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Nonexistence results for semi-linear Moore-Gibson-Thompson equation with nonlocal operator

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Abstract

We study the nonexistence of global weak solutions to the following semi-linear Moore - Gibson- Thompson equation with the nonlinearity of derivative type, namely,

$$\begin{cases} u_{ttt}+u_{tt}-\Delta u-(-\Delta)^{\frac{\alpha}{2}}u_t=|u_t|^p, \quad x\in \mathbb{R}^n, \quad t>0,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \quad u_{tt}(0,x)=u_2(x) \quad x\in\mathbb{R}^n, \end{cases}$$

where $\alpha\in(0,2]$, $p>1$, and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order $\frac{\alpha}{2}$. Then, this result is extended to the case of a weakly coupled system. We intend to apply the method of a modified test function to establish nonexistence results and to overcome some difficulties as well caused by the well-known fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$.The results obtained in this paper extend several contributions in this field.

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2010 MSC: 35B33, 35B44, 35R11, 35L76.

1. Introduction

The main goal of this paper is to discuss the nonexistence of global weak solutions to the following semi-linear Moore-Gibson-Thompson equation

$$\begin{cases} u_{ttt}+u_{tt}-\Delta u-(-\Delta)^{\frac{\alpha}{2}}u_t=|u_t|^p, \quad x\in \mathbb{R}^n, \quad t>0,\\ u(0,x)=u_0(x), \quad u_t(0,x)=u_1(x), \quad u_{tt}(0,x)=u_2(x), \quad x\in\mathbb{R}^n, \end{cases} \tag{1}$$

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where $p > 1, n \geq 1$, $\alpha \in (0, 2]$, and $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian operator of order $\frac{\alpha}{2}$. We extend our analysis to the case of a weakly coupled system, more precisely,

$$\begin{cases} u_{ttt} + u_{tt} - \Delta u - (-\Delta)^{\frac{\alpha}{2}} u_t = |v_t|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ v_{ttt} + v_{tt} - \Delta v - (-\Delta)^{\frac{\beta}{2}} v_t = |u_t|^q, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u_{tt}(0,x) = u_2(x), \quad x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), v_t(0,x) = v_1(x), v_{tt}(0,x) = v_2(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{2}$$

Recently, the nonexistence of global (in time) solutions to the following system

$$\begin{cases} u_{tt} - \Delta u + (-\Delta)^{\delta_1} u_t = |v|^p, \quad x \in \mathbb{R}^n, \quad t > 0, \\ v_{tt} - \Delta v + (-\Delta)^{\delta_2} v_t = |u|^q, \quad x \in \mathbb{R}^n, \quad t > 0, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n, \\ v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x) \quad x \in \mathbb{R}^n. \end{cases} \tag{3}$$

is investigated in [10]. It was shown that if $\delta_1, \delta_2 \in \left[0, \frac{1}{2}\right], u_0 = u_1 = 0$ and $u_1, v_1 \in \mathbb{L}^1(\mathbb{R}^n)$ satisfy

$$\int_{\mathbb{R}^n} u_1(x)dx > \varepsilon_1, \quad \int_{\mathbb{R}^n} u_1(x)dx > \varepsilon_2,$$

and

$$\frac{n}{2} \leq \frac{1 + q\frac{1-\delta_2}{1-\delta_1} + (pq-1)\delta_2}{(q-1)\frac{\delta_1 - \delta_2}{1-\delta_2} + (pq-1)} \quad \text{if} \quad \delta_1 \geq \delta_2,$$

$$\frac{n}{2} \leq \frac{1 + p\frac{1-\delta_1}{1-\delta_2} + (pq-1)\delta_2}{(p-1)\frac{\delta_2 - \delta_1}{1-\delta_1} + (pq-1)} \quad \text{if} \quad \delta_2 \geq \delta_1,$$

then there is no global (in time) Sobolev solution $(u, v) \in \mathcal{C}\left([0, \infty) \times \mathbb{L}^2(\mathbb{R}^n)\right) \times \mathcal{C}\left([0, \infty) \times \mathbb{L}^2(\mathbb{R}^n)\right)$ to (3). The critical exponent to the following structurally damped wave equation with the power nonlinearity $|u_t|^p$:

$$\begin{cases} u_{tt} - \Delta u + \mu(-\Delta)^{\frac{\alpha}{2}} u_t = |u_t|^p, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x) \quad x \in \mathbb{R}^n, \end{cases} \tag{4}$$

has been studied by Tuan Anh Dao and Ahmad Z. Fino in [11]. It was shown in [11] that if

$$1 < p \leq 1 + \frac{\tilde{\alpha}}{n} \quad \text{where} \quad \tilde{\alpha} = min\{1, \alpha\},$$

