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Computing the Forgotten Topological Index for Zero Divisor Graphs of MV-Algebras

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ABSTRACT: Topological indices are numerical values in graphs. Recently, they have been an attractive topic and multidisciplinary research area in mathematics, computer science, chemistry, pharmacy, and etc. In this work, we investigate the Forgotten index of zero divisor graphs of MV-algebra. Firstly, using isomorphism between L_n and n-element MV-chain, we generalize the Forgotten topological index for zero divisor graph of L_n where $n > 5$. Then, the Forgotten index of zero divisor graph for MV-algebras which have cardinality 5,6,7,8 is computed with supporting samples.

Keywords: Forgotten index, topological index, MV-algebra, zero divisor graph, graph theory

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INTRODUCTION

Recently, logical algebraic structures have been examined in new mathematical works. Researchers deeply concern with keen these structures since these structures have been tackled as an impressive instrument for data systems and various sections in computer sciences, such as fuzzy knowledge with soft and rough concepts (Ansari et al., 2018). In the last thirty years, one of the exciting research topics has been to study algebraic structures by using graphs. Zero divisor graphs of commutative rings as the area of algebraic combinatorics were studied firstly by I. Beck (Beck, 1988). He gives the definition of the zero-divisor graph such that two elements $x, y \in R$ where this graph has vertex set on R are adjacent whenever $xy = 0$. In (Anderson and Livingston, 1999), this definition of a zero-divisor graph of a commutative ring was later modified. After the presentation of zero-divisor graphs, different types of graphs related to commutative rings go out such as annihilating-ideal graphs, comaximal graphs, total graphs (Anderson and Badawi, 2008; Akbari et al., 2009; Shaveisi, 2016; Ye and Wu, 2012; Wang, 2008; Tehranian and Maimani, 2011; Aijaz and Pirzada, 2020; Salehifar et al., 2017; Sinha and Rao, 2018). More recently, Gan and Yang have studied on zero-divisor graphs of MV-algebras concerning both logical structure and graphs (Gan and Yang, 2020a). They prove that this graph G is connected with the $diam(G) \leq 3$. Then, using the zero divisor graph, they categorize all MV-algebras having a cardinality is less than eight. Lastly, as a continuation of this work, Gan and Yang have also introduced an annihilator graph of an MV-algebra (Gan and Yang, 2020b). They show that the annihilator graph includes the zero divisor graph as a spanning graph. Moreover, the girth $gr(AG(A)) = \{3, 4, \infty\}$ is obtained by them.

Researches on the topological indices of zero-divisor graphs attract many authors since before time. In chemistry, graph theory uses to make predicts the chemical properties of the molecule such that every molecule can be showed as a graph. These graphs present atoms as vertices and chemical bonds as edges. Topological indices are used in studies such as the structure-dependency of total π-electron energy, physico-chemical applicability, etc. These indices can be classified into distance-based and degree-based. The Wiener index which is a distance-based index was studied for zero-divisor graphs in (Asir and Rabikka, 2021; Singh and Bhat, 2020). The Zagreb index which is a degree-based index was studied in (Singh and Bhat, 2020). Nowadays, the Sombor index is introduced by Gutman in (Gutman, 2021a). This topological index is degree-based and has attracted researchers, and many kinds of research have been conducted about this index (Das et al., 2021; Gutman, 2021b; Ghanbari and Alikhani, 2021; Reti et al., 2021; Zhou et al., 2021).

It is very important that determining the pharmacological, chemical and biological characteristics in sciences based on molecular such as medicine manufacture, chemistry, nanomaterials, and pharmacy. For this reason, topological index computation is widely applied. In the last 50 years, many topological indices, such as Sombor index, Forgotten index, Zagreb index, and PI index have been introduced to determine and measure the characters of molecules. These topological indices play an important role in medicine mathematical modeling. Because of that, the motivation of this paper is to research the forgotten topological index.

In this paper, we examine the Forgotten topological index for zero-divisor graphs of MV-algebras. Initially, we calculate this index for zero divisor graphs of MV-algebras having the cardinality of less than 5. Then, we generalize the Forgotten topological index for zero divisor graph of L_n where $n > 5$ using isomorphism between L_n and n-element MV-chain. Lastly, while the Forgotten index of zero divisor graph for MV-algebras which have cardinality 5, 6, 7, and 8 is computed, theorems presented are strengthened by supporting examples.

MATERIALS AND METHODS

The notion of MV-algebra was presented as Lukasiewicz's many-valued logic by C. C. Chang (Chang, 1958). This definition is used recently as follows (Chartrand et al., 2016):

Definition 1.1 The $M = \langle M, \oplus, \neg, 0\rangle$ is an MV-algebra if it satisfies the following axioms:
(MV1) $x \oplus 0 = x$
(MV2) $\neg\neg x = x$
(MV3) $x \oplus y = y \oplus x$
(MV4) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$
(MV5) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

Let $[0,1] = \{x \in \mathbb{R} | 0 \leq x \leq 1\}$ be a real unit interval and $\oplus$ operation is defined by $x \oplus y =_{def} \min(1, x + y)$ and $\neg x =_{def} 1 - x$. So, $\langle [0,1], \oplus, \neg, 0 \rangle$ is an MV-algebra.

Definition 1.2 Let N be a subset of an MV-algebra M such that containing the zero element of M. If N is closed under the operations of M and equipped with the restriction to of N these operations then it is called subalgebra of an MV-algebra M.

On an MV-algebra M, the constant 1 and the operations could be defined as:
$\neg 1 = 0$ and $x \odot y =_{def} \neg(\neg x \oplus \neg y)$ and $x \ominus y =_{def} x \odot \neg y$.

