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Stability and convergence analysis of hybrid algorithms for Berinde contraction mappings and its applications

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Abstract

In this paper, we construct a new hybrid iteration, called SR-iteration, and prove its stability and convergence analysis for weak contraction mappings in a Banach space. We compare the rate of convergence between the SR-iteration and other iterations. Moreover, we provide numerical comparisons for supporting our main theorem and apply our main result to prove the existence problem of mixed type Volterra-Fredholm functional nonlinear integral equation.

Keywords: SR-iteration, rate of convergence, weak contraction, Banach space.
2020 MSC: 41A25, 47H09, 47H10.

1. Introduction and preliminaries

Let C be a nonempty convex subset of a Banach space X, and $T : C \to C$ be a mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T) = \{x \in C : x = Tx\}$. Fixed point theory plays very important role in solving various nonlinear equations. Many iteration methods were introduced extensively by a huge number of mathematicians for approximating solutions of the studied problems, see [1, 2, 3]. For a class of contraction mappings, Picard iteration is a powerful and efficient method for approximating a fixed point of a contraction mapping. Renowned Picard iteration [4] is formulated as follows: $x_1 \in C$ and

$$x_{n+1} = Tx_n,$$

for all $n \in \mathbb{N}$.

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In 1953, Mann [5] introduced an iteration as follows: $x_1 \in C$ and

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

In 1974, Ishikawa [6] introduced an iteration as follows: $x_1 \in C$ and

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n, \end{aligned}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$.

In 2000, Noor [7] introduced a there-step iteration as follows: $h_1 \in C$ and

$$\begin{aligned} l_n &= (1 - \gamma_n)h_n + \gamma_n T h_n, \\ k_n &= (1 - \beta_n)h_n + \beta_n T l_n, \\ h_{n+1} &= (1 - \alpha_n)h_n + \alpha_n T k_n, \end{aligned} \tag{1}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$.

In 2011, Phuengrattana and Suantai [8] introduced an iteration, called SP-iteration, and studied comparison of the rate of convergence between this method and Mann, Ishikawa, Noor iterations. After that they [9] also introduced a new iteration as follows: $w_1 \in C$ and

$$\begin{aligned} u_n &= (1 - \gamma_n)w_n + \gamma_n T w_n, \\ v_n &= (1 - \beta_n)u_n + \beta_n T u_n, \\ w_{n+1} &= (1 - \alpha_n - \lambda_n)v_n + \alpha_n T v_n + \lambda_n T u_n, \end{aligned} \tag{2}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, and $\{\alpha_n + \lambda_n\}$ are sequences in $[0, 1]$. We will call it here Phuengrattana iteration.

Remark 1.1. *The Phuengrattana iteration reduces to the SP-iteration when we take $\lambda_n = 0$ for all $n \in \mathbb{N}$.*

A mapping $T : C \to C$ is said to satisfy condition $(*)$ if there exist $\theta \in (0, 1)$ and some $L_1 \geq 0$ such that

$$\|Tx - Ty\| \leq \theta\|x - y\| + L_1\|x - Tx\|, \tag{3}$$

for each $x, y \in C$.

Remark 1.2. *Note that, by the symmetry of the distance, (3) is satisfied for all $x, y \in C$ if and only if*

$$\|Tx - Ty\| \leq \theta\|x - y\| + L_1\|y - Ty\|.$$

also holds, for all $x, y \in C$.

In 1995, Osilike [10] proved several stability results of some iteration methods for a class of mappings satisfying the condition $(*)$. Those results are generalizations and extensions of those of Rhoades [11].

In order to study the order of convergence of a real sequence $\{a_n\}$ converging to a, we usually use the well-known terminology in numerical analysis, see [12], for example.

Definition 1.3. [12] *Suppose $\{a_n\}$ is a sequence that converges to a, with $a_n \neq a$ for all n. If positive constants λ and α exist with*

$$\lim_{n\to\infty} \frac{|a_{n+1} - a|}{|a_n - a|^\alpha} = \lambda,$$

then $\{a_n\}$ converges to a of order α, with asymptotic error constant λ. If $\alpha = 1$ (and $\lambda < 1$), the sequence is linearly convergent and if $\alpha = 2$, the sequence is quadratically convergent.

