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Some results on generalized Euler-type integrals related to the four parameters Mittag-Leffler function

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Abstract —Special functions such as hypergeometric, zeta, Bessel, Whittaker, Struve, Airy, Weber-Hermite and Mittag-Leffler functions are obtained as a solution to complex differential equations in engineering, science and technology. In this work, generalized Euler-type integrals involving four parameters Mittag-Leffler function are proposed. Some special cases of this type of generalized integrals that are corresponding to well-known results in the literature are also inferred.

Subject Classification (2020): 33B15, 33B90.

1. Introduction

Throughout this article, $\mathbb{N}$ and $\mathbb{C}$ represent the sets of natural and complex numbers, respectively. In 1729, Leonard Euler studied the classical gamma function by extending the factorial function from the domain of natural numbers, to the region in the right half of the complex plane given as follows [1-2]:

$$\Gamma(\sigma) = \int_0^1 t^{\sigma-1} exp(-t)dt, \quad (\text{Re }(\sigma) > 0) \tag{1.1}$$

A year later, He established classical beta function, $B(\sigma, \mathcal{E})$ for a pair of complex numbers σ and $\mathcal{E}$ with positive real parts through the integrand which is given by [3-4].

$$B(\sigma, \mathcal{E}) = \int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1}\, dt, \qquad (\text{Re }(\sigma) > 0, \text{Re }(\mathcal{E}) > 0) \tag{1.2}$$

Carl Friedrich Gauss [5] generalized the geometric series in the following classical Gauss hypergeometric function:

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$$F(\sigma,\mathcal{E};\eth;z)=\sum_{n_2}^{\infty}\frac{(\sigma)_{n_2}(\mathcal{E})_{n_2}}{(\eth)_{n_2}}\frac{z^{n_2}}{n_2!}=\sum_{n_2}^{\infty}(\sigma)_{n_2}\frac{B(\mathcal{E}+n_2,\eth-\mathcal{E})}{B(\mathcal{E},\ \eth-\mathcal{E})}\frac{z^{n_2}}{n_2!},\qquad (|z|<1,\mathrm{Re}\,(\eth)>\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.3)$$

and it can reduced to the Kumar confluent hyper geometric function defined by Ernst Eduard Kumar in [6].

$$\Phi(\mathcal{E};\eth;z)=\sum_{n_2}^{\infty}\frac{(\mathcal{E})_{n_2}}{(\eth)_{n_2}}\frac{z^{n_2}}{n_2!}\frac{z^{n_2}}{n_2!}=\sum_{n_2}^{\infty}\frac{B(\mathcal{E}+n_2,\eth-\mathcal{E})}{B(\mathcal{E},\ \eth-\mathcal{E})}\frac{z^{n_2}}{n_2!},\qquad (\mathrm{Re}\,(\eth)>\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.4)$$

Here $(\sigma)_{n_2}$represent classical pochhammer symbol defined as [7-8].

$$(\sigma)_{n_2}=\begin{cases}\sigma(\sigma+1)(\sigma+2)(\sigma+3)\cdots(\sigma+n_2-1), & (n_2\geq 0,\sigma\neq 0)\\ 1\qquad, & (n_2=0)\end{cases}$$

Other properties of Gauss hypergeometric and confluent hypergeometric such integral representation, transformation formulas, summation formulas and contiguity relations can be found in [9].

Chaudhry and Zubair [10-11] extended classical gamma function in (1.1) by using exponential kernel as follows [12-14]:

$$\Gamma_{\mathfrak{J}}(\sigma)=\int_0^1 t^{\sigma-1}exp\left(-t-\frac{\mathfrak{J}}{t}\right)dt,\qquad (\mathrm{Re}\,(\sigma)>0,\mathrm{Re}\,(\mathfrak{J})>0)$$

Chaudhry et al., [15-17] introduced the following extension of beta function as an extension of classical beta function in (1.2):

$$B_{\mathfrak{J}}(\sigma,\mathcal{E})=\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1}\,exp\left(-\frac{\mathfrak{J}}{t(1-t)}\right)dt,\qquad (\mathrm{Re}\,(\mathfrak{J})>0,\mathrm{Re}\,(\sigma)>0,\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.5)$$