then there is no global (in time) weak solution to (4). Note that one of the most typical important methods to verify critical exponent is well-known test function method (see [13]). Concretely, this method is used to prove the nonexistence of global solutions by a contradiction argument. However, standard test function method seems difficult to be applied to (1) containing pseudo-differential operators $(-\Delta)^{\frac{\alpha}{2}}$ for any $\alpha \in (0, 2]$. The difficulty is caused by the nonlocal property of the fractional Laplacian operator. D' Abbicco and Reissig in [2] investigated the structurally damped wave equation with the power nonlinearity $|u|^p$. The critical exponent has been studied and they proposed to distinguish between (parabolic like models) in the case $\sigma \in (0, 1]$, the so-called effective damping, and (hyperbolic like models) in the remaining case $\sigma \in (1, 2]$, the so-called noneffective damping according to expected decay estimates (see more [3]). In the former case, they proved the existence of global (in time) solutions when

$$p > p_c = 1 + \frac{2}{(n - \sigma)_+}$$

for the small initial data and low space dimensions $2 \leq n \leq 4$ by using the energy estimates. Last years, the Moore-Gibson-Thompson (MGT) equation, a linearization of a model for wave propagation in viscous thermally relaxing fluids has been studied by several authors (see [14],[6], [7], [16], [8],[17] and references therein). This model is realized through the third order hyperbolic partial differential equation

$$\tau u_{ttt} + u_{tt} - c^2\Delta u - b\Delta u_t = 0,$$

where the unknown function u denotes a scalar acoustic velocity, c denotes the speed of sound and τ denotes the thermal relaxation. Besides, the coefficient $b = \beta c^2$ is related to the diffusivity of the sound with $\tau \in (0, \beta]$. Let us underline that, to our knowledge, the MGT equation has not been widely investigated in the case of presence of non-local operators. For other contributions related to the semi-linear Moore-Gibson-Thompson equation with the power nonlinearity of derivative type we refer the reader to [7],[8] and references therein.

Motivated by the above contributions, our goal in this paper is to investigate problems (1) and (2). The paper is organized as follows. In the next section, we give some auxiliary results and formulate our main results. In Section 3, we prove our main results.

2. Auxiliary Results

Before to formulate our main results, we need the following definitions.

Definition 2.1. *([15],[18]) Let $s \in (0,1)$. Let X be a suitable set of functions defined on $\mathbb{R}^n$. Then, the fractional Laplacian $(-\Delta)^s$ in $\mathbb{R}^n$is a non-local operator given by*

$$(-\Delta)^s : f \in X \to (-\Delta)^s f(x) = C_{n,s}\ P.V \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2s}}\, dy,$$

as long as the right-hand side exists. Here P.V stands for the Cauchy's principal value and $C_{n,s} = \dfrac{4^s\Gamma\left(\frac{n}{2}+s\right)}{\pi^{\frac{n}{2}}\Gamma(-s)}$ is the normalization constant and Γ denotes the Gamma function.

Definition 2.2. *(Weak solution for (1)) Let $T > 0, p > 1$, and $(u_0, u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \times \mathbb{L}^2(\mathbb{R}^n)$. We say that $u \in \mathcal{C}\left([0,T), H^2(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,T), H^1(\mathbb{R}^n)\right) \cap \mathcal{C}^2\left([0,T), \mathbb{L}^2(\mathbb{R}^n)\right)$, satisfying $u_t \in \mathbb{L}^p_{loc}([0,T) \times \mathbb{R}^n)$, is a local weak solution to (1) if*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi(t,x)dxdt + \int_{\mathbb{R}^n} (u_1(x) + u_2(x))\, \varphi(0,x)dx \\
&- \int_{\mathbb{R}^n} u_1(x)\varphi_t(0,x)dx = \int_0^T \int_{\mathbb{R}^n} u_t(x,t)\varphi_{tt}(t,x)dxdt \\
&- \int_0^T \int_{\mathbb{R}^n} u_t(x,t)\varphi_t(t,x)dxdt - \int_0^T \int_{\mathbb{R}^n} u_t(x,t)(-\Delta)^{\frac{\alpha}{2}}\varphi(x,t)dxdt \\
&- \int_0^T \int_{\mathbb{R}^n} u(x,t)\Delta\varphi(x,t)dxdt,
\end{aligned} \tag{5}$$

for any test function $\varphi \in \mathcal{C}_0^\infty([0,T) \times \mathbb{R}^n)$ such that its support in time is compact and $\varphi(x,T) = \varphi_t(x,T) = \varphi_{tt}(x,T) = 0$ for all $x \in \mathbb{R}^n$. If $T = \infty$, we say that u is a global weak solution to (1).