In 2002, Berinde [13] employed above concept for comparing the rate of convergence between the two iterative methods as follows:

Definition 1.4. [13] *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive numbers that converge to a, b, respectively. Assume there exists*

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = l.$$

(i) If $l = 0$, then it is said that the sequence $\{a_n\}$ converges to a faster than the sequence $\{b_n\}$ to b.
(ii) If $0 < l < \infty$, then we say that the sequence $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Definition 1.5. [13, 9] *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a mapping. Suppose $\{x_n\}$ and $\{w_n\}$ are two iterations which converge to a fixed point p of T. We say that $\{x_n\}$ converges faster than $\{w_n\}$ to p if*

$$\lim_{n\to\infty} \frac{\|x_n - p\|}{\|w_n - p\|} = 0.$$

He also introduced a new class of operators which is more general than that of Zamfirescu operators.

Definition 1.6. [13] *Let C be a nonempty closed convex subset of a Banach space X. A mapping $T : C \to C$ is said to be weak contraction if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|y - Tx\|,$$

for all $x, y \in C$.

The following existence and uniqueness results can be found in [13].

Proposition 1.7. [13] *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak contraction with condition $(*)$. Then T has a unique fixed point. Further, the Picard iteration converges to a unique fixed point of T.*

Remark 1.8. *It is known that only weak contraction does not guarantee the uniqueness of fixed point of T. But if T also satisfies the condition $(*)$, its fixed point must be unique.*

In 2013, Phuengrattana and Suantai [9] proved the strong convergence of the Phuengrattana iteration to a fixed point of a weak contraction, and this iteration converges faster than Mann, Ishikawa and Noor iterations.

Recently, Gürsoy [14] used a Picard-S iteration, which was introduced by Gürsoy and Karakaya [15] in 2014, to approximate the unique solution of mixed type Volterra-Fredholm functional nonlinear integral equation. Many contributions on fixed point outcomes with different contractive conditions have recently been published, see also [17, 18, 19].

In this work, we introduce a new iteration, called the SR-iteration, as follows:

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n, \\ y_n &= (1-\beta_n)T z_n + \beta_n T^2 z_n, \\ x_{n+1} &= (1-\alpha_n)T y_n + \alpha_n T^2 y_n, \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$, where $x_1 \in C$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$. We prove strong convergence theorems of the SR-iteration for approximating fixed points of weak contractions in a Banach space, and also compare the rate of convergence of this iteration with Phuengrattana and Noor iterations. Moreover, we prove the stability result of the SR-iteration for a weak contraction and apply the SR-iteration to estimate the unique solution of mixed type Volterra-Fredholm functional nonlinear integral equation.

2. Convergence theorems

In this section, we prove the following strong convergence theorems to fixed points of weak contractions in a Banach space.

Theorem 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak contraction with condition $(*)$. Suppose that the sequence $\{x_n\}$ is defined by the SR-iteration, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0, 1]$ which satisfy one of the following conditions:*

$(C1)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$; $(C2)$ $\sum_{n=1}^{\infty} \beta_n = \infty$; $(C3)$ $\sum_{n=1}^{\infty} \gamma_n = \infty$.

Then $\{x_n\}$ converges strongly to a unique fixed point of T.

Proof. Let $p \in F(T)$. Then by Proposition 1.7, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1 - \alpha_n)Ty_n + \alpha_n T^2 y_n - p\| \\
&\leq (1 - \alpha_n)\|Ty_n - p\| + \alpha_n \|T^2 y_n - p\| \\
&\leq (1 - \alpha_n)\theta\|y_n - p\| + \alpha_n \theta^2 \|y_n - p\| \\
&= (1 - \alpha_n(1 - \theta))\theta\|y_n - p\|,
\end{aligned}$$