From above the definitions, $\langle [0,1], \odot, \neg, 0 \rangle$ is an MV-algebra such that $x \odot y = max\{0, x + y - 1\}$.

For each $n \geq 2$ integer number,

$$L_n =_{def} \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\right\} \quad (1)$$

indicates an n-element subalgebra of $[0,1]$.

Now, we give the well known following conclusions.

Lemma 1.3 (Singh and Bhat, 2020) Let $\langle M, \oplus, \neg, 0\rangle$ be an MV-algebra. The following conditions
(1) $\langle M, \odot, \neg, 0\rangle$ is an MV-algebra,
(2) The mapping $\neg : \langle M, \oplus, \neg, 0\rangle \to \langle M, \odot, \neg, 0\rangle$ is an isomorphism,
(3) $x \oplus 1 = 1$,
(4) $x \oplus \neg x = 1$,
(5) $x \oplus y = \neg(\neg x \odot \neg y)$,
(6) $x \odot \neg x = 0$
are verified.

Lemma 1.4 Suppose that $\langle M, \oplus, \neg, 0\rangle$ is an MV-algebra. The following conditions
(1) $x \leq y$,
(2) There is an element $m \in M$ such that $x \oplus m$=y,
(3) $x \oplus m \leq y \oplus m$ for all $m \in M$,
(4) $\neg x \oplus y = 1$,
(5) $\neg y \leq \neg x$,
(6) $x \odot \neg y = 0$,
(7) $x \odot m \leq y \odot m$ for all $m \in M$,
(8) $x \vee y = y$
are equivalent to each other.

Lemma 1.5 (Chang, 1958; Mundici, 2007) The following conditions
(1) $(x \oplus y) \oplus (x \odot y) = (x \oplus y)$,
(2) If $x \odot y = x \odot z$ and $x \oplus y = x \oplus z$, then $y = z$,
(3) $x \odot y \leq x \wedge y \leq x$, $y \leq x \vee y \leq x \oplus y$,
(4) $x \oplus (y \wedge z) = (x \oplus y) \wedge (x \oplus z)$,

(5) $x\odot(y \vee z) = (x\odot y) \vee (x\odot z)$
are verified for every $x, y, z \in M$.

Lemma 1.6 (Chang, 1958; Cignoli et al., 2013) The below conditions are equivalent to each other for $x, y \in M$:
(1) $x\oplus y = y$,
(2) $x\odot y = x$,
(3) $x \wedge \neg y = 0$,
(4) $\neg x \vee y = 1$.

All types of MV-algebras are produced from the MV-algebra $[0,1]$. For this reason, the MV-algebra $[0,1]$ is significant. The Chang Completeness Theorem supports this situation where an equation provides in $[0,1]$ if and only if it provides in each MV-algebra. Moreover, L_n is isomorphic to MV-chain which has n elements (Mundici, 2007).

Now, we recall some notations in graph theory. $G = (V, E)$ indicates a graph where $V(G)$ expresses the set of vertices and $E(G)$ expresses the set of edges. A graph G' is named a subgraph of G if $V(G') \subseteq V(G)$, $E(G') \subseteq E(G)$ and $u, v \in V(G')$ while $(u, v) \in E(G')$. Moreover, G is a connected graph if G contains no isolated vertices i.e. there must a path among any two vertices u and v for $u, v \in V(G)$ and $u \neq v$ (Chartrand et al., 2016; Harary, 1999).

In a graph G, the set of all neighbors of vertex $v \in V(G)$ is demonstrated by $N_G(v)$ and $N_G(v) = \{u |\, (u, v) \in E(G)\}$. A degree of a vertex $v \in V(G)$ is the number of edges connected to v and indicated by d_v or $deg(v)$ where $d_v = |N_G(v)|$.

If G is a graph with an edge between every pair of distinct vertices, then G is called a complete graph. Also, G is called to be a path graph if G has a sequence of vertices as $v_1, v_2, \dots, v_{n-1}, v_n$ where $d_{v_1} = d_{v_n} = 1$ and $d(v_i) = 2$ for $i = 2,3, \dots, n-1$ and $E(G) = \bigcup_{j=2,3,\dots,n-1}(v_j, v_{j+1})$. A graph G having n vertices and $E(G) = \bigcup_{i=2,3,\dots,n}(v_1, v_i)$ edge set is called star graph, where $d_{v_1} = n-1, d_{v_i} = 1$ for $i = 2, 3, \dots, n$. (Chartrand et al., 2016; Harary, 1999)

If $V(G)$ is able to be divided into two subsets V_1 and V_2 where $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V(G)$ and $u \in V_1$ and $v \in V_2$ for every $(u, v) \in E(G)$, then G is said to be a bipartite graph. In addition to that, if every vertex in V_1 is connected to every vertex in V_2, then G is called a complete bipartite graph.

For $u, v \in V(G)$, the distance indicates the length of the shortest path between u and v and is denoted by $d(u, v)$. Besides, the diameter of G is the maximum length of the shortest paths in G, that is, $diam(G) = max\{d(u, v) | u, v \in V(G)\}$.

Any two graphs G and H are isomorphic if a bijective mapping $f: V(G) \to V(H)$ exists where $(f(u), f(v)) \in E(H)$ when $(u, v) \in E(G)$, and indicated by $G \cong H$. Otherwise, G and H are called non-isomorphic.

During this paper, $\Gamma(M)$ expresses the zero divisor graph produced by an MV-algebra M where $V(\Gamma(M)) = \{x \in M | \exists\, y \in M\backslash\{0\}, x \odot y = 0\}$, and $E\big(\Gamma(M)\big) = \{(x, y) \,| x \neq y, x, y \in M, x \odot y = 0\}$.