Definition 2.3. *(Weak solution for (2)) Let $p, q > 1$ and $T > 0$. We say that (u,v) is a local weak solution to the problem (2) if $(u_t, v_t) \in \mathbb{L}^q_{loc}([0,T) \times \mathbb{R}^n) \times \mathbb{L}^p_{loc}([0,T) \times \mathbb{R}^n)$ and satisfies the equations*

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi(t,x)dxdt + \int_{\mathbb{R}^n} (u_1(x) + u_2(x))\, \varphi(0,x)dx \\
&- \int_{\mathbb{R}^n} u_1(x)\varphi_t(0,x)dx = \int_0^T \int_{\mathbb{R}^n} u_t(x,t)\varphi_{tt}(t,x)dxdt \\
&- \int_0^T \int_{\mathbb{R}^n} u_t(x,t)\varphi_t(t,x)dxdt - \int_0^T \int_{\mathbb{R}^n} u_t(x,t)(-\Delta)^{\frac{\alpha}{2}}\varphi(x,t)dxdt \\
&- \int_0^T \int_{\mathbb{R}^n} u(x,t)\Delta\varphi(x,t)dxdt,
\end{aligned} \tag{6}$$

and

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi(t,x)dxdt + \int_{\mathbb{R}^n} (v_1(x)+v_2(x))\,\varphi(0,x)dx \\ &- \int_{\mathbb{R}^n} v_1(x)\varphi_t(0,x)dx = \int_0^T \int_{\mathbb{R}^n} v_t(x,t)\varphi_{tt}(t,x)dxdt \\ &- \int_0^T \int_{\mathbb{R}^n} v_t(x,t)\varphi_t(t,x)dxdt - \int_0^T \int_{\mathbb{R}^n} v_t(x,t)(-\Delta)^{\frac{\beta}{2}}\varphi(x,t)dxdt \\ &- \int_0^T \int_{\mathbb{R}^n} v(x,t)\Delta\varphi(x,t)dxdt, \end{aligned} \tag{7}$$

for any test function $\varphi \in \mathcal{C}_0^\infty([0,T)\times\mathbb{R}^n)$ *such that its support in time is compact and* $\varphi(x,T) = \varphi_t(x,T) = \varphi_{tt}(x,T) = 0$ *for all* $x \in \mathbb{R}^n$. *If* $T = \infty$, *we say that* (u,v) *is a global weak solution to (2).*

Now, we are ready to state the main results of this paper.

Theorem 2.4. *Let* $\alpha \in (0,2]$ *and* $\tilde{\alpha} = min\{1,\alpha\}$. *We assume that* $(u_0,u_1,u_2) \in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ *satisfy the following condition:*

$$\int_{\mathbb{R}^n} (u_1(x)+u_2(x))\,\varphi(0,x)dx > 0. \tag{8}$$

If

$$1 < p \le 1 + \frac{\tilde{\alpha}}{n}, \tag{9}$$

then there is no global (in time) weak solution to problem (1). Moreover, the sharp behavior of the lifespan T_ε *of local (in time) solutions to (1) with respect to a sufficiently small parameter* $\varepsilon > 0$ *is given by*

$$T_\varepsilon \le C\varepsilon^{-\frac{\tilde{\alpha}(p-1)}{\tilde{\alpha}-(p-1)n}}, \quad \text{for all small positive constant} \quad \varepsilon. \tag{10}$$

Theorem 2.5. *Let* $\alpha,\beta \in (0,2]$, $\tilde{\alpha} = min\{1,\alpha\}$, *and* $\tilde{\beta} = min\{1,\beta\}$. *We assume that* $(u_0,u_1,u_2) \in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ *and* $(v_0,v_1,v_2) \in H^2(\mathbb{R}^n)\times H^1(\mathbb{R}^n)\times\mathbb{L}^2(\mathbb{R}^n)$ *satisfy the following conditions:*

$$\begin{cases} \displaystyle\int_{\mathbb{R}^n} (u_1(x)+u_2(x))\,\varphi(0,x)dx > 0, \\ \displaystyle\int_{\mathbb{R}^n} (v_1(x)+v_2(x))\,\varphi(0,x)dx > 0. \end{cases} \tag{11}$$

If

$$n \le \frac{1}{pq-1}\max\Big\{\tilde{\beta}+\tilde{\alpha}p, \tilde{\alpha}+\tilde{\beta}q\Big\}, \tag{12}$$

then there is no global (in time) weak solution to (2). Moreover, the blow-up time T_ε *is estimated by*

$$T_\varepsilon \le C\varepsilon^{-\frac{\tilde{\alpha}}{\frac{\tilde{\alpha}+\tilde{\beta}q}{pq-1}-n}} \quad \text{for all small positive constants} \quad \varepsilon. \tag{13}$$

The proofs of our main results are given in the next section. For the proofs of Theorems 2.4 and 2.5, we shall use the nonlinear capacity method combined with the following pointwise estimate (see Dao and Reissig [12]).

Lemma 2.6. *([12]) Let* $\langle x\rangle = \big(1+(|x|-1)^4\big)^{\frac{1}{4}}$. *Let* $s \in (0,1)$ *and* $\phi : \mathbb{R}^n \to \mathbb{R}$ *be the function defined by*

$$\phi(x) = \begin{cases} \langle x\rangle^{-n-2s} & \text{if} \quad |x| \ge 1, \\ 1 & \text{if} \quad |x| \le 1. \end{cases} \tag{14}$$

Then $\phi \in \mathcal{C}^2(\mathbb{R}^n)$, *and the following estimate holds*

$$|(-\Delta)^s \phi(x)| \leq C\phi(x), x \in \mathbb{R}^n, \tag{15}$$

where C *is a constant independent of* x.