$$\begin{aligned}
\|y_n - p\| &\leq (1 - \beta_n)\|Tz_n - p\| + \beta_n \|T^2 z_n - p\| \\
&\leq (1 - \beta_n)\theta\|z_n - p\| + \beta_n \theta^2 \|z_n - p\| \\
&= (1 - \beta_n(1 - \theta))\theta\|z_n - p\|,
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\| &\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n \|Tx_n - p\| \\
&\leq (1 - \gamma_n)\|x_n - p\| + \gamma_n \theta \|Tx_n - p\| \\
&= (1 - \gamma_n(1 - \theta))\|x_n - p\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \theta^2 (1 - \alpha_n(1 - \theta))(1 - \beta_n(1 - \theta))(1 - \gamma_n(1 - \theta))\|x_n - p\| \\
&\vdots \\
&\leq \theta^{2n} \prod_{k=1}^{n} (1 - \alpha_k(1 - \theta))(1 - \beta_k(1 - \theta))(1 - \gamma_k(1 - \theta))\|x_1 - p\|. \qquad (5)
\end{aligned}$$

By the assumption, we can conclude that $\{x_n\}$ converges to p. $\square$

Theorem 2.2. [9] *Assume X, C, T are as in Theorem 2.1. Suppose that the sequence $\{w_n\}$ is defined by the Phuengrattana iteration and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, and $\{\alpha_n + \lambda_n\}$ are in $[0, 1]$ which satisfy one of the conditions $(C1)$, $(C2)$, $(C3)$ in Theorem 2.1. Then $\{w_n\}$ converges strongly to a unique fixed point of T.*

Theorem 2.3. [9] *Assume X, C, T are as in Theorem 2.1. Suppose that the sequence $\{h_n\}$ is defined by the Noor iteration and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then $\{h_n\}$ converges strongly to a unique fixed point of T.*

3. Rate of convergence

In this section, we compare the rate of convergence between the SR-iteration and Phuengrattana iteration and Noor iteration.

Theorem 3.1. *Assume X, C, T are as in Theorem 2.1. Suppose $\{x_n\}$, $\{w_n\}$ and $\{h_n\}$ are sequences generated by SR-iteration, Phuengrattana iteration, and Noor iteration, respectively, where $x_1 = w_1 = h_1 \in C$, and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\lambda_n\}$, and $\{\alpha_n + \lambda_n\}$ are in $[0, 1]$ and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ satisfy one of the conditions $(C1)$, $(C2)$, $(C3)$ in Theorem 2.1. Then*

(i) If $\lim_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) = 0 = \lim_{n\to\infty} \gamma_n$, then $\{x_n\}$ converges faster than $\{w_n\}$ to a unique fixed point of T.

(ii) If $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}$ converges faster than $\{h_n\}$ to a unique fixed point of T.

Proof. By Theorems 2.1, 2.2 and 2.3, the sequence $\{x_n\}$, $\{w_n\}$ and $\{h_n\}$ converge to a unique fixed point of T, say p.

(i) Assume that $\lim_{n\to\infty} \alpha_n + \beta_n + \lambda_n = 0 = \lim_{n\to\infty} \gamma_n$. From Phuengrattana iteration, we have

$$
\begin{aligned}
&\|w_{n+1} - p\| = \|(1 - \alpha_n - \lambda_n)v_n + \alpha_n T v_n + \lambda_n T u_n - p\| \\
&\geq (1 - \alpha_n - \lambda_n)\|v_n - p\| - \alpha_n\|T v_n - p\| - \lambda_n\|T u_n - p\| \\
&\geq (1 - \alpha_n - \lambda_n)\|v_n - p\| - \alpha_n\theta\|v_n - p\| - \lambda_n\theta\|u_n - p\| \\
&= (1 - \alpha_n(1 + \theta) - \lambda_n)\|v_n - p\| - \lambda_n\theta\|u_n - p\| \\
&\geq (1 - \alpha_n(1 + \theta) - \lambda_n)(1 - \beta_n - \beta_n\theta)\|u_n - p\| - \lambda_n\theta\|u_n - p\| \\
&= [(1 - \alpha_n(1 + \theta) - \lambda_n)(1 - \beta_n(1 + \theta)) - \lambda_n\theta]\|u_n - p\| \\
&= [1 - \beta_n(1 + \theta) - \alpha_n(1 + \theta)(1 - \beta_n(1 + \theta)) - \lambda_n(1 - \beta_n(1 + \theta) - \lambda_n\theta]\|u_n - p\| \\
&= [1 - \alpha_n(1 + \theta)(1 - \beta_n(1 + \theta)) - \beta_n(1 + \theta)(1 - \lambda_n) - \lambda_n(1 + \theta)]\|u_n - p\| \\
&\geq (1 - (\alpha_n + \beta_n + \lambda_n)(1 + \theta))(1 - \gamma_n(1 + \theta))\|w_n - p\| \\
&\vdots \\
&\geq \prod_{k=1}^{n}(1 - (\alpha_k + \beta_k + \lambda_k)(1 + \theta))(1 - \gamma_k(1 + \theta))\|w_1 - p\|.
\end{aligned}
$$