Proposition 1.7 For any MV-algebra M satisfies $V\big(\Gamma(\mathrm{M})\big) = \mathrm{M}\backslash\{0, 1\}$ where $V\big(\Gamma(\mathrm{M})\big)$ indicates the set of vertices in zero divisor graph of M.

Gan and Yang characterized MV-algebra M such that $diam\big(\Gamma(M)\big) \in \{0, 1\}$. Moreover, $|M|$ shows the cardinal number of M (Gan and Yang, 2020a).

Theorem 1.8 Let M be an MV-algebra. The below statements are equivalent.

(1) $diam(\Gamma(M)) = 0$,
(2) $|M| = 3$,
(3) $M \cong L_3$,
(4) $\Gamma(M)$ is an empty graph.

Theorem 1.9 Suppose that M is an MV algebra. Then, the below statements are equivalent:
(1) $diam(\Gamma(M)) = 1$,
(2) $\Gamma(M) \cong K_2$,
(3) $|M| = 4$,
(4) $M \cong L_4$ or $M \cong B_4$ is the 4-element Boolean algebra.

In the same paper, Gan and Yang prove that if $|M| = n \neq 4$ and $\Gamma(M) \cong \Gamma(L_n)$, then $M \cong L_n$. However, for this important theorem, the following Lemma should be given.

Lemma 1.10 Suppose that $n \geq 5$ be an integer number. For vertex degrees of $\Gamma(L_n)$,
(1) If n is an odd number, then

$$\deg\left(\frac{i}{n-1}\right) = \begin{cases} n-1-i & if \quad \frac{n-1}{2} < i < n-2 \\ n-2-i & if \quad 1 \leq i \leq \frac{n-1}{2} \end{cases}. \tag{2}$$

(2) If n is an even number, then

$$\deg\left(\frac{i}{n-1}\right) = \begin{cases} n-1-i & if \quad \frac{n}{2} < i < n-2 \\ n-2-i & if \quad 1 \leq i \leq \frac{n}{2} \end{cases}. \tag{3}$$

Theorem 1.11 If $|M| = n \geq 5$ and $\deg(\Gamma(M)) = \deg(\Gamma(L_n))$, then $M \cong L_n$.

Corollary 1.12 Suppose that n is an integer number and $|M| = n \neq 4$. Then $\Gamma(M) \cong \Gamma(L_n)$ if and only if $M \cong L_n$.

Then, the authors classified MV-algebras as having cardinality up to seven where $n \in \{5, 6, 7\}$. They used the zero divisor graph of MV-algebras (Gan and Yang, 2020a).

Theorem 1.13 Let M be an MV-algebra such that
i. $|M| = 5$, then $M \cong L_5$,
ii. $|M| = 6$, then $M \cong L_6$ or $M \cong L_2 \times L_3$,
iii. $|M| = 7$, then $M \cong L_7$.

Topological indices, also known as molecular structure descriptors in natural sciences, are numerical values in graph theory, and these indices are inter disciplinary research areas related to mathematics, natural sciences, computer science and etc. The Forgotten topological index is also a degree-based topological index and is described as

$$FT(G) = \sum_{u \in V(G)} d_u^{\,3} = \sum_{uv \in E(G)} \sqrt{d_u^{\,2} + d_v^{\,2}} \tag{4}$$

where $G = (V(G), E(G))$ represents a graph (Furtula and Gutman, 2015).

RESULTS AND DISCUSSION

In this section, we calculate the Forgotten index of zero divisor graphs in MV-algebras. While doing these calculations, we use $\Gamma(A) \cong \Gamma(L_n) \Leftrightarrow A \cong L_n$, where L_n is the MV-chain containing n elements. During this paper, A will be considered as an MV-algebra.

Theorem 2.1 If an MV-algebra A holds one of the below properties, then the Forgotten index of $\Gamma(A)$ equals zero.

i. $\text{diam}(\Gamma(A)) = 0$,
ii. $|A| = 3$,
iii. $A \cong L_3$,
iv. $\Gamma(A)$ is a null graph.

Proof We assume that $|A| = 3$. Then, we have $|V(\Gamma(A))| = 1$ from the definition of zero divisor graph of MV-algebra. Therefore, we obtain

$$FT(\Gamma(A)) = \sum_{u \in V(\Gamma(A))} {d_u}^3 = 0^3 = 0.$$

The other statement's proof is clear because of Theorem 1.8.

Theorem 2.2 If an MV-algebra A holds one of the following properties, then the Forgotten index of $\Gamma(A)$ equals 2.

i. $\text{diam}(\Gamma(A)) = 1$,
ii. $|A| = 4$,
iii. $\Gamma(A) \cong K_2$,
iv. $A \cong L_4$ or $A \cong B_4$ where $B_4$is the 4-element Boolean algebra.

Proof We assume that $\Gamma(A) \cong K_2$. In this case, we have a graph such as figure 1.

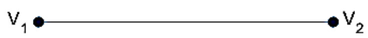


Figure 1. Complete graph K_2

Then, the Forgotten index of this graph K_2 is

$$FT(\Gamma(A)) = \sum_{u \in V(K_2)} {d_u}^3 = {d_{u_1}}^3 + {d_{u_2}}^3 = 1^3 + 1^3 = 2.$$

All statements are equivalent to each other since Theorem 1.9.