Lemma 2.7. *([12]) Let* $s \in (0,1)$. *Let* ψ *be a smooth function satisfying* $\partial_x^2 \psi \in \mathbb{L}^\infty(\mathbb{R}^n)$. *For any* $R > 0$, *let* ψ_R *be a function defined by*

$$\psi_R(x) = \psi\left(\frac{x}{R}\right), \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Then, $(-\Delta)^s \psi_R$ *satisfies the following scaling properties:*

$$(-\Delta)^s(\psi_R)(x) = R^{-2s}(-\Delta)^s \psi\left(\frac{x}{R}\right) \quad \textit{for all} \quad x \in \mathbb{R}^n.$$

Remark 2.8. *Throughout,* C *denotes a positive constant, whose value may change from line to line.*

2.1. Proof of Theorem 2.4

Let u be a global weak solution to (1), then for all $\varphi \in \mathcal{C}\left([0,\infty); H^2(\mathbb{R}^n)\right) \cap \mathcal{C}^1\left([0,\infty); \mathbb{L}^2(\mathbb{R}^n)\right)$, one has

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi(t,x)dxdt + \int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\varphi(0,x)dx \\
&- \int_{\mathbb{R}^n} u_1(x)\varphi_t(0,x)dx = \int_0^{+\infty}\int_{\mathbb{R}^n} u_t(x,t)\varphi_{tt}(t,x)dxdt \\
&- \int_0^{+\infty}\int_{\mathbb{R}^n} u_t(x,t)\varphi_t(t,x)dxdt - \int_0^{+\infty}\int_{\mathbb{R}^n} u(x,t)\Delta\varphi(x,t)dxdt \\
&- \int_0^{+\infty}\int_{\mathbb{R}^n} u_t(x,t)(-\Delta)^{\frac{\alpha}{2}}\varphi(x,t)dxdt.
\end{aligned} \tag{16}$$

Now, we introduce the function $\phi = \phi(x)$, defined in (14) with $s = \frac{\alpha}{2}$, and the function $\eta = \eta(t)$ having the following properties:

1. $\eta \in \mathcal{C}_0^\infty([0,\infty))$ and $\begin{cases} 1 & \text{if} \quad 0 \leq t \leq \frac{1}{2}, \\ \text{decreasing} & \text{if} \quad \frac{1}{2} \leq t \leq 1, \\ 0 & \text{if} \quad t \geq 1. \end{cases}$
2. $\eta^{-\frac{1}{p}}(t)\left(|\eta(t)| + |\eta'(t)| + |\eta''(t)|\right) \leq C$ for any $t \in [\frac{1}{2}, 1]$.

Let R be a large parameter in $[0,\infty)$. We define the following test function:

$$\varphi_R(x,t) = \eta_R(t)\phi_R(x),$$

where $\eta_R(t) = \eta(R^{-\tilde{\alpha}}t))$ and $\phi_R(x) = \phi(R^{-1}K^{-1}x)$ for some $K \geq 1$ which will be fixed later. Moreover, we check easily that $supp(\eta) \subset [0, R^{\tilde{\alpha}}]$. We define the functionals

$$I_1 = \int_0^{+\infty}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(t,x)dxdt = \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(t,x)dxdt,$$

and

$$I_2 = \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(t,x)dxdt, \quad I_3 = \int_0^{R^{\tilde{\alpha}}}\int_{\{|x| \geq RK\}} |u_t(x,t)|^p \varphi_R(t,x)dxdt.$$

From (16), one obtains

$$I_1+\int_{\mathrm{IR}^n}(u_1(x)+u_2(x))\,\phi_R(x)dx=\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}u_t(x,t)\eta_R''(t)\phi_R(x)dxdt$$
$$-\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}u_t(x,t)\eta_R'(t)\phi_R(x)dxdt-\int_0^{R^{\tilde\alpha}}\int_{\{|x|\ge RK\}}u(x,t)\eta_R(t)\Delta\phi_R(x)dxdt$$
$$-\int_0^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}u_t(x,t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)dxdt.$$

Using integrating by parts, one has

$$\begin{aligned}
&I_1+\int_{\mathrm{IR}^n}(u_1(x)+u_2(x))\,\phi_R(x)dx+\int_{\mathrm{IR}^n}u_0(x)\Psi_R(0)\Delta\phi_R(x)dx\\
&=\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}u_t(x,t)\eta_R''(t)\phi_R(x)dxdt\\
&-\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}u_t(x,t)\eta_R'(t)\phi_R(x)dxdt\\
&+\int_0^{R^{\tilde\alpha}}\int_{\{|x|\ge RK\}}u_t(x,t)\Psi_R(t)\Delta\phi_R(x)dxdt\\
&-\int_0^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}u_t(x,t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)dxdt=J_1-J_2+J_3-J_4,
\end{aligned}\tag{17}$$

