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Existence of an initial value problem for time-fractional Oldroyd-B fluid equation using Banach fixed point theorem

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Abstract

In this paper, we study the initial boundary value problem for time-fractional Oldroyd-B fluid equation. Our model contains two Riemann-Liouville fractional derivatives which have many applications, for example, in viscoelastic flows. For the linear case, we obtain regularity results under some different assumptions of the initial data and the source function. For the non-linear case, we obtain the existence of a unique solution using Banach's fixed point theorem.

Keywords: Time-fractional Oldroyd-B fluid problem; Riemman-Liouville; Regularity; Banach fixed point theory.
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1. Introduction

In recent years, fractional calculus has been widely applied in many different fields science and technology. The generalized fractional Oldroyd-B fluid model is a special case of non-Newtonian fluids that is of paramount importance in a large number of industries and applied sciences. Therefore, the number of publications on this topic is very abundant with many different detailed research directions. There are currently several definitions for fraction derivatives and fraction integrals, such as Riemann-Liouville, Caputo, Hadamard, Riesz, etc. We can refer the reader to some papers [2–4, 17–25]. The study of the exact analytical solutions have been found in some papers [6–8, 11, 12]. Numerical solutions for time fractional Oldroyd-B

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model are studied in [5, 9, 10]. Let $\mathcal{D} \subset \mathbb{R}^N$ be a bounded domain with sufficiently smooth boundary $\partial\mathcal{D}$. We investigate the following initial problem for the time-fractional Oldroyd-B fluid equation

$$\begin{cases} (1 + a\partial_t^\alpha)u_t(x,t) = \mu(1 + b\partial_t^\beta)\Delta u(x,t) + F(x,t,u(x,t)), & x \in \mathcal{D},\ 0 < t \le T, \\ u = 0,\ (x,t) \in \partial\mathcal{D} \times (0,T), \\ u(x,0) = u_0(x),\ I^{1-\alpha}u_t(x,0) = 0,\ x \in \mathcal{D}. \end{cases} \tag{1.1}$$

where ∂_t^α is the Riemann-Liouville fractional derivative [13]

$$\partial_t^\alpha v(t) := \frac{\partial}{\partial t}\int_0^t \mu_{1-\alpha}(s)v(t-s,x)ds, \qquad \mu_\beta(s) := \frac{1}{\Gamma(\beta)}s^{\beta-1}, \quad (\beta > 0), \tag{1.2}$$

Here u_0 is called the initial data and F is the source functions which are defined later. Noting that if $a = 0$ and $b > 0$ then (1.1) describes a Rayleigh-Stokes problem for a generalized fractional second-grade fluid. The problem with $b = 0$ and $a > 0$ express in general fractional Maxwell model [14–16] and if $a = b = 0$, we get immediately that classical Newtonian fluids.

The problem (1.1) was first mentioned by two authors E. Bazhlekova and I. Bazhlekov [1]. However, the properties and existence of solutions were not investigated carefully in this paper. Continuation of work [21, 26–30], in recent paper [5], M. Al-Maskari and S. Karaa considered the regularity result for homogeneous linear case, i.e, $F = 0$. So far, there has not been any work related to the qualitative solution of the problem (1.1) for both cases $F = F(x,t)$ and $F = F(x,t,u)$. Motivated by this reason, we try to solve the above problem to consider the well-posedness of this problem. The two main results detailed in the paper are given below

- The first major result concerns the mild solution of the problem in the linear case. We investigate the regularity of the solution with two different cases of of the smoothness of input data.
- The second major result proves the global existence of a mild solution of the problem (1.1) in the nonlinear case. Using Banach's fixed point theorem, we have proved the problem has only one solution. The difficulty that we face is choosing some suitable solution spaces.

2. Preliminaries

We recall the Hilbert scale space, which is given as follows

$$\mathcal{H}^s(\mathcal{D}) = \left\{ f \in L^2(\mathcal{D}),\ \sum_{j=1}^{\infty} \lambda_j^s \langle f, e_j\rangle^2_{L^2(\mathcal{D})} < \infty \right\},$$

for any $s \ge 0$. Here the symbol $\langle\cdot,\cdot\rangle_{L^2(\mathcal{D})}$ denotes the inner product in $L^2(\mathcal{D})$. It is well-known that $\mathcal{H}^r(\mathcal{D})$ is a Hilbert space corresponding to the norm $\|f\|_{\mathcal{H}^s(\mathcal{D})} = \sqrt{\sum_{j=1}^\infty \lambda_j^s \langle f, e_j\rangle^2_{L^2(\mathcal{D})}}$, $f \in \mathcal{H}^s(\mathcal{D})$. In view of $\mathcal{H}^\nu(\Omega) \equiv D((-\mathbb{L})^\nu)$ is a Hilbert space. Then $D((-\mathbb{L})^{-\nu})$ is a Hilbert space with the norm

$$\|v\|_{D((-\mathbb{L})^{-\nu})} = \left(\sum_{j=1}^{\infty} |\langle v, e_j\rangle|^2 \lambda_j^{-2\nu} \right)^{\frac{1}{2}},$$

where $\langle\cdot,\cdot\rangle$ in the latter equality denotes the duality between $D((-\mathbb{L})^{-\nu})$ and $D((-\mathbb{L})^{\nu})$.

Lemma 2.1. *The following inclusions hold true:*

$$\left.\begin{aligned} L^p(\Omega) &\hookrightarrow D(\mathcal{A}^\sigma), \quad \text{if} \quad -\frac{N}{4} < \sigma \le 0, \quad p \ge \frac{2N}{N-4\sigma}, \\ D(\mathcal{A}^\sigma) &\hookrightarrow L^p(\Omega), \quad \text{if} \quad 0 \le \sigma < \frac{N}{4}, \quad p \le \frac{2N}{N-4\sigma}. \end{aligned}\right\} \tag{2.3}$$

3. Linear inhomogeneous source

In this section, we consider the (1.1) problem in the linear case, that is, the source function has the simple form $F = F(x,t)$. Applying eigenfunction decomposition, the solution u of Problem (1.1) has the form of Fourier series $u(x,t) = \sum_{j=1}^{\infty} u_j(t)e_j(x)$. Let us denote by $u_j(t) = \langle u(x,t), e_j\rangle$. Then we get the following equation

$$(1 + aD_t^{\alpha})\frac{du_j(t)}{dt} = -\lambda_j\mu\,(1 + bD_t^{\alpha})\,u_j(t) + F_j(t),\ \ u_j(0) = \langle u_0(x), e_j\rangle\,. \tag{3.4}$$

Our next step is to solving this equation. By applying the Laplace transform, we obtain the formal eigen expansion of solution $u_j(t)$ as follows

$$u_j(t) = \mathbf{K}_j(t)\,\langle u_0, e_j\rangle + \int_0^t \mathbf{G}_j(t-\tau)\,\langle F(\tau), e_j\rangle\, d\tau, \tag{3.5}$$

which allows us to get that the explicit formula of the solution u

$$u(x,t) = \sum_{j=1}^{\infty}\mathbf{K}_j(t)\,\langle u_0, e_j\rangle\, e_j(x) + \sum_{j=1}^{\infty}\left(\int_0^t \mathbf{G}_j(t-\tau)\,\langle F(\tau), e_j\rangle\, d\tau\right)e_j(x). \tag{3.6}$$

Here two functions $\mathbf{K}_j$ and $\mathbf{G}_j$ have the following Laplace transform

$$\mathcal{L}(\mathbf{K}_j)(s) = \frac{1 + as^{\alpha}}{s(1+as^{\alpha}) + \mu\lambda_j\,(1+bs^{\alpha})},\ \ \mathcal{L}(\mathbf{G}_j)(s) = \frac{1}{s(1+as^{\alpha}) + \mu\lambda_j\,(1+bs^{\alpha})}. \tag{3.7}$$

Thanks for the results from the work of E. Bazhlekova and I. Bazhlekov [1], we have the following lemma right away

Lemma 3.1. *Two expressions $\mathbf{K}_j$ and $\mathbf{G}_j$ have the following properties*

$$\mathbf{K}_j(0) = 1,\ \ \mathbf{G}_j(0) = 0,\ \ |\mathbf{K}_j(t)| \le C_1,\ \ t \ge 0 \tag{3.8}$$

$$|\mathbf{K}_j(t)| \le \frac{C_2\left(t^{\beta-1} + at^{\beta-\alpha-1}\right)}{\lambda_j},\ \ \int_0^t |\mathbf{G}_j(\tau)|d\tau \le \frac{C_3}{\lambda_j}, \tag{3.9}$$

where the constants C_1, C_2, C_3 are independent of n and t.

Theorem 3.1. *Let the source function $F \in L^{\infty}(0,T;\mathcal{H}^{\theta}(\mathcal{D}))$.*
a) If $u_0 \in H^s(\mathcal{D}))$ then

$$\big\|u\big\|^2_{L^{\infty}(0,T;\mathcal{H}^s(\mathcal{D}))} \le 2C_1\|u_0\|^2_{\mathcal{H}^s(\mathcal{D})} + 2\overline{C}_1\overline{C}_2(s,\theta,N)C_3^2\big\|F\big\|^2_{L^{\infty}(0,T;\mathcal{H}^{\theta}(\mathcal{D}))}. \tag{3.10}$$

Here s,θ satisfies the condition $4 + 4\theta - 4s > N$.
b) If $u_0 \in H^{s-1}(\mathcal{D}))$ then we get

$$\big\|u(.,t)\big\|_{\mathcal{H}^s(\mathcal{D})} \le 2C_2\left(t^{\beta-1} + at^{\beta-\alpha-1}\right)\|u_0\|_{\mathcal{H}^{s-1}(\mathcal{D})} + \sqrt{2\overline{C}_1\overline{C}_2(s,\beta,N)}C_3\big\|F\big\|_{L^{\infty}(0,T;\mathcal{H}^{\theta}(\mathcal{D}))}. \tag{3.11}$$

Remark 3.1. *From part 2 of the above theorem, we notice that if $u_0 \in H^{s-1}(\mathcal{D}))$ then $t^{\gamma}\big\|u(.,t)\big\|_{\mathcal{H}^s(\mathcal{D})}$ belongs to the space $L^{\infty}(0,T;\mathcal{H}^s(\mathcal{D}))$ with $\gamma \ge 1 + \alpha - \theta$.*

Remark 3.2. *Let us assume that $u_0 \in L^p(\mathcal{D})$ for $1 \le p < 2$. Then using Lemma (2.1), we find that $u_0 \in \mathcal{H}^{\sigma}(\mathcal{D})$ for $-\frac{N}{4} < \sigma \le \frac{(p-2)N}{4p}$. Let us choose $\sigma = \frac{(p-2)N}{4p}$ then if $F \in L^{\infty}(0,T;\mathcal{H}^{\theta}(\mathcal{D}))$ for $\theta > \frac{1}{4}\left(N - \frac{2N}{p} - 3\right)$ from Theorem (3.1), we can deduce that $u \in L^{\infty}\big(0,T;\mathcal{H}^{\frac{(p-2)N}{4p}}(\mathcal{D})\big)$.*

Proof. Using Parseval's equality, we find that the following estimate

$$\begin{aligned}
&\Big\|u(.,t)\Big\|^2_{\mathcal{H}^s(\mathcal{D})}\\
&\le 2\Big\|\sum_{j=1}^{\infty}\mathbf{K}_j(t)\langle u_0,e_j\rangle e_j(x)\Big\|^2_{\mathcal{H}^s(\mathcal{D})}+2\Big\|\sum_{j=1}^{\infty}\bigg(\int_0^t\mathbf{G}_j(t-\tau)\langle F(\tau),e_j\rangle d\tau\bigg)e_j(x)\Big\|^2_{\mathcal{H}^s(\mathcal{D})}\\
&\le 2\sum_{j=1}^{\infty}\lambda_j^{2s}|\mathbf{K}_j(t)|^2\langle u_0,e_j\rangle^2+2\sum_{j=1}^{\infty}\lambda_j^{2s}\bigg(\int_0^t\mathbf{G}_j(t-\tau)\langle F(\tau),e_j\rangle d\tau\bigg)^2=\mathcal{J}_1+\mathcal{J}_2. \qquad (3.12)
\end{aligned}$$

For the term $\mathcal{J}_2$, we first give the following bound by using Hölder inequality

$$\begin{aligned}
\lambda_j^{2s}\bigg(\int_0^t\mathbf{G}_j(t-\tau)\langle F(\tau),e_j\rangle d\tau\bigg)^2&\le\lambda_j^{2s}\bigg(\int_0^t\mathbf{G}_j(t-\tau)d\tau\bigg)\bigg(\int_0^t\mathbf{G}_j(t-\tau)\langle F(\tau),e_j\rangle^2 d\tau\bigg)\\
&\le C_3\lambda_j^{2s-1}\bigg(\int_0^t\mathbf{G}_j(t-\tau)\langle F(\tau),e_j\rangle^2 d\tau\bigg). \qquad (3.13)
\end{aligned}$$

It is not difficult to realize that

$$\lambda_j^{2s-1}\bigg(\int_0^t\mathbf{G}_j(t-\tau)\langle F(\tau),e_j\rangle^2 d\tau\bigg)=\lambda_j^{2s-2-2\theta}\bigg(\int_0^t\lambda_j\mathbf{G}_j(t-\tau)\lambda_j^{2\theta}\langle F(\tau),e_j\rangle^2 d\tau\bigg). \qquad (3.14)$$

By the definition of the function F on the space $L^\infty(0,T;\mathcal{H}^{s-1}(\mathcal{D}))$, we find that

$$\big\|F\big\|^2_{L^\infty(0,T;\mathcal{H}^\beta(\mathcal{D}))}=\sup_{0\le\tau\le T}\big\|F(\tau)\big\|^2_{\mathcal{H}^\beta(\mathcal{D})}\ge\lambda_j^{2\theta}\langle F(\tau),e_j\rangle^2 \qquad (3.15)$$

which allows us to obtain that

$$\begin{aligned}
\bigg(\int_0^t\lambda_j\mathbf{G}_j(t-\tau)\lambda_j^{2\theta}\langle F(\tau),e_j\rangle^2 d\tau\bigg)&\le\big\|F\big\|^2_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}\Big(\int_0^t\lambda_j\mathbf{G}_j(t-\tau)d\tau\Big)\\
&\le C_3\big\|F\big\|^2_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}. \qquad (3.16)
\end{aligned}$$

Combining (3.13), (3.14), and (3.16), we obtain that

$$\mathcal{J}_2\le 2C_3^2\big\|F\big\|^2_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}\sum_{j=1}^{\infty}\lambda_j^{2s-2-2\theta}. \qquad (3.17)$$

It is well-known that to recall $\lambda_j\le\overline{C}_1 j^{2/N}$, for N is the dimensional number of the domain $\mathcal{D}$. Therefore, we arrive at the following estimate $\sum_{j=1}^{\infty}\lambda_j^{2s-2-2\theta}\le\overline{C}\sum_{j=1}^{\infty}j^{\frac{4s-4-4\theta}{N}}$. Since the condition $4+4\theta-4s>N$, we know that the infinite series $\sum_{j=1}^{\infty}j^{\frac{4s-4-4\theta}{N}}$ is convergent. Let us assume that $\sum_{j=1}^{\infty}j^{\frac{4s-4-4\theta}{N}}=\overline{C}_2(s,\theta,N)$ then we follows from (3.17) that

$$\mathcal{J}_2\le 2\overline{C}_1\overline{C}_2(s,\theta,N)C_3^2\big\|F\big\|^2_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}. \qquad (3.18)$$

For considering the first term $\mathcal{J}_1$, we divide two cases.
Case 1. Let us assume that $u_0\in\mathcal{H}^s(\mathcal{D})$. Under this case, we can bound the quantity $\mathcal{J}_1$ as follows

$$\mathcal{J}_1=2\sum_{j=1}^{\infty}\lambda_j^{2s}|\mathbf{K}_j(t)|^2\langle u_0,e_j\rangle^2\le 2C_1\sum_{j=1}^{\infty}\lambda_j^{2s}\langle u_0,e_j\rangle^2=2C_1\|u_0\|^2_{\mathcal{H}^s(\mathcal{D})}. \qquad (3.19)$$

Combining (3.18) and (3.19), we arrive at

$$\Big\|u(.,t)\Big\|^2_{\mathcal{H}^s(\mathcal{D})} \le \mathcal{J}_1 + \mathcal{J}_2 \le 2C_1\|u_0\|^2_{\mathcal{H}^s(\mathcal{D})} + 2\overline{C}_1\overline{C}_2(s,\beta,N)C_3^2\Big\|F\Big\|^2_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}. \tag{3.20}$$

The right hand side of the above expression is independent of t, so we can deduce that $u \in L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))$. We also give the following regularity result

$$\Big\|u\Big\|^2_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))} \le \mathcal{J}_1 + \mathcal{J}_2 \le 2C_1\|u_0\|^2_{\mathcal{H}^s(\mathcal{D})} + 2\overline{C}_1\overline{C}_2(s,\beta,N)C_3^2\Big\|F\Big\|^2_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}. \tag{3.21}$$

Case 2. Let us assume that $u_0 \in \mathcal{H}^{s-1}(\mathcal{D})$. Under this case, we give the following estimation for $\mathcal{J}_1$ in the following

$$\begin{aligned}\mathcal{J}_1 = 2\sum_{j=1}^\infty \lambda_j^{2s}|\mathbf{K}_j(t)|^2 \langle u_0, e_j\rangle^2 &\le 2C_2^2\left(t^{\beta-1} + at^{\beta-\alpha-1}\right)^2 \sum_{j=1}^\infty \lambda_j^{2s-2}\langle u_0, e_j\rangle^2 \\ &= 2C_2^2\left(t^{\beta-1} + at^{\beta-\alpha-1}\right)^2 \|u_0\|^2_{\mathcal{H}^{s-1}(\mathcal{D})}.\end{aligned} \tag{3.22}$$

Combining (3.18) and (3.22), we arrive at

$$\begin{aligned}\Big\|u(.,t)\Big\|_{\mathcal{H}^s(\mathcal{D})} &\le \sqrt{\mathcal{J}_1} + \sqrt{\mathcal{J}_2} \\ &\le 2C_2\left(t^{\beta-1} + at^{\beta-\alpha-1}\right)\|u_0\|_{\mathcal{H}^{s-1}(\mathcal{D})} + \sqrt{2\overline{C}_1\overline{C}_2(s,\theta,N)}C_3\Big\|F\Big\|_{L^\infty(0,T;\mathcal{H}^\theta(\mathcal{D}))}.\end{aligned} \tag{3.23}$$

□

4. Nonlinear time-fractional Oldroyd-B fluid equation

In this section, we consider the following nonlinear problem

$$\begin{cases}(1 + a\partial_t^\alpha)u_t(x,t) = \mu(1 + b\partial_t^\beta)\Delta u(x,t) + F(u(x,t)), \quad x \in \mathcal{D},\ 0 < t \le T, \\ u = 0,\ (x,t) \in \partial\mathcal{D} \times (0,T), \\ u(x,0) = u_0(x),\ I^{1-\alpha}u_t(x,0) = 0,\ x \in \mathcal{D}.\end{cases} \tag{4.24}$$

By using a similar explanation as in previous section, we derive that

$$u(x,t) = \sum_{j=1}^\infty \mathbf{K}_j(t)\langle u_0, e_j\rangle e_j(x) + \sum_{j=1}^\infty\left(\int_0^t \mathbf{G}_j(t-\tau)\langle F(u(\tau)), e_j\rangle d\tau\right)e_j(x). \tag{4.25}$$

Theorem 4.1. *Let the initial datum $u_0 \in \mathcal{H}^s(\mathcal{D}))$. Let F satisfies that $F(0) = 0$ and*

$$\Big\|F(w_1) - F(w_2)\Big\|_{\mathcal{H}^\beta(\mathcal{D})} \le K_f\Big\|w_1 - w_2\Big\|_{\mathcal{H}^s(\mathcal{D})}, \tag{4.26}$$

for K_f is a positive constant. Then if K_f enough small then problem (4.24) has a unique solution $u \in L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))$.

Remark 4.1. *In the above theorem, we need to assume the condition of the function F with a sufficiently small Lipschitz coefficient K_f. We don't have much information about $\mathbf{G}_j$ so unbinding K_f is a thorny and challenging issue. There is only one information about $\mathbf{G}_j$ then the best method in this case is Banach fixed point theorem applied to the solution space $L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))$. We will try to investigate it in another future article.*

Proof. Set the following function

$$\mathcal{Q}w(t) = \sum_{j=1}^{\infty} \mathbf{K}_j(t) \langle u_0, e_j \rangle e_j(x) + \sum_{j=1}^{\infty} \left(\int_0^t \mathbf{G}_j(t-\tau) \langle F(w(\tau)), e_j \rangle d\tau \right) e_j(x). \quad (4.27)$$

If $w = 0$ then since the condition $F(0) = 0$, we know that $\mathcal{Q}w(t) = \sum_{j=1}^{\infty} \mathbf{K}_j(t) \langle u_0, e_j \rangle e_j(x)$. Since the fact that $|\mathbf{K}_j(t)| \le C_1$ as in Lemma (3.1) and the initial datum $u_0 \in \mathcal{H}^s(\mathcal{D})$), we can easily to obtain that $\mathcal{Q}w \in L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))$. Take any functions $w_1, w_2 \in L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))$. It follows from (4.27) that

$$\mathcal{Q}w_1(t) - \mathcal{Q}w_2(t) = \sum_{j=1}^{\infty} \left(\int_0^t \mathbf{G}_j(t-\tau) \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle d\tau \right) e_j(x). \quad (4.28)$$

By looking closely at the above expression and using Parseval's equality and Hölder inequality, we get the following result by some calculations

$$\begin{aligned} &\Big\| \mathcal{Q}w_1(t) - \mathcal{Q}w_2(t) \Big\|_{\mathcal{H}^s(\mathcal{D})}^2 \\ &= \sum_{j=1}^{\infty} \lambda_j^{2s} \left(\int_0^t \mathbf{G}_j(t-\tau) \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle d\tau \right)^2 \\ &\le \sum_{j=1}^{\infty} \lambda_j^{2s} \left(\int_0^t \Big| \mathbf{G}_j(t-\tau) \Big| d\tau \right) \left(\int_0^t \Big| \mathbf{G}_j(t-\tau) \Big| \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \right) \\ &\le C_3 \sum_{j=1}^{\infty} \lambda_j^{2s-1} \left(\int_0^t \Big| \mathbf{G}_j(t-\tau) \Big| \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \right). \end{aligned} \quad (4.29)$$

It is easy to see that

$$\Big\| \mathcal{Q}w_1(t) - \mathcal{Q}w_2(t) \Big\|_{\mathcal{H}^s(\mathcal{D})}^2 \le C_3 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\beta} \left(\int_0^t \lambda_j \mathbf{G}_j(t-\tau) \lambda_j^{2\beta} \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \right). \quad (4.30)$$

Let us continue to deal with the integral term on the right hand side of the above expression. By looking at the globally Lipschitz condition of F as in (4.26), we infer that

$$\begin{aligned} \lambda_j^{2\theta} \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 &\le \big\| F(w_1(\tau)) - F(w_2(\tau)) \big\|_{\mathcal{H}^\beta(\mathcal{D})}^2 \\ &\le K_f \big\| w_1(\tau) - w_2(\tau) \big\|_{\mathcal{H}^\beta(\mathcal{D})}^2 \le K_f \sup_{0 \le \tau \le T} \big\| w_1(\tau) - w_2(\tau) \big\|_{\mathcal{H}^s(\mathcal{D})}^2 \\ &\le K_f \big\| w_1 - w_2 \big\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))}^2. \end{aligned} \quad (4.31)$$

By combining the two evaluations (4.30) and (4.31), we have immediately the result of the upper bound of the integral on the right hand side of (4.30)

$$\int_0^t \lambda_j \mathbf{G}_j(t-\tau) \lambda_j^{2\beta} \langle F(w_1(\tau)) - F(w_2(\tau)), e_j \rangle^2 d\tau \le K_f \big\| w_1 - w_2 \big\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))}^2 \Big(\int_0^t \lambda_j \mathbf{G}_j(t-\tau) d\tau \Big). \quad (4.32)$$

Hence, from some above observations, we can derive that

$$\begin{aligned} \Big\| \mathcal{Q}w_1(t) - \mathcal{Q}w_2(t) \Big\|_{\mathcal{H}^s(\mathcal{D})}^2 &\le K_f C_3 \big\| w_1 - w_2 \big\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))}^2 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\beta} \Big(\int_0^t \lambda_j \mathbf{G}_j(t-\tau) d\tau \Big) \\ &\le K_f C_3 \big\| w_1 - w_2 \big\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))}^2 \sum_{j=1}^{\infty} \lambda_j^{2s-2-2\beta} \\ &\le K_f C_4 \big\| w_1 - w_2 \big\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))}^2 \end{aligned} \quad (4.33)$$

where we note that the infinite series $\sum_{j=1}^{\infty} \lambda_j^{2s-2-2\theta}$ is convergent. This latter inequality leads to

$$\left\| \mathcal{Q}w_1 - \mathcal{Q}w_2 \right\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))} \leq \sqrt{K_f C_4} \left\| w_1 - w_2 \right\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))}. \tag{4.34}$$

With the help of Banach Fixed Point Theorem and noting that $K_f C_4 < 1$, if K_f is small enough, we immediately conclude that $\mathcal{Q}$ has a fixed point $u \in L^\infty(0,T;\mathcal{H}^s(\mathcal{D}))$.

□

5. Conclusion

In this work, we focus on the time-fractional Oldroyd-B fluid equation with the initial boundary value problem. Here, the Riemann-Liouville fractional derivatives have many applications where we consider two cases. Firstly, we obtain regularity results under some different assumptions of the initial data and the source function for the linear problem. Secondly, for the non-linear problem, we obtain the existence of a unique solution using Banach's fixed point theorem.

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