Theorem 2.3 Let L_n be the n-element MV-chain such that $n \geq 5$. Then, Forgotten index of $\Gamma(L_n)$ is

$$FT(\Gamma(L_n)) = \begin{cases} \frac{(n-3)^2(2n-5)(n-1)}{8}, & n \text{ is odd} \\ \frac{(n-2)^2(2n^2-11n+16)}{8}, & n \text{ is even} \end{cases}. \quad (5)$$

Proof: Firstly, we prove that the degree of each vertex in $\Gamma(L_n)$. We have

$$L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\right\} \text{ and } x \odot y = \frac{i}{n-1} \odot \frac{j}{n-1} = max\left\{0, \frac{i+j}{n-1} - 1\right\}$$

for any x and $y \in V(\Gamma(L_n))$ such that $x = \frac{i}{n-1}$, $y = \frac{j}{n-1}$, for $n \geq 5$.

We suppose that n is an odd integer number. For each $\frac{i}{n-1}$ vertex of $\Gamma(L_n)$ where $i > 0$ and $i \leq n-2$ is adjacent to $\frac{j}{n-1}$ vertex where $i+j \leq n-1$. So, the degree of vertex $\frac{i}{n-1}$ is $n-2-i$ where $i = 1, \dots, \frac{n-1}{2}$. On the other hand, the degree of vertex $\frac{i}{n-1}$ is $n-1-i$ where $i = \frac{n+1}{2}, \dots, n-2$. So, we obtain Forgotten index of $\Gamma(L_n)$ is

$$FT\big(\Gamma(L_n)\big) = \sum_{i=1}^{\frac{n-1}{2}} (n-2-i)^3 + \sum_{i=\frac{n+1}{2}}^{\frac{n-2}{2}} (n-1-i)^3$$

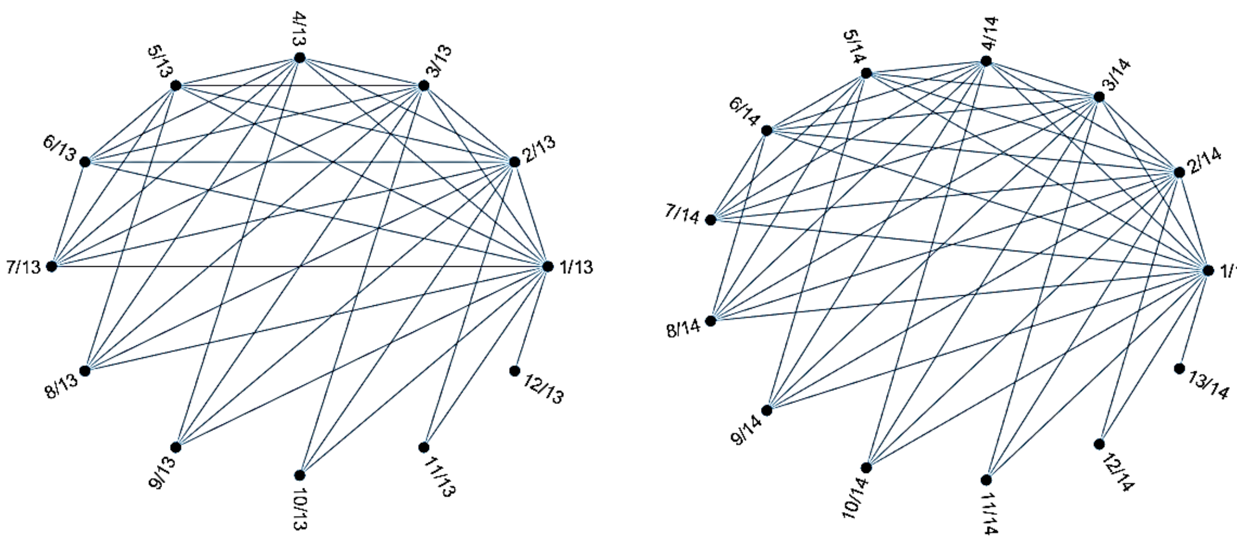
$$= \frac{(n-3)^2(2n-5)(n-1)}{8}.$$

Now, we suppose that n is an even integer number. For each $\frac{i}{n-1}$ vertex of $\Gamma(L_n)$ where $i > 0$ and $i \leq n-2$ is also adjacent to $\frac{j}{n-1}$ vertex where $i+j \leq n-1$. Therefore, the degree of the vertex $\frac{i}{n-1}$ is $n-2-i$ where $i = 1, \dots, \frac{n}{2}-1$. On the other hand, the degree of the vertex $\frac{i}{n-1}$ is $n-1-i$ where $i = \frac{n}{2}, \dots, n-2$. So, we obtain

$$FT\big(\Gamma(L_n)\big) = \sum_{i=1}^{\frac{n}{2}-1} (n-2-i)^3 + \sum_{i=\frac{n}{2}}^{n-2} (n-1-i)^3$$

$$= \frac{(n-2)^2(2n^2-11n+16)}{8}.$$

Example 2.4 The graphs $\Gamma(L_{14})$ and $\Gamma(L_{15})$ as follows:



(a) (b)

Figure 2. The graphs $\Gamma(L_{14})$ and $\Gamma(L_{15})$

The degrees of vertices of the graphs $\Gamma(L_{14})$ and $\Gamma(L_{15})$ are presented in Table 1 and Table 2.

Table 1. Vertices and their degrees of $\Gamma(L_{14})$.

Vertex	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{3}{13}$	$\frac{4}{13}$	$\frac{5}{13}$	$\frac{6}{13}$	$\frac{7}{13}$	$\frac{8}{13}$	$\frac{9}{13}$	$\frac{10}{13}$	$\frac{11}{13}$	$\frac{12}{13}$
Degree	11	10	9	8	7	6	6	5	4	3	2	1

Table 2. Vertices and their degrees of $\Gamma(L_{15})$.