where

$$\Psi_R(t)=\int_t^{R^{\tilde\alpha}}\eta_R(\tau)d\tau.$$

Applying Hölder's inequality with $\frac{1}{p}+\frac{1}{p'}=1$, we can proceed the estimate for J_1 as follows:

$$\begin{aligned}
|J_1|&\le C\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}|u_t(x,t)||\eta_R''(t)|\phi_R(x)dxdt\\
&\le\left(\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}\left(|u_t(x,t)|\varphi_R^{\frac{1}{p}}(t,x)\right)^p dxdt\right)^{\frac{1}{p}}\\
&\times\left(\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}\left(|\eta_R''(t)|\phi_R(x)\varphi_R^{-\frac{1}{p}}(t,x)\right)^{p'}dxdt\right)^{\frac{1}{p'}}\\
&\le CI_2^{\frac{1}{p}}\left(\int_{\frac{R^{\tilde\alpha}}{2}}^{R^{\tilde\alpha}}\int_{\mathrm{IR}^n}\eta_R^{-\frac{p'}{p}}(t)|\eta_R''(t)|^{p'}\phi_R(x)dxdt\right)^{\frac{1}{p'}}.
\end{aligned}$$

Using change of variables $\tilde t=R^{-\tilde\alpha}t$ and $\tilde x=R^{-1}K^{-1}x$, we get

$$|J_1|\le CI_2^{\frac{1}{p}}R^{-2\tilde\alpha+\frac{n+\tilde\alpha}{p'}}K^{\frac{n}{p'}}\left(\int_{\mathrm{IR}^n}\langle\tilde x\rangle^{-n-\alpha}d\tilde x\right)^{\frac{1}{p'}}\le CI_2^{\frac{1}{p}}R^{-2\tilde\alpha+\frac{n+\tilde\alpha}{p'}}K^{\frac{n}{p'}}.\tag{18}$$

Now, let us turn to estimate J_2, J_3, and J_4. Applying Hölder 's inequality again, as we estimated J_1, leads to

$$\begin{aligned}
|J_3|&\le CI_3^{\frac{1}{p}}\left(\int_0^{R^{\tilde\alpha}}\int_{\{|x|\ge RK\}}\Psi_R^{p'}(t)\eta_R^{-\frac{p'}{p}}(t)\phi_R^{-\frac{p'}{p}}(x)|\Delta\phi_R(x)|^{p'}dxdt\right)^{\frac{1}{p'}}\\
&\le CI_3^{\frac{1}{p}}R^{-2+\tilde\alpha+\frac{n+\tilde\alpha}{p'}}K^{-2+\frac{n}{p'}},
\end{aligned}\tag{19}$$

$$
\begin{aligned}
|J_2| &\le C I_2^{\frac{1}{p}} \left(\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}} \int_{\mathrm{IR}^n} \eta_R^{-\frac{p'}{p}}(t) |\eta_R'(t)|^{p'} \phi_R(x) dx dt \right)^{\frac{1}{p'}} \\
&\le C I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} \left(\int_{\mathrm{IR}^n} \langle \tilde{x} \rangle^{-n-\alpha} d\tilde{x} \right)^{\frac{1}{p'}} \le C I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}},
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
|J_4| &\le C I_1^{\frac{1}{p}} \left(\int_0^{R^{\tilde{\alpha}}} \int_{\mathrm{IR}^n} \eta_R(t) \phi_R^{-\frac{p'}{p}}(x) |(-\Delta)^{\frac{\alpha}{2}} \phi_R(x)|^{p'} dx dt \right)^{\frac{1}{p'}} \\
&\le C I_1^{\frac{1}{p}} R^{-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}}.
\end{aligned}
\tag{21}
$$

Combining the estimates from (18) to (21) we may arrive at

$$
\begin{aligned}
I_1 &+ \int_{\mathrm{IR}^n} (u_1(x) + u_2(x)) \phi_R(x) dx \le \int_{\mathrm{IR}^n} |u_0(x)| |\Psi_R(0)| |\Delta \phi_R(x)| dx \\
&+ C \Big(I_2^{\frac{1}{p}} R^{-2\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} \\
&+ I_1^{\frac{1}{p}} R^{-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} \Big).
\end{aligned}
$$

Moreover, it is clear that

$$
\Psi_R(t) = \int_t^{R^{\tilde{\alpha}}} \eta_R(\tau) d\tau = R^{\tilde{\alpha}} - t \quad \text{then} \quad \Psi_R(0) = R^{\tilde{\alpha}}.
$$

We can easily check that $|\Delta \phi_R(x)| \le R^{-2} \phi_R(x)$. Therefore, this implies that

$$
\begin{aligned}
I_1 &+ \int_{\mathrm{IR}^n} (u_1(x) + u_2(x)) \phi_R(x) dx \le R^{\tilde{\alpha}-2} \int_{\mathrm{IR}^n} |u_0(x)| \phi_R(x) dx \\
&+ C \Big(I_2^{\frac{1}{p}} R^{-2\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} \\
&+ I_1^{\frac{1}{p}} R^{-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} \Big).
\end{aligned}
\tag{22}
$$

