γ_α the Kuratowski measure of non-compactness on $C_\alpha([0,\infty), E)$. Let us first show that γ_α satisfies the following equality

$$\gamma_\alpha(NV) = \sup\left\{\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right),\ t \in (0,\infty)\right\},\ \text{for all } V \subset D. \tag{17}$$

Remark 3.5. *From the definitions of $C_\alpha([0,\infty), E)$, we see that*

$$\gamma_\alpha(\Omega) = \gamma_\varphi(\Omega_\alpha), \quad \textit{for all bounded subset } \Omega \textit{ of } C_\alpha([0,\infty), E).$$

We show first $\gamma_\alpha(NV) \leq \sup_{(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right)$.
Let ε be a strictly positive real number. From the equiconvergence of NV_α, there exists $A > 0$ such that

$$\|\frac{t_2^{2-\alpha}Ny(t_2)}{1+t_2^\alpha} - \frac{t_1^{2-\alpha}Ny(t_1)}{1+t_1^\alpha}\| < \varepsilon, \quad t_1, t_2 > A. \tag{18}$$

Let $NV_\alpha|_K$ be the restriction of NV_α on the interval $K = [0, A]$, by using Lemma 2.8 and the third step, we get

$$\gamma_\alpha(NV_\alpha|_K) = \sup_K \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) \leq \sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right),$$

this implies that there exists a finite partition NV_α^i of NV_α so that $NV_\alpha = \cup_i NV_\alpha^i$ and

$$diam(NV_\alpha^i|_K) < \sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) + \varepsilon, \quad i = 0, 1, \cdots, k. \tag{19}$$

Consequently, using inequalities (18) and (19), for all Ny_1, Ny_2 of NV_i and $t \geq A$, we have

$$\begin{aligned}\|\frac{t^{2-\alpha}Ny_2(t)}{1+t^\alpha} - \frac{t^{2-\alpha}Ny_1(t))}{1+t^\alpha}\| &\leq \|\frac{t^{2-\alpha}Ny_2(t)}{1+\psi_\alpha(t,0)} - \frac{A^{2-\alpha}Ny_2(A)}{1+A^\alpha}\| \\ &+ \|\frac{A^{2-\alpha}Ny_2(A)}{1+A^\alpha} - \frac{A^{2-\alpha}Ny_1(A)}{1+A^\alpha}\| + \|\frac{t^{2-\alpha}Ny_1(t)}{1+\psi_\alpha(t,0)} - \frac{A^{2-\alpha}Ny_2(A)}{1+A^\alpha}\| \\ &< 3\varepsilon + \sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right).\end{aligned}$$

So,

$$\|\frac{t^{2-\alpha}Ny_2(t)}{1+t^\alpha} - \frac{t^{2-\alpha}Ny_1(t))}{1+t^\alpha}\| \leq 3\varepsilon + \sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right). \tag{20}$$

From (18) and (19), we obtain

$$diam(NV_i) < \sup_{t\in(0,\infty)} \gamma\left(\frac{\psi_t^{2-\alpha}NV(t)}{1+t^\alpha}\right) + 3\varepsilon, \quad i = 0, 1, \cdots, k.$$

Thus,

$$\gamma_\alpha(NV) < \sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) + 3\varepsilon.$$

Since ε is arbitrary, this leads us to the desired result.
Conversely, we show that $\sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) \leq \gamma_\alpha(NV)$. According to the definition of Kuratowski MNC,

we have, for all $\varepsilon > 0$, we can find a finite partition $NV_\alpha = \cup_i NV_\alpha^i$ such that $diam(NV_\alpha^i) < \gamma_\alpha(NV) + \varepsilon$, then for all $y_1, y_2 \in V$ and $t \in (0, \infty)$, we obtain,

$$\|\frac{t^{2-\alpha} Ny_2(t)}{1+t^\alpha} - \frac{t^{2-\alpha} Ny_1(t)}{1+t^\alpha}\| \leq \|Ny_2 - Ny_1\|_\alpha < \gamma_\alpha(NV) + \varepsilon.$$

According to $NV_\alpha(t) = \cup_i NV_\alpha^i(t)$, we get $\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) < \gamma_\alpha(NV) + \varepsilon$, since ε is arbitrary, we then have $\gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) \leq \gamma_\alpha(NV)$. So,

$$\sup_{t\in(0,\infty)} \gamma\left(\frac{t^{2-\alpha}NV(t)}{1+t^\alpha}\right) \leq \gamma_\alpha(NV).$$