Vertex	$\frac{1}{14}$	$\frac{2}{14}$	$\frac{3}{14}$	$\frac{4}{14}$	$\frac{5}{14}$	$\frac{6}{14}$	$\frac{7}{14}$	$\frac{8}{14}$	$\frac{9}{14}$	$\frac{10}{14}$	$\frac{11}{14}$	$\frac{12}{14}$	$\frac{13}{14}$
Degree	12	11	10	9	8	7	6	6	5	4	3	2	1

And using the above tables, one can conclude the Forgotten index of these graphs as follows:

$$FT\big(\Gamma(L_{14})\big) = \sum_{u\in\Gamma(L_{14})} {d_u}^3 = \frac{(14-2)^2(2\cdot 14^2 - 11\cdot 14 + 16)}{8} = 4572$$

$$FT\big(\Gamma(L_{15})\big) = \sum_{u\in\Gamma(L_{15})} {d_u}^3 = \frac{(15-3)^2(2\cdot 15 - 5)(15-1)}{8} = 6300$$

Theorem 2.5 Suppose that A is an MV-algebra and $|A| = n \neq 4$ and $n \in \mathbb{Z}$. If $A \cong L_n$ then $FT\big(\Gamma(A)\big) = FT\big(\Gamma(L_n)\big)$.

Proof We assume that $\mathrm{A} \cong L_n$ and $|A| = n \neq 4$ .where $L_n = \left\{0, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, 1\right\}$ and $A = \{0, a_1, a_2, \dots, a_{n-2}, 1\}$. Since the function $f = 0 \mapsto 0, \dots, a_i \mapsto \frac{i}{n-1}, \dots, 1 \mapsto 1$ where $i = 1, \dots, n-2$ is an isomorphism from L_n to A. If $\Gamma(A) \cong \Gamma(L_n)$, then we have a bijection from $V(\Gamma(A))$ to $V(\Gamma(L_n))$. So that, for any $a_i, a_j \in V(\Gamma(A))$, if there exists an edge between vertices a_i and a_j, then there must be an edge between vertices $f(a_i)$ and $f\big(a_j\big)$ where $f(a_i), f\big(a_j\big) \in \Gamma(L_n)$.

If $n = 2$ or $n = 3$, then $\Gamma(A) \cong \Gamma(L_n)$, obviously.

We suppose that $n \geq 5$ and $a_i, a_j \in V(\Gamma(A))$. For $f(a_i), f\big(a_j\big) \in \Gamma(L_n)$, we have $f(a_i) \odot f\big(a_j\big) = \frac{i}{n-1} \odot \frac{j}{n-1}$ and from the definition of L_n we obtain $max\left\{0, \frac{i+j}{n-1} - 1\right\} = max\left\{0, \frac{i+j-n+1}{n-1}\right\}$. Now, let $\frac{i+j-n+1}{n-1} > 0$, then $i + j - n + 1 > 0$ and we attain $j > n - i - 1$. This is contradiction. Therefore, $i + j - n + 1 \leq 0$ and $max\left\{0, \frac{i+j-n+1}{n-1}\right\} = 0$. So, we have an edge between vertices $f(a_i)$ and $f\big(a_j\big)$, and $\Gamma(A) \cong \Gamma(L_n)$.

Since we have $\Gamma(A) \cong \Gamma(L_n)$, then $deg\big(\Gamma(A)\big) = deg(\Gamma(L_n))$. Therefore,

$$FT\big(\Gamma(A)\big) = \sum_{a_i\in V(\Gamma(A))} {d_{a_i}}^3 = \sum_{f(a_i)\in V(\Gamma(L_n))} {d_{f(a_i)}}^3 = FT\big(\Gamma(L_n)\big).$$

Theorem 2.6 Assume that A is an MV-algebra and $|A| = 5$. Then the Forgotten index of $\Gamma(A)$ is $FT\big(\Gamma(A)\big) = 10$.

Proof If $|A| = 5$, then $\mathrm{A} \cong L_5$ and $\Gamma(A) \cong \Gamma(L_5)$ from Corollary 1.12. For $n = 5$, $deg\big(\Gamma(L_5)\big) = \{1,1,2\}$. Therefore, we obtain

$$FT\big(\Gamma(L_5)\big) = \sum_{u\in\Gamma(L_5)} {d_u}^3 = \frac{(n-3)^2(2n-5)(n-1)}{8} = 10$$

for $n = 5$.

Theorem 2.7 Assume that A is an MV-algebra and $|A| = 6$. Then the Forgotten index of $\Gamma(A)$ is $FT(\Gamma(A)) = 44$ or $FT(\Gamma(A)) = 18$.

Proof If $|A| = 6$, then we get $A \cong L_6$ or $A \cong L_2 \times L_3$ and $\Gamma(A) \cong \Gamma(L_6)$ or $\Gamma(A) \cong \Gamma(L_2 \times L_3)$ from Theorem 1.13. For $n = 6$, $deg(\Gamma(L_6)) = \{1, 2, 2, 3\}$. Therefore, we obtain

$$FT(\Gamma(L_6)) = \sum_{u \in \Gamma(L_6)} {d_u}^3 = \frac{(n-2)^2(2n^2 - 11n + 16)}{8} = 44$$

for $n = 6$.

On the other hand, $deg(\Gamma(L_2 \times L_3)) = \{1, 1, 2, 2\}$. Therefore, we obtain

$FT(\Gamma(L_2 \times L_3)) = 1^3 + 1^3 + 2^3 + 2^3 = 18$.

Theorem 2.8 Assume that A is an MV-algebra and $|A| = 7$. Then the Forgotten index of $\Gamma(A)$ is $FT(\Gamma(A)) = 10$.