Chaudhry et al., [18] proposed the following extended of Gauss hypergeometic and confluent hyper geometric functions by using extended beta function in (1.3) and (1.4) as

$$F_{\mathfrak{J}}(\sigma,\mathcal{E};\eth;z)=\sum_{n_2}^{\infty}(\sigma)_{n_2}\frac{B_{\mathfrak{J}}(\mathcal{E}+n_2,\eth-\mathcal{E})}{B(\mathcal{E},\ \eth-\mathcal{E})}\frac{z^{n_2}}{n_2!},\qquad (\mathfrak{J}\geq 0;\ |z|<1,\mathrm{Re}\,(\eth)>\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.6)$$

and

$$\Phi_{\mathfrak{J}}(\mathcal{E};\eth;z)=\sum_{n_2}^{\infty}\frac{B_{\mathfrak{J}}(\mathcal{E}+n_2,\eth-\mathcal{E})}{B(\mathcal{E},\ \eth-\mathcal{E})}\frac{z^{n_2}}{n_2!},\qquad (\mathfrak{J}\geq 0;\ \mathrm{Re}\,(\eth)>\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.7)$$

Lee et al., [19-20] presented and investigated the following extension of beta function as an extension of (1.5):

$$B_{\mathfrak{J}}^{\eth}(\sigma,\mathcal{E})=\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1}\,exp\left(-\frac{\mathfrak{J}}{t^{\eth}(1-t)^{\eth}}\right)dt, (\mathrm{Re}\,(\mathfrak{J})>0,\mathrm{Re}\,(\sigma)>0,\mathrm{Re}\,(\mathcal{E})>0,\mathrm{Re}\,(\eth)>0)\qquad (1.8)$$

They also [19] extended Gauss and confluent hyper geometric functions in (1.6) and (1.7) as follows:

$$F_{\mathfrak{I}}^{\eth}(\sigma,\mathcal{E};\eth;z)=\sum_{n_2}^{\infty}(\sigma)_{n_2}\frac{B_{\mathfrak{I}}^{\eth}(\mathcal{E}+n_2,\eth-\mathcal{E})}{B(\mathcal{E},\ \eth-\mathcal{E})}\frac{z^{n_2}}{n_2!},\qquad (\mathfrak{I}\geq 0;\ |z|<1,\mathrm{Re}\,(\eth)>\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.9)$$

and

$$\Phi_{\mathfrak{I}}^{\eth}(\mathcal{E};\eth;z)=\sum_{n_2}^{\infty}\frac{B_{\mathfrak{I}}^{\eth}(\mathcal{E}+n_2,\eth-\mathcal{E})}{B(\mathcal{E},\ \eth-\mathcal{E})}\frac{z^{n_2}}{n_2!},\qquad (\mathfrak{I}\geq 0;\ \mathrm{Re}\,(\eth)>\mathrm{Re}\,(\mathcal{E})>0)\qquad (1.10)$$

Luo et al., [21] presented the following extension of beta function as a generalization of beta function in (1.10):

$$B_{\mathfrak{I}}^{\eth,\lambda}(\sigma,\mathcal{E})=\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1}\,exp\left(-\frac{\mathfrak{I}}{t^{\eth}(1-t)^{\lambda}}\right)dt\qquad (1.11)$$

$$(\mathrm{Re}\,(\mathfrak{I})>0,\mathrm{Re}\,(\sigma)>0,\mathrm{Re}\,(\mathcal{E})>0,\mathrm{Re}\,(\eth)>0,\mathrm{Re}\,(\lambda)>0)$$

The classical Mittag-Leffler function was first studied by G.M. Mittag-Leffler [22-24] as an extension of exponential function as shown below:

$$E_{\omega}(z)=\sum_{n_2=0}^{\infty}\frac{z^{n_2}}{\Gamma(\omega n_2+1)},\qquad (\omega,z\in\mathbb{C},\mathrm{Re}\,(\omega)>0)\qquad (1.12)$$