Since $u_0 \in \mathbb{L}^1(\mathrm{IR}^n)$, it implies immediately that

$$
\lim_{R \to \infty} \left[R^{\tilde{\alpha}-2} \int_{\mathrm{IR}^n} |u_0(x)| \phi_R(x) dx \right] = 0.
$$

Invoking the assumption (8), one obtains

$$
R^{\tilde{\alpha}-2} \int_{\mathrm{IR}^n} |u_0(x)| \phi_R(x) dx < \frac{1}{2} \int_{\mathrm{IR}^n} (u_1(x) + u_2(x)) \phi_R(x) dx.
$$

From (22), we easily see that

$$
\begin{aligned}
I_1 &+ \frac{1}{2} \int_{\mathrm{IR}^n} (u_1(x) + u_2(x)) \phi_R(x) dx \le C \Big(I_2^{\frac{1}{p}} R^{-2\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} \\
&+ I_2^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} K^{-2+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{-\alpha+\frac{n+\tilde{\alpha}}{p'}} K^{-\alpha+\frac{n}{p'}} \Big).
\end{aligned}
\tag{23}
$$

By choosing $K = 1$ and noticing the relations $I_2 \leq I_1$ and $I_3 \leq I_1$, we may arrive at

$$I_1 + \frac{1}{2}\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\phi_R(x)dx \leq C\Big(I_1^{\frac{1}{p}} R^{-2\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} + I_1^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} + I_1^{\frac{1}{p}} R^{-2+\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} + I_1^{\frac{1}{p}} R^{-\alpha+\frac{n+\tilde{\alpha}}{p'}}\Big) \leq C I_1^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}}. \tag{24}$$

Thanks to the following ε-Young's inequality:

$$ab \leq \varepsilon a^p + C(\varepsilon) b^{p'}, \quad \text{for all} \quad a, b > 0 \quad \text{and} \quad \text{for any} \quad \varepsilon > 0,$$

we conclude

$$C I_1^{\frac{1}{p}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{p'}} \leq \varepsilon I_1 + C(\varepsilon) R^{-\tilde{\alpha}p'+n+\tilde{\alpha}}.$$

Consequently, from (24) we derive

$$(1-\varepsilon) I_1 + \frac{1}{2}\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\phi_R(x)dx \leq C(\varepsilon) R^{-\tilde{\alpha}p'+n+\tilde{\alpha}},$$

which follows that

$$I_1 \leq C R^{-\tilde{\alpha}p'+n+\tilde{\alpha}}, \tag{25}$$

and

$$\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\phi_R(x)dx \leq C R^{-\tilde{\alpha}p'+n+\tilde{\alpha}}. \tag{26}$$

It is clear that the assumption (9) is equivalent to $-\tilde{\alpha}p' + n + \tilde{\alpha} \leq 0$. For this reason, we will split our consideration into two cases.

Case 1:In the subcritical case $-\tilde{\alpha}p' + n + \tilde{\alpha} < 0$, letting $R \to \infty$ in (26), we easily deduce

$$\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\phi_R(x)dx \leq 0,$$

which contradicts the assumption (8).

Case 2: For the critical case $-\tilde{\alpha}p' + n + \tilde{\alpha} = 0$, from (25), we can see that $I_1 \leq C$. Using Beppo Levi's theorem on monotone convergence, one obtains

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^n} |u_t(x,t)|^p dxdt &= \lim_{R\to\infty} \int_0^{R^{\tilde{\alpha}}} \int_{\mathbb{R}^n} |u_t(x,t)|^p \varphi_R(x,t) dxdt \\ &= \lim_{R\to\infty} I_1 \leq C.\end{aligned}$$

We conclude that $u_t \in \mathbb{L}^p((0,\infty) \times \mathbb{R}^n)$. By the absolute continuity of the Lebesgue integral, it follows that $I_2 \to 0$ and $I_3 \to 0$, as $R \to \infty$. Using again the fact that $\tilde{\alpha} = \frac{n+\tilde{\alpha}}{p'}$, we obtain from (23) the following estimate:

$$I_1 + \frac{1}{2}\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\phi_R(x)dx \leq C\Big(I_2^{\frac{1}{p}} R^{-\tilde{\alpha}} K^{\frac{n}{p'}} + I_2^{\frac{1}{p}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2+2\tilde{\alpha}} K^{-2+\frac{n}{p'}} + I_1^{\frac{1}{p}} R^{-\alpha+\tilde{\alpha}} K^{-\alpha+\frac{n}{p'}}\Big), \tag{27}$$

for all $K \geq 1$.