Proof If $|A| = 7$, then $A \cong L_7$ and $\Gamma(A) \cong \Gamma(L_7)$. For $n = 7$, $deg(\Gamma(L_7)) = \{1,1,2\}$. Therefore, we obtain

$$FT(\Gamma(L_7)) = \sum_{u \in \Gamma(L_7)} {d_u}^3 = \frac{(n-3)^2(2n-5)(n-1)}{8} = 108$$

for $n = 7$.

Theorem 2.9 Assume that A is an MV-algebra and $|A| = 8$. Then, the Forgotten index of $\Gamma(A)$ can only be equal to one of the followings:

- $FT(\Gamma(A)) = 252$
- $FT(\Gamma(A)) = 128$
- $FT(\Gamma(A)) = 84$

Proof When enumerating all 8-element MV-algebras, we get 22 different samples in three cases (Fig.3, Fig.5, and Fig.7).

Case 1: In this case, we have 6 different samples for MV-algebra with 8-element.

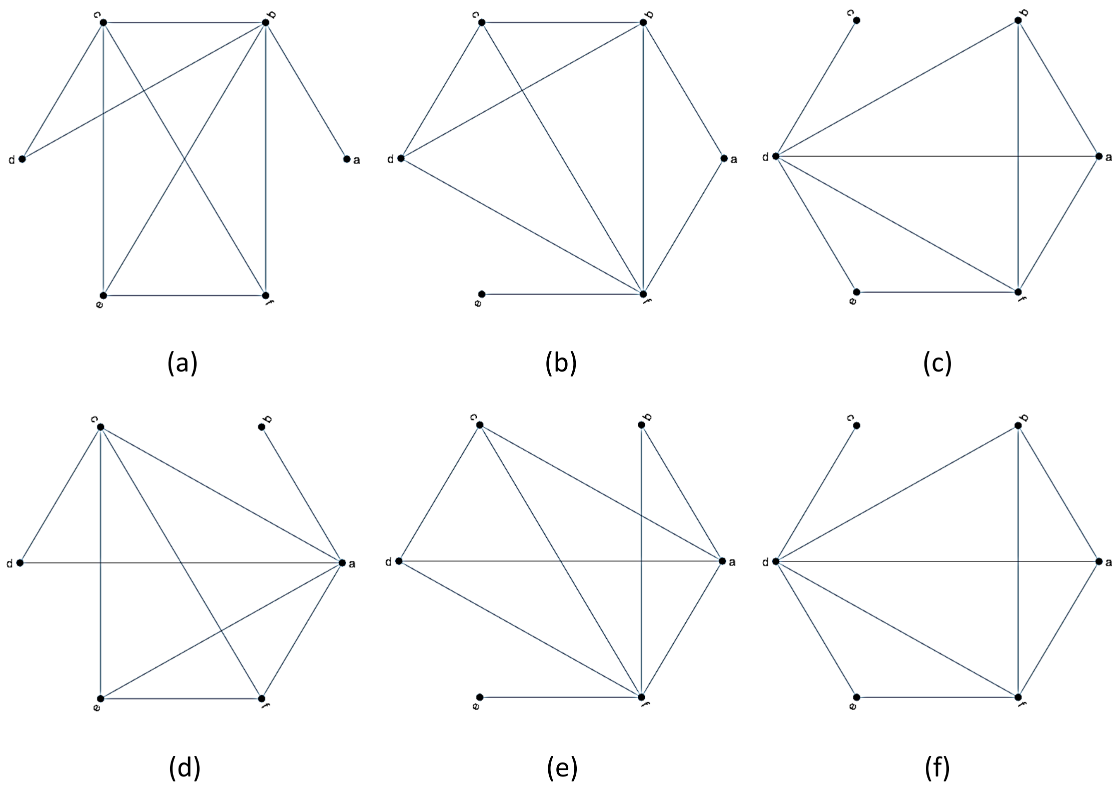


Figure 3. MV-algebras with 8-element in case 1

Also, zero divisor graphs $\Gamma(A)$ in Fig.3.a-f are isomorphic to $\Gamma(L_8)$ in Fig.4. Therefore, $FT\big(\Gamma(A)\big) = FT(\Gamma(L_8))$ and

$$FT\big(\Gamma(L_8)\big) = \sum_{u\in\Gamma(L_8)} {d_u}^3 = 252.$$

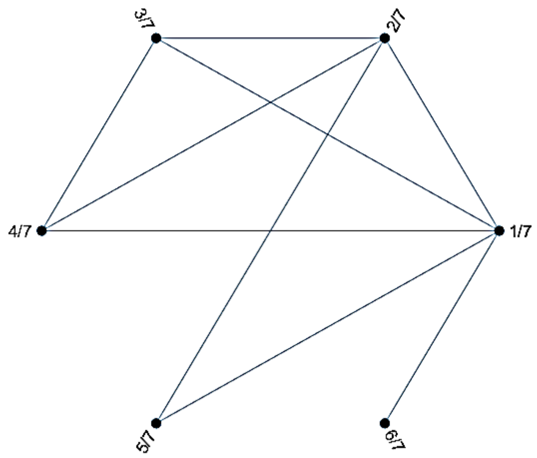


Figure 4. $\Gamma(L_8)$.

Case 2: In this case, we have 12 different samples for MV-algebra with 8-element.