Wiman [25-26] extended classical Mittag-Leffler function in (1.12) by introducing two parameters Mittag-Leffler function as follows:

$$E_{\omega,\varpi}(z)=\sum_{n_2=0}^{\infty}\frac{z^{n_2}}{\Gamma(\omega n_2+\varpi)},\qquad (\omega,\varpi,z\in\mathbb{C}\ Re\,(\omega)>0,Re(\varpi)>0)\qquad (1.13)$$

Prabhakar [27] investigated three parameters Mittag-Leffler function as a generalization of (1.13) as follows:

$$E_{\omega,\varpi}^{\eta}(z)=\sum_{n_2=0}^{\infty}\frac{(\eta)_{n_2}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2},\qquad (\omega,\varpi,\eta,z\in\mathbb{C},\mathrm{Re}\,(\omega)>0,\mathrm{Re}\,(\varpi)>0,\mathrm{Re}\,(\eta)>0)\qquad (1.14)$$

Shukla and Prajapati [28] investigated four parameters Mittag-Leffler function as an extension of (1.14) as follows:

$$E_{\omega,\varpi}^{\eta,q}(z)=\sum_{n_2=0}^{\infty}\frac{(\eta)_{qn_2}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2},\qquad (\omega,\varpi,\eta,z\in\mathbb{C},\mathrm{Re}\,(\omega)>0,\mathrm{Re}\,(\varpi)>0,\mathrm{Re}\,(\eta)>0,q\in(0,1)\cup\mathbb{N})\qquad (1.15)$$

where $(\eta)_{qn_2}$ represents generalized pochhammer symbol defined by [29]

$$(\eta)_{qn_2}=\frac{\Gamma(\eta+qn_2)}{\Gamma(\eta)}$$

Related literature is also available in [30-34].

2. Main Result

The generalized Euler-type integrals involving the four parameters Mittag-Leffler function are presented in the following theorems and corollaries:

Theorem 2.1. If $\omega, \varpi, \eta, \sigma, \mathcal{E}, \Im \in \mathbb{C}$, $Re(\omega) > 0, Re(\varpi) > 0$, $Re(\eta) > 0$, $Re(\sigma) > 0$, $Re(\mathcal{E}) > 0$, $Re(\Im) > 0$ and $q \in \mathbb{N}$, then

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt = \sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2 + \varpi)} \frac{z^{n_2}}{n_2!} B_{\Im}^{\eth,\lambda}(\sigma + \omega n_2, \mathcal{E}) \quad (2.1)$$

Proof.

Let represent left-hand side of (2.1) by A_1, we have

$$A_1 = \int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt$$

Applying (1.15), we obtain

$$A_1 = \int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) \left\{\sum_{n=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2 + \varpi)} \frac{z^{n_2}}{n_2!} t^{\omega n_2}\right\} dt$$

Interchanging the order of summation and integration, gives

$$A_1 = \sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2 + \varpi)} \frac{z^{n_2}}{n_2!} \int_0^1 t^{\sigma+\omega n_2 - 1}(1-t)^{\mathcal{E}-1} \, exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) dt$$

Considering (1.11), we have

$$A_1 = \sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2 + \varpi)} \frac{z^{n_2}}{n_2!} B_{\Im}^{\eth,\lambda}(\sigma + \omega n_2, \mathcal{E})$$