Thus

$$
\frac{1}{\|w_{n+1} - p\|} \leq \frac{1}{\prod_{k=1}^{n}(1 - (\alpha_k + \beta_k + \lambda_k)(1 + \theta))(1 - \gamma_k(1 + \theta))\|w_1 - p\|} \tag{6}
$$

By inequalities (5), (6) and the assumption, we have

$$
\frac{\|x_{n+1} - p\|}{\|w_{n+1} - p\|} \leq \frac{\theta^{2n}\prod_{k=1}^{n}(1 - \alpha_k(1 - \theta))(1 - \beta_k(1 - \theta))(1 - \gamma_k(1 - \theta))}{\prod_{k=1}^{n}(1 - (\alpha_k + \beta_k + \lambda_k)(1 + \theta))(1 - \gamma_k(1 + \theta))}
$$

Setting $\sigma_n = \frac{\theta^{2n}\prod_{k=1}^{n}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))}{\prod_{k=1}^{n}(1-(\alpha_k+\beta_k+\lambda_k)(1+\theta))(1-\gamma_k(1+\theta))}$, we get

$$
\begin{aligned}
\frac{\sigma_{n+1}}{\sigma_n} &= \frac{\theta^{2(n+1)}\prod_{k=1}^{n+1}(1 - \alpha_k(1 - \theta))(1 - \beta_k(1 - \theta))(1 - \gamma_k(1 - \theta))}{\prod_{k=1}^{n+1}(1 - (\alpha_k + \beta_k + \lambda_k)(1 + \theta))(1 - \gamma_k(1 + \theta))} \\
&\quad \times \frac{\prod_{k=1}^{n}(1 - (\alpha_k + \beta_k + \lambda_k)(1 + \theta))(1 - \gamma_k(1 + \theta))}{\theta^{2n}\prod_{k=1}^{n}(1 - \alpha_k(1 - \theta))(1 - \beta_k(1 - \theta))(1 - \gamma_k(1 - \theta))} \\
&= \frac{\theta^2(1 - \alpha_{n+1}(1 - \theta))(1 - \beta_{n+1}(1 - \theta))(1 - \gamma_{n+1}(1 - \theta))}{(1 - (\alpha_{n+1} + \beta_{n+1} + \lambda_{n+1})(1 + \theta))(1 - \gamma_{n+1}(1 + \theta))},
\end{aligned}
$$

and so $\lim_{n\to\infty} \frac{\sigma_{n+1}}{\sigma_n} = \theta^2 < 1$. By the ratio test, it implies that $\sum_{n=1}^{\infty} \sigma_n < \infty$. So, $\lim_{n\to\infty} \sigma_n = 0$, we conclude that $\{x_n\}$ converges faster than $\{w_n\}$.

(ii) Assume that $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$. By the proof of Theorem 2.4. in [9], we have

$$\|h_{n+1} - p\| \geq \prod_{k=1}^{n} (1 - \alpha_k(1+\theta))\|h_1 - p\|.$$

Hence

$$\frac{1}{\|h_{n+1} - p\|} \leq \frac{1}{\prod_{k=1}^{n}(1 - \alpha_k(1+\theta))\|h_1 - p\|}. \tag{7}$$

It follows from (5) and (7) that

$$\frac{\|x_{n+1} - p\|}{\|h_{n+1} - p\|} \leq \frac{\theta^{2n} \prod_{k=1}^{n}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))}{\prod_{k=1}^{n}(1-\alpha_k(1+\theta))}.$$