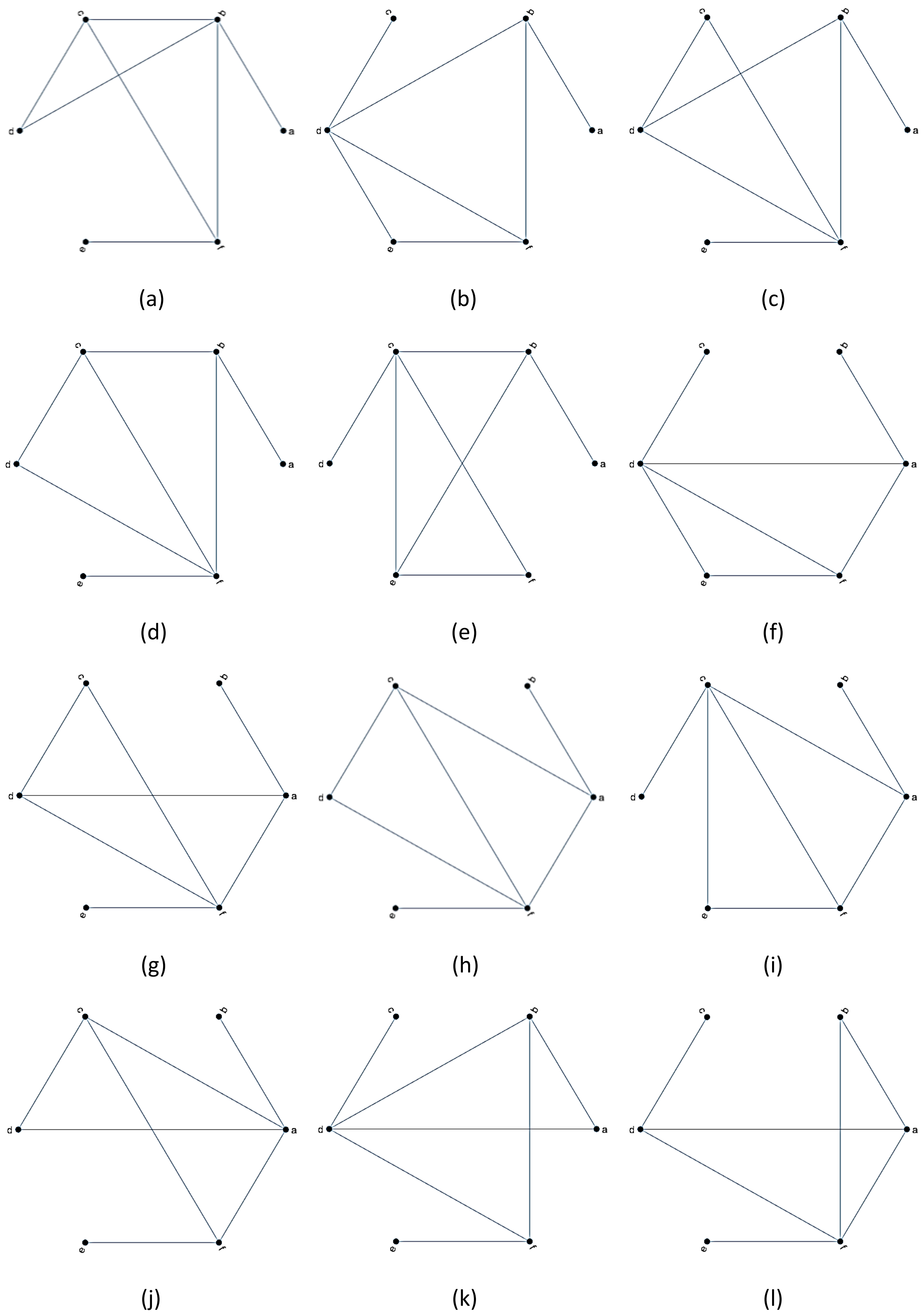


Figure 5. MV-algebras with 8-element in case 2

In addition, zero divisor graphs $\Gamma(A)$ in Fig.5.a-l are isomorphic to $\Gamma(L_2 \times L_4)$ in Fig.6. Therefore, $FT\big(\Gamma(A)\big) = FT(\Gamma(L_2 \times L_4))$ and

$$FT\big(\Gamma(L_2 \times L_4)\big) = \sum_{u \in \Gamma(L_2 x L_4)} {d_u}^3 = 128.$$

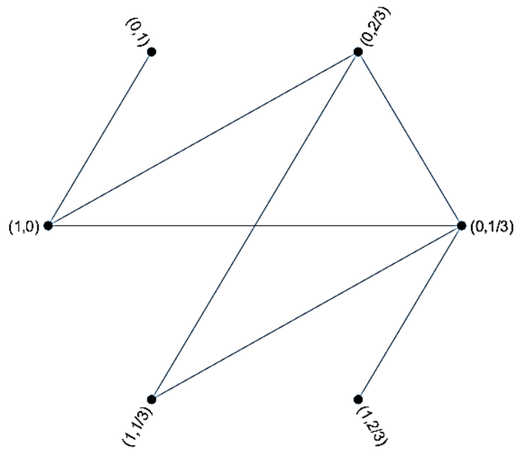


Figure 6. $\Gamma(L_2 x L_4)$

Case 3: In the final case, we have the remaining 4 different samples for MV-algebra with 8-element.