Lemma 2.2. The following result holds [35]:

$$\int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{g} dt = B(\sigma, \mathcal{E})(b-a)^{\sigma+\mathcal{E}-1}(av+h)^{g} {}_2F_1\left(\sigma, -g; \sigma+\mathcal{E}; -\frac{v(b-a)}{(av+h)}\right) \quad (2.2)$$

$$(Re(\sigma) > 0, Re(\mathcal{E}) > 0; \; |arg((bv+h)(av+h)^{-1})| < \pi)$$

Theorem 2.3. If $\omega, \varpi, \eta, \sigma, \mathcal{E}, \Im \in \mathbb{C}$, $Re(\omega) > 0$, $Re(\varpi) > 0$, $Re(\eta) > 0$, $Re(\sigma) > 0$, $Re(\mathcal{E}) > 0$, $Re(\Im) > 0$, $|arg((bv+h)(av+h)^{-1})| < \pi$ and $q \in \mathbb{N}$, then

$$\int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{\mathscr{g}} exp\left(-\frac{\mathfrak{J}}{(t-a)^{\eth}(b-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}\left(z(b-t)^{\ell}\right)dt$$

$$= (av+h)^{\mathscr{g}} \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \frac{(-\mathfrak{J})^{n_1}}{n_1} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!} B(\sigma-\eth n_1, \mathcal{E}-\lambda n_1+\kappa n_2) \tag{2.3}$$

$$\times (b-a)^{\sigma+\mathcal{E}-\eth n_1-\lambda n_1+\kappa n_2-1} {}_2F_1\left(\sigma-\eth n_1, -\mathscr{g}; \sigma+\mathcal{E}-\eth n-\lambda n_1+\kappa n_{2_1}; -\frac{v(b-a)}{(av+h)}\right)$$

Proof.

Let donate left-hand side of (2.3) by A_2, we obtain

$$A_2 = \int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{\mathscr{g}} exp\left(-\frac{\mathfrak{J}}{(t-a)^{\eth}(b-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}\left(z(b-t)^{\ell}\right)dt$$

Using (1.11), we obtain

$$A_2 = \int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{\mathscr{g}} exp\left(-\frac{\mathfrak{J}}{t^{\eth}(1-t)^{\lambda}}\right)\left\{\sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{\left(z(b-t)^{\ell}\right)^{n_2}}{n_2!}\right\}dt$$

Considering the definition of exponential function

$$A_2 = \int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{\mathscr{g}} \left\{\sum_{n_1=0}^{\infty} \frac{(-\mathfrak{J})^{n_1}}{(t-a)^{n_1\eth}(b-t)^{n_1\lambda} n_1!}\right\}$$

$$\times \left\{\sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{\left(z(b-t)^{\ell}\right)^{n_2}}{n_2!}\right\}dt$$

Interchanging the order of integration and summations, yields

$$A_2 = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{(-\mathfrak{J})^{n_1}}{n_1!} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!} \int_a^b (t-a)^{\sigma-n_1\eth-1}(b-t)^{\mathcal{E}-n_1\lambda+n_2\ell-1}(tv+h)^{\mathscr{g}} dt$$

Applying (2.2) and simplifying, we get

$$A_2 = (av+h)^{\mathscr{g}} \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \frac{(-\mathfrak{J})^{n_1}}{n_1} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!} B(\sigma-\eth n_1, \mathcal{E}-\lambda n_1+\kappa n_2)$$

$$\times (b-a)^{\sigma+\mathcal{E}-\eth n_1-\lambda n_1+\kappa n_2-1} {}_2F_1\left(\sigma-\eth n_1, -\mathscr{g}; \sigma+\mathcal{E}-\eth n-\lambda n_1+\kappa n_{2_1}; -\frac{v(b-a)}{(av+h)}\right)$$

Corollary 2.4. Substituting $\Im = 0$, the following result can be obtained

$$\int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{g} exp\left(-\frac{\Im}{(t-a)^{\eth}(b-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}\left(z(b-t)^{\ell}\right)dt$$

$$= (av+h)^{g} \sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!} B(\sigma, \mathcal{E}+\kappa n_2)\,(b-a)^{\sigma+\mathcal{E}+\kappa n_2-1}$$

$$\times\, {}_2F_1\left(\sigma, -g; \sigma+\mathcal{E}+\kappa n_2; -\frac{v(b-a)}{(av+h)}\right)$$