1. If $\alpha \in (0, 1]$, then $\alpha = \tilde{\alpha}$. Consequently, from (27), we have

$$I_1 + \frac{1}{2}\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\,\phi_R(x)dx \leq C\Big(I_2^{\frac{1}{p}} R^{-\alpha} K^{\frac{n}{p'}} + I_2^{\frac{1}{p}} K^{\frac{n}{p'}} + I_3^{\frac{1}{p}} R^{-2(1-\alpha)} K^{-2+\frac{n}{p'}} + I_1^{\frac{1}{p}} K^{-\alpha+\frac{n}{p'}}\Big). \tag{28}$$

Letting $R \to \infty$ in (28), we get

$$\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\, \phi_R(x)dx \leq CK^{-\alpha+\frac{n}{p'}} \quad \text{for all} \quad K \geq 1. \tag{29}$$

It is obvious that $-\alpha + \frac{n}{p'} < 0$. We can fix a sufficiently large constant $K \geq 1$ in (29) to gain a contradiction to (8).

2. If $\alpha \in (1, 2]$, then $\tilde{\alpha} = 1$. As a result, choosing $K = 1$, we may conclude from (27) that

$$I_1 + \frac{1}{2}\int_{\mathbb{R}^n} (u_1(x) + u_2(x))\, \phi_R(x)dx \leq C\Big(I_2^{\frac{1}{p}} R^{-1} + I_2^{\frac{1}{p}} + I_3^{\frac{1}{p}} + I_1^{\frac{1}{p}} R^{1-\alpha}\Big). \tag{30}$$

Since $\alpha > 1$, letting $R \to \infty$ in (30) we obtain a contradiction to (8) again.

Let us now consider the case of subcritical exponent to prove the estimate for lifespan T_ε of solutions to (1). We assume that $u = u(x,t)$ is a local solution to (1). In order to prove the lifespan estimate, we replace the initial data $(0, u_1, u_2)$ by $(0, \varepsilon f_1, \varepsilon f_2)$ with a small constant $\varepsilon > 0$, where $(f_1, f_2) \in H^1(\mathbb{R}^n) \times \mathbb{L}^2(\mathbb{R}^n)$ satisfy the assumption (8). Invoking the fact that

$$\int_{\mathbb{R}^n} (f_1(x) + f_2(x))\phi_R(x)dx \geq c > 0,$$

and repeating the steps in the above proofs we arrive at the following estimate:

$$\varepsilon \leq CR^{-\tilde{\alpha}p'+n+\tilde{\alpha}}.$$

Let $R = T^{\frac{1}{\tilde{\alpha}}}$, then a standard calculation lead to

$$T_\varepsilon \leq \varepsilon^{-\frac{\tilde{\alpha}(p-1)}{\tilde{\alpha}-(p-1)n}}.$$

Summarizing, the proof of the Theorem 2.4 is completed.

2.2. Proof of Theorem 2.5

First, we introduce the same test function as in the proof of Theorem 2.4. Let us assume that (u, v) is the global weak solution to (2). We define the functionals

$$J_1 = \int_0^{+\infty}\int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(t,x)dxdt = \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(t,x)dxdt,$$

and

$$J_2 = \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |u_t(x,t)|^q \varphi_R(t,x)dxdt, \quad J_3 = \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} |u_t(x,t)|^q \varphi_R(t,x)dxdt,$$

$$I_1 = \int_0^{+\infty}\int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi_R(t,x)dxdt = \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi_R(t,x)dxdt,$$

and

$$I_2 = \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} |v_t(x,t)|^p \varphi_R(t,x)dxdt, \quad I_3 = \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} |v_t(x,t)|^p \varphi_R(t,x)dxdt.$$

From (6) and (7), one has

$$\begin{aligned} I_1 + \int_{\mathbb{R}^n} (u_1(x) + u_2(x))\, \varphi(0,x)dx &= \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} u_t(x,t)\eta_R''(t)\phi_R(x)dxdt \\ &- \int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} u_t(x,t)\eta_R'(t)\phi_R(x)dxdt - \int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}} u(x,t)\eta_R(t)\Delta\phi_R(x)dxdt \\ &- \int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n} u_t(x,t)\eta_R(t)(-\Delta)^{\frac{\alpha}{2}}\phi_R(x)dxdt, \end{aligned}$$

and

$$J_1+\int_{\mathbb{R}^n}(v_1(x)+v_2(x))\,\varphi(0,x)dx=\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}v_t(x,t)\eta_R''(t)\phi_R(x)dxdt$$
$$-\int_{\frac{R^{\tilde{\alpha}}}{2}}^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}v_t(x,t)\eta_R'(t)\phi_R(x)dxdt-\int_0^{R^{\tilde{\alpha}}}\int_{\{|x|\geq RK\}}v(x,t)\eta_R(t)\Delta\phi_R(x)dxdt$$
$$-\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}v_t(x,t)\eta_R(t)(-\Delta)^{\frac{\beta}{2}}\phi_R(x)dxdt.$$

Repeating the steps of the proof from (18) to (24), we may conclude the following estimates:

$$I_1\leq J_1^{\frac{1}{q}}R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}}. \tag{31}$$

In the analogous way, one obtains

$$J_1\leq I_1^{\frac{1}{p}}R^{-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}}. \tag{32}$$

From (31) and (32), we obtain

$$I_1^{\frac{pq-1}{pq}}\leq R^{\left(-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}\right)\frac{1}{q}-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}}=R^{\delta_1}, \tag{33}$$

$$J_1^{\frac{pq-1}{pq}}\leq R^{\left(-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}\right)\frac{1}{p}-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}}=R^{\delta_2}. \tag{34}$$

It is clear that the assumption (12) is equivalent to $\max\{\delta_1,\delta_2\}\leq 0$. For this reason, we will split our consideration into two cases.