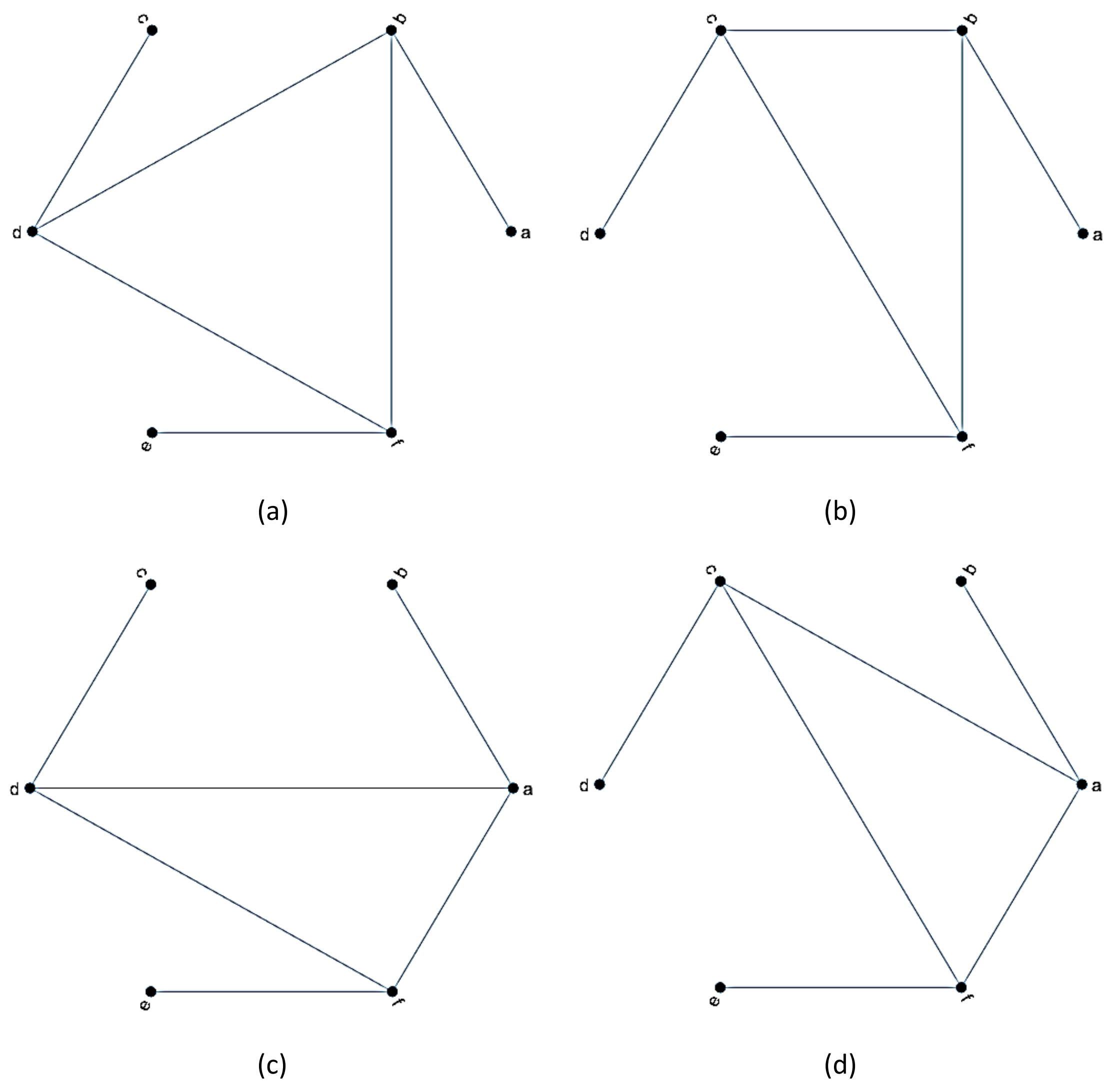


Figure 7. MV-algebras with 8-element in case 3

Besides, zero divisor graphs $\Gamma(A)$ in Fig.7.a-d are isomorphic to $\Gamma(L_2 \times L_2 \times L_2)$ in Fig.8. Therefore, $FT\big(\Gamma(A)\big) = FT(\Gamma(L_2 \times L_2 \times L_2))$ and

$$FT\big(\Gamma(L_2 \times L_2 \times L_2)\big) = \sum_{u\in\Gamma(L_2 \mathrm{x} L_2 \mathrm{x} L_2)} {d_u}^3 = 84.$$

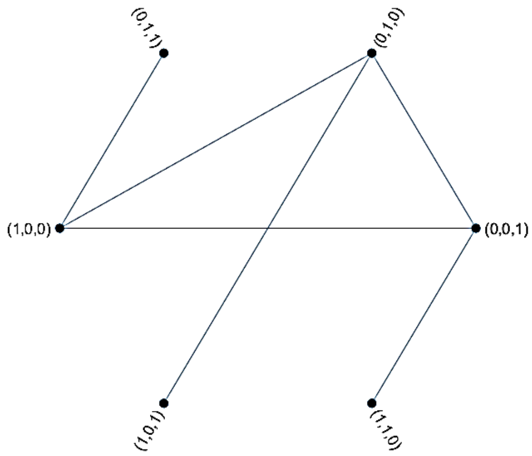


Figure 8. $\Gamma(L_2 \mathrm{x} L_2 \mathrm{x} L_2)$

CONCLUSION

Topological indices, which are also known as molecular structure descriptors, are numerical values in graphs, and they are multidisciplinary research topics that can be concerned with the chemical constitution for correlation of chemical structure with various physical characteristics, biological activity, or chemical reactivity. The forgotten index is a degree-based topological index and is described as the sum of cubes of the degrees of each vertex in a graph. In this study, it is investigated the Forgotten topological index for zero divisor graphs of MV-algebras having a cardinality of less than 5. Also, the Forgotten topological index for zero divisor graph of L_n where $n > 5$ using isomorphism between L_n and n-element MV-chain is concluded. Moreover, the Forgotten index of zero divisor graph for MV-algebras which have cardinality 5, 6, 7, and 8 is examined.

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