Corollary 2.5. Putting $a = 0$ and $b = 1$, the following formula can be obtained

$$\int_0^1 (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{g} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}\left(z(1-t)^{\ell}\right)dt$$

$$= h^{g} \sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!}\, B(\sigma-\eth n_1, \mathcal{E}-\lambda n_1+\kappa n_2)\, {}_2F_1\left(\sigma, -g; \sigma+\mathcal{E}+\kappa n_2; -\frac{v}{h}\right)$$

Theorem 2.6. If $\omega,\ \varpi,\ \eta,\ \sigma,\ \mathcal{E},\ \Im \in \mathbb{C},\ Re(\omega) > 0,\ Re(\varpi) > 0,\ Re(\eta) > 0, Re(\sigma) > 0,\ Re(\mathcal{E}) > 0,$ $Re(\Im) > 0;\ \alpha,\ \beta \geq 0$ and $q \in \mathbb{N}$, then

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\left\{1-vt^{\alpha}(1-t)^{\beta}\right\}^{-\gamma} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt$$

$$= \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \frac{v^{n_1}(\gamma)_{n_1}}{n_1} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!} B_{\Im}^{\eth,\lambda}(\sigma+\alpha n_1+\omega n_2, \mathcal{E}-\sigma+\beta n_1) \tag{2.4}$$

Proof.

Let represent left-hand side of (2.4) by A_3, we have

$$A_3 = \int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\left\{1-vt^{\alpha}(1-t)^{\beta}\right\}^{-\gamma} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt$$

Applying (1.15), we have

$$A_3 = \int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\left\{1-vt^{\alpha}(1-t)^{\beta}\right\}^{-\gamma} exp\left(-\frac{\Im}{t^{\eth}(1-t)^{\lambda}}\right)\left\{\sum_{n_2=0}^{\infty} \frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)} \frac{z^{n_2}}{n_2!} t^{\omega n_2}\right\}dt$$

Using binomial theorem, we get

$$A_3=\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\left\{\sum_{n_1=0}^{\infty}\frac{v^{n_1}t^{\alpha n_1}(1-t)^{\beta n_1}}{n_1!}(\gamma)_{n_1}\right\}exp\left(-\frac{\mathfrak{I}}{t^{\eth}(1-t)^{\lambda}}\right)$$

$$\times\left\{\sum_{n_2=0}^{\infty}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}t^{\omega n_2}\right\}dt$$

Interchanging the order of integration and summation, we have

$$A_3=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{v^{n_1}(\gamma)_{n_1}}{n_1}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}\int_0^1 t^{\sigma+\alpha n_1+\omega n_2-1}(1-t)^{\mathcal{E}-\sigma+\beta n_1-1}exp\left(-\frac{\mathfrak{I}}{t^{\eth}(1-t)^{\lambda}}\right)dt$$

Re-written this equation using (1.11), gives

$$A_3=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{v^{n_1}(\gamma)_{n_1}}{n_1}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}B_{\mathfrak{I}}^{\eth,\lambda}(\sigma+\alpha n_1+\omega n_2,\mathcal{E}-\sigma+\beta n_1)$$

Corollary 2.7. Setting $\gamma=0$ in (2.4), we have

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}exp\left(-\frac{\mathfrak{I}}{t^{\eth}(1-t)^{\lambda}}\right)\mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt=\sum_{n_2}^{\infty}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}B_{\mathfrak{I}}^{\eth,\lambda}(\sigma+\omega n_2,\mathcal{E}-\sigma)$$

Corollary 2.8. Setting $\mathfrak{I}=\gamma=0$ in (2.4), we get

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\,\mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt=\sum_{n_2}^{\infty}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}B(\sigma+\omega n_2,\mathcal{E}-\sigma)$$

3. Conclusion

In this work, we have proposed some generalized Euler type integrals involving four parameters Mittag-Leffler function of the form $\mathrm{E}_{\omega,\varpi}^{\eta,q}(z)$ (refer to, [28]). In some special cases of this new generalized Euler type integrals includes:

Setting $\eth=\lambda$ in (2.1), (2.3) and (2.4), we obtained the following Euler-type integrals that are in [36]:

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1}exp\left(-\frac{\mathfrak{I}}{t^{\eth}(1-t)^{\eth}}\right)\mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt=\sum_{n=0}^{\infty}\frac{(\eta)_{nq}}{\Gamma(\omega n+\varpi)}\frac{z^n}{n!}B_{\mathfrak{I}}^{\eth}(\sigma+\omega n,\mathcal{E})$$

$$\int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{g}exp\left(-\frac{\mathfrak{I}}{(t-a)^{\eth}(b-t)^{\eth}}\right)\mathrm{E}_{\omega,\varpi}^{\eta,q}\left(z(b-t)^{\ell}\right)dt$$

$$=(av+h)^{g}\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{(-\mathfrak{I})^{n_1}}{n_1}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}B(\sigma-\eth n_1,\mathcal{E}-\eth n_1+\kappa n_2)$$

$$\times (b-a)^{\sigma+\mathcal{E}-2\eth n_1+\kappa n_2-1} F\left(\sigma-\eth n_1,-\mathscr{g};\sigma+\mathcal{E}-2\eth n+\kappa n_{2_1};-\frac{v(b-a)}{(av+h)}\right)$$

and

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\{1-vt^{\alpha}(1-t)^{\beta}\}^{-\gamma} exp\left(-\frac{\mathfrak{J}}{t^{\eth}(1-t)^{\eth}}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt$$

$$=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{v^{n_1}(\gamma)_{n_1}}{n_1}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}B_{\mathfrak{J}}^{\eth}(\sigma+\alpha n_1+\omega n_2,\mathcal{E}-\sigma+\beta n_1)$$

Finally, putting $\eth = \lambda = 1$ in (2.1), (2.3) and (2.4), we obtained the following Euler-type integrals that are in [37]:

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-1} exp\left(-\frac{\mathfrak{J}}{t(1-t)}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt=\sum_{n=0}^{\infty}\frac{(\eta)_{nq}}{\Gamma(\omega n+\varpi)}\frac{z^n}{n!}B_{\mathfrak{J}}(\sigma+\omega n,\mathcal{E})$$

$$\int_a^b (t-a)^{\sigma-1}(b-t)^{\mathcal{E}-1}(tv+h)^{\mathscr{g}} exp\left(-\frac{\mathfrak{J}}{(t-a)(b-t)}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}\left(z(b-t)^{\ell}\right)dt$$

$$=(av+h)^{\mathscr{g}}\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{(-\mathfrak{J})^{n_1}}{n_1}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}B(\sigma-n_1,\mathcal{E}-n_1+\kappa n_2)$$

$$\times (b-a)^{\sigma+\mathcal{E}-2n_1+\kappa n_2-1} F\left(\sigma-n_1,-\mathscr{g};\sigma+\mathcal{E}-2n+\kappa n_{2_1};-\frac{v(b-a)}{(av+h)}\right)$$

and

$$\int_0^1 t^{\sigma-1}(1-t)^{\mathcal{E}-\sigma-1}\{1-vt^{\alpha}(1-t)^{\beta}\}^{-\gamma} exp\left(-\frac{\mathfrak{J}}{t(1-t)}\right) \mathrm{E}_{\omega,\varpi}^{\eta,q}(zt^{\omega})dt$$

$$=\sum_{n_1}^{\infty}\sum_{n_2}^{\infty}\frac{v^{n_1}(\gamma)_{n_1}}{n_1}\frac{(\eta)_{n_2 q}}{\Gamma(\omega n_2+\varpi)}\frac{z^{n_2}}{n_2!}B_{\mathfrak{J}}(\sigma+\alpha n_1+\omega n_2,\mathcal{E}-\sigma+\beta n_1)$$

Other form of special cases of this generalized Euler-type integral that are in the form of exponential, Classical Mittag-Leffler, Wiman and Prabhakar functions that are in [36-37] also follows.

Conflicts of Interest

The authors declare no conflict of interest.

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