Finally we need to prove the following implication

$$\forall \varepsilon > 0,\ \exists \varrho(\varepsilon) : \varepsilon \leq \gamma(V) < \varepsilon + \varrho \Longrightarrow \gamma_{(\alpha,\psi)}(NV) < \varepsilon, \text{ for any } V \subset D. \tag{21}$$

Let ε be a strictly positive real number, $V \subset D$ and $t \in (0, \infty)$, for all $\kappa \in \mathbb{R}_+^*$ satisfying $t \leq \kappa$, we define the auxiliary operator N_κ by

$$\begin{aligned} N_\kappa(y)(t) &= \frac{y_\infty}{\Gamma(\alpha)}[t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2}] + \frac{Tt^{\alpha-2}}{\Gamma(\alpha)} \sum_{i=1}^m \lambda_i \int_0^{\tau_i} (\tau_i - s)^{\alpha-1} f(s, y(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t [t^{\alpha-1} + T(\sum_{i=1}^m \lambda_i \tau_i^{\alpha-1})t^{\alpha-2} - (t-s)^{\alpha-1}] f(s, y(s))ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_t^\kappa (t-s)^{\alpha-1} f(s, y(s))ds. \end{aligned}$$

Then from$(\mathbf{H_1})$, we obtain

$$\begin{aligned} \frac{t^{2-\alpha}}{1+t^\alpha}\|N_\kappa(y)(t) - N(y)(t)\| &\leq \frac{1 + |T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})}{\Gamma(\alpha)} \int_\kappa^\infty \|f(t, y(t))\|dt \\ &\leq \frac{1 + |T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})}{\Gamma(\alpha)} \left(\int_\kappa^\infty a(t)dt + R\int_n^\infty (1+t^\alpha))b(t)dt\right), \end{aligned}$$

this shows that $H_d\left(\frac{t^{2-\alpha}N_{\xi,n}(V)(t)}{1+t^\alpha}, \frac{t^{2-\alpha}N(V)(t)}{1+t^\alpha}\right) \to 0$ as $\xi \to 0$ and $n \to \infty$, $t \in J$. Where H_d denotes the Hausdorff metric in space E. By the property of non-compactness measure, we get

$$\lim_{\kappa\to\infty} \gamma\left(\frac{t^{2-\alpha}N_\kappa(V)(t)}{1+t^\alpha}\right) = \gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^\alpha}\right). \tag{22}$$

By a similar argument as the one of third step, we show that the $N_\kappa V_\alpha$ is equicontinuous and bounded on $[0, \kappa]$. From Lemmas 2.7,2.8,2.9, $(\mathbf{H_3})$ and the previous steps, it follows, that there exists a sequence $\{u_n\}_{n=0}^\infty \subset V$ such that

$$\begin{aligned} \gamma\left(\frac{t^{2-\alpha}N_\kappa V(t)}{1+t^\alpha}\right) &\leq \frac{\varepsilon}{2} + \frac{2 + 2|T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})}{\Gamma(\alpha)} \int_0^\kappa \gamma\left\{f(s, u_n(s)),\ n \in \mathbb{N}\right\} ds \\ &\leq \frac{\varepsilon}{2} + \frac{2 + 2|T|(\sum\limits_{i=1}^m \lambda_i \tau_i^{\alpha-1})}{\Gamma(\alpha)} \int_0^\kappa (1+s^\alpha)\ell(s)\gamma_\alpha(N(V))ds. \end{aligned}$$

From (22), we know that

$$\gamma\left(\frac{t^{2-\alpha}N(V)(t)}{1+t^{\alpha}}\right) \leq \frac{\varepsilon}{2} + \frac{2[1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\gamma_{\alpha}(N(V))}{\Gamma(\alpha)} \int_0^{\infty}(1+s^{\alpha})\ell(s)ds.$$

Thus,

$$\gamma_{\alpha}(N(V)) \leq \frac{\varepsilon}{2} + \frac{2[1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\vartheta_{\alpha}(N(V))}{\Gamma(\alpha)} \int_0^{\infty}(1+s^{\alpha})\ell(s)ds.$$

If

$$\gamma_{\alpha}(N(V)) \leq \frac{\varepsilon}{2} + \frac{2[1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\gamma_{\alpha}(N(V))}{\Gamma(\alpha)} \int_0^{\infty}(1+s^{\alpha})\ell(s)ds < \varepsilon,$$

this implies that

$$\gamma_{\alpha}(N(V)) < \frac{\Gamma(\alpha)}{4[1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\int_0^{\infty}(1+s^{\alpha})\ell(s)ds}\varepsilon,$$

so that implication (21) is fulfilled, we take

$$\varrho = \frac{\Gamma(\alpha) - 4[1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\int_0^{\infty}(1+s^{\alpha})\ell(s)ds}{4[1+|T|(\sum_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})]\int_0^{\infty}(1+s^{\alpha})\ell(s)ds}\varepsilon.$$