Case 1:In the subcritical case $\max\{\delta_1,\delta_2\}<0$, letting $R\to\infty$ in (33)and (34) we easily deduce

$$\int_{\mathbb{R}^n}(v_1(x)+v_2(x))\,\phi_R(x)dx\leq 0\quad\text{and}\quad\int_{\mathbb{R}^n}(u_1(x)+u_2(x))\,\phi_R(x)dx\leq 0,$$

which contradicts the assumption (11).

Case 2: For the critical case $\delta_2=0$, from (25) we can see that $J_1\leq C$. Using Beppo Levi's theorem on monotone convergence, one obtains

$$\begin{aligned}\int_0^{\infty}\int_{\mathbb{R}^n}|u_t(x,t)|^q dxdt&=\lim_{R\to\infty}\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}|u_t(x,t)|^q\varphi_R(x,t)dxdt\\&=\lim_{R\to\infty}J_1\leq C.\end{aligned} \tag{35}$$

Repeating the steps of the proof from (22) to (24), we may conclude the following estimates:

$$J_1+\frac{1}{2}\int_{\mathbb{R}^n}(v_1(x)+v_2(x))\,\phi_R(x)dx\leq C\left(\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}|v_t(x,t)|^p\varphi_R(t,x)dxdt\right)^{\frac{1}{p}}R^{-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}},$$

and

$$I_1+\frac{1}{2}\int_{\mathbb{R}^n}(u_1(x)+u_2(x))\,\phi_R(x)dx\leq C\left(\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}|u_t(x,t)|^q\varphi_R(t,x)dxdt\right)^{\frac{1}{q}}R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}}.$$

Since $\delta_2=0$ and invoking the above estimates, we easily deduce that

$$\begin{aligned}&J_1+\frac{1}{2}\int_{\mathbb{R}^n}(v_1(x)+v_2(x))\,\phi_R(x)dx\\&\leq\left(\int_0^{R^{\tilde{\alpha}}}\int_{\mathbb{R}^n}|u_t(x,t)|^q\varphi_R(t,x)dxdt\right)^{\frac{1}{pq}}.\end{aligned} \tag{36}$$

Letting $R \to \infty$ in (36) and using (37), one obtains

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |u_t(x,t)|^q dxdt + \int_{\mathbb{R}^n} (v_1(x) + v_2(x))\, \phi_R(x)dx = 0,$$

which is a contradiction to (11). In the case $\delta_1 = 0$ we repeat the same arguments as in $\delta_2 = 0$.
Let us now consider the case of subcritical exponent to prove the estimate for lifespan T_ε of solutions to (2). We assume that $(u, v) = (u(x,t), v(x,t))$, is a local solution to (2). In order to prove the lifespan estimate, we replace the initial data $(0, u_1, u_2), (0, v_1, v_2)$ by $(0, \varepsilon f_1, \varepsilon f_2), (0, \varepsilon g_1, \varepsilon g_2)$ with a small constant $\varepsilon > 0$, where $(f_1, f_2), (g_1, g_2) \in H^1(\mathbb{R}^n) \times \mathbb{L}^2(\mathbb{R}^n)$ satisfy the assumption (11). Repeating the steps in the above proofs, we arrive at the following estimate:

$$I_1 + c\varepsilon \leq J_1^{\frac{1}{q}} R^{-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}}, \tag{37}$$

and

$$J_1 + c\varepsilon \leq I_1^{\frac{1}{p}} R^{-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}}. \tag{38}$$

If we plug (37) in (38), we find

$$J_1 + c\varepsilon \leq C J_1^{\frac{1}{pq}} R^{\left(-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}\right)+\left(-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}\right)\frac{1}{p}}. \tag{39}$$

We easily obtains that

$$c\varepsilon \leq C J_1^{\frac{1}{pq}} R^{\left(-\tilde{\beta}+\frac{n+\tilde{\beta}}{p'}\right)+\left(-\tilde{\alpha}+\frac{n+\tilde{\alpha}}{q'}\right)\frac{1}{p}} - J_1,$$

which leads to

$$\varepsilon \leq C R^{-\left[\frac{\tilde{\alpha}+\tilde{\beta}q}{pq-1} - n\right]}.$$

Let $R = T^{\frac{1}{\tilde{\alpha}}}$. Then with a standard calculation, one has

$$T_\varepsilon \leq \varepsilon^{-\frac{\tilde{\alpha}(pq-1)}{\tilde{\alpha}+\tilde{\beta}q-n(pq-1)}}.$$

Summarizing, the proof of the Theorem 2.5 is completed.

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