Setting $\tau_n = \frac{\theta^{2n} \prod_{k=1}^{n}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))}{\prod_{k=1}^{n}(1-\alpha_k(1+\theta))}$, we obtain

$$\begin{aligned}\frac{\tau_{n+1}}{\tau_n} &= \frac{\theta^{2(n+1)} \prod_{k=1}^{n+1}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))}{\prod_{k=1}^{n+1}(1-\alpha_k(1+\theta))} \\ &\quad \times \frac{\prod_{k=1}^{n}(1-\alpha_k(1+\theta))}{\theta^{2n} \prod_{k=1}^{n}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))} \\ &= \frac{\theta^2(1-\alpha_{n+1}(1-\theta))(1-\beta_{n+1}(1-\theta))(1-\gamma_{n+1}(1-\theta))}{(1-\alpha_{n+1}(1+\theta))},\end{aligned}$$

and so $\lim_{n\to\infty} \frac{\tau_{n+1}}{\tau_n} = \theta^2 < 1$. By the ratio test, it implies that $\sum_{n=1}^{\infty} \tau_n < \infty$. So, $\lim_{n\to\infty} \tau_n = 0$, we conclude that $\{x_n\}$ converges faster than $\{h_n\}$. □

4. Stability

In this section, we prove the stability result for the SR-iteration defined by (4) for a weak contraction with condition $(*)$. We recall the concept of stability.

Definition 4.1. *Let X be a Banach space and $T : C \to C$ be a mapping. Suppose a point $x_1 \in X$ and define a fixed point iteration procedure by a general relation of the form*

$$x_{n+1} = f(T, x_n),$$

for all $n \in \mathbb{N}$, and $\{x_n\}$ converge to a fixed point p of T. Let $\{y_n\}$ be an arbitrary sequence in X and set

$$\varepsilon_n = \|y_{n+1} - f(T, y_n)\|.$$

The sequence $\{x_n\}$ is T-stable (or stable with respect to T) if

$$\lim_{n\to\infty} \varepsilon_n = 0 \text{ if and only if } \lim_{n\to\infty} y_n = p.$$

We now prove the stability result of the SR-iteration.

Theorem 4.2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak contraction with condition $(*)$. Suppose that the sequences $\{x_n\}$ is defined by the SR-iteration and the sequence $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $[0,1]$ which satisfy one of the conditions $(C1)$, $(C2)$, $(C3)$ in Theorem 2.1. Then the sequence $\{x_n\}$ is T-stable.*

Proof. From Theorem 2.1, the sequence $\{x_n\}$ converge to a unique fixed point of T, say p. Let $\{p_n\}$ be an arbitrary sequence in C and define

$$\begin{aligned} q_n &= (1-\gamma_n)p_n + \gamma_n T p_n, \\ r_n &= (1-\beta_n)Tq_n + \beta_n T^2 q_n, \\ \varepsilon_n &= \|p_{n+1} - \big((1-\alpha_n)Tr_n + \alpha_n T^2 r_n\big)\|, \end{aligned}$$

for all $n \in \mathbb{N}$. By the same proof of Theorem 2.1, we obtain

$$\begin{aligned} &\|(1-\alpha_n)Tr_n + \alpha_n T^2 r_n - p\| \\ &\le \theta^2(1-\alpha_n(1-\theta))(1-\beta_n(1-\theta))(1-\gamma_n(1-\theta))\|p_n - p\| \\ &\ \vdots \\ &\le \theta^{2n}\prod_{k=1}^{n}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))\|p_1 - p\|. \end{aligned}$$

Next, assume that $\lim_{n\to\infty} \varepsilon_n = 0$. By above inequality, we have

$$\begin{aligned} &\|p_{n+1} - p\| \\ &\le \|p_{n+1} - \big((1-\alpha_n)Tr_n + \alpha_n T^2 r_n\big)\| + \|(1-\alpha_n)Tr_n + \alpha_n T^2 r_n - p\| \\ &\le \varepsilon_n + \theta^{2n}\prod_{k=1}^{n}(1-\alpha_k(1-\theta))(1-\beta_k(1-\theta))(1-\gamma_k(1-\theta))\|p_1 - p\|. \end{aligned}$$

It follows from above inequality and our assumptions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ that $\lim_{n\to\infty} p_n = p$.

Conversely, assume that $\lim_{n\to\infty} p_n = p$, then

$$\begin{aligned} \varepsilon_n