So, N is a Meir-Keeler condensing operator via $\gamma_{(\alpha,\psi)}$, thus all the hypotheses of the Theorem 2.11 are fulfilled. Then, the problem $(1)-(3)$ is non-empty and compact. □

4. Example

As an application of our results, we consider the following fractional differential equation.

$$^{\mathcal{RL}}\mathcal{D}^{\frac{3}{2}}y(t) = \left(\frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{10t}} + \frac{\sin(t)}{1+t^2}\right)_{n=1}^{\infty}, \quad t \in J = (0,+\infty), \tag{23}$$

$$\mathcal{I}_{0^+}^{\frac{1}{2}}y(t) = \frac{1}{2}y(1) + y(4), \tag{24}$$

$$^{\mathcal{RL}}\mathcal{D}_{0^+}^{\frac{1}{2}}y(\infty) = y_{\infty}. \tag{25}$$

Let

$$E = \{(y_1, y_2, \ldots, y_n, \ldots) : \sup|y_n| < \infty\},$$

with the norm $\|y\| = \sup_n |y_n|$, then E is a Banach space and problem (23)-(25) can be regaded as an abstract problem (1)-(3), with

$$\alpha = \frac{3}{2},\ T \simeq 0.5642 \text{ and } f(t, y(t)) = (f(t, y_1(t)), \ldots, f(t, y_n(t)), \ldots),$$

where

$$f(t, y_n(t)) = \frac{\sqrt{t}y_n(t)}{(1+t^{\frac{3}{2}})e^{10t}} + \frac{\sin(t)}{1+t^2},\ n \in \mathbb{N}^*.$$

We shall verify the conditions $(\mathbf{H_1})-(\mathbf{H_3})$. Evidently, f is continuous in $J\times E$ and

$$\|f(t,y(t))\|\leq \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{10t}}\|y(t)\|+\frac{1}{1+t^2}.$$

With the help of simple computation, we find that

$$\int_0^\infty e^{-10t}dt=\frac{1}{10}<\frac{\Gamma(\alpha)}{3(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}\simeq 0.2451 \text{ and } \int_0^\infty \frac{1}{1+t^2}dt=\frac{\pi}{2}<\infty.$$

Finally, we verify condition $(\mathbf{H_3})$. For any bounded set $B\subset E$, we have

$$f(t,B(t))=\frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{10t}}B(t)+\frac{\sin(t)}{1+t^2}.$$

Then

$$\gamma(f(t,B(t))\leq \frac{\sqrt{t}}{(1+t^{\frac{3}{2}})e^{10t}}\gamma(B(t)).$$

Since

$$\int_0^\infty e^{-10t}dt=\frac{1}{10}<\frac{\Gamma(\alpha)}{4(1+|T|\sum\limits_{i=1}^{m}\lambda_i\tau_i^{\alpha-1})}\simeq 0.3676,$$

we conclude that condition $(\mathbf{H_3})$ is satisfied. Therefore, Theorem 3.4 ensures that problem (23)-(25) is non-empty and compact.

Conclusion

In this work, we deal with the problem concerning existence of solution sets and its topological structure for non-local Riemman-Liouville fractional differential equation modeled by equation (1)-(3) on the half line with Riemann-Liouville fractional integral and derivative boundary conditions involving the discontinuity of the state y at 0^+. Our main result is to prove the existence of solution sets and its topological structure for the problem (1)-(3) on unbounded domain with the non-local conditions. To overcome the difficulty of the problem, we have defined a special weight space of continuous functions $C_\alpha(J,E)$. The constructed space is in a natural way, in the sense that this space is endowed with a Banach structure.
As far as we know, in our opinion, this problem has not been studied in the literature.
The assumed hypotheses have as goals:
i) In this work we have assumed a more general growth condition (H_1) unlike the affine condition.
ii) Hypothesis (H_2) being supposed to overcome the equiconvergence at infinity.
iii) Conditions (H_3) and (H_4) ensure the veracity of the Meir-Keeler fixed point theorem for condensing operator.
These conditions are optimal in the sense that no condition implies the other. We make use in our approach the Meir-Keeler fixed point theorem combined with tools from classical functional analysis and measure of non-compactness. The paper concludes with an example to illustrate the feasibility of our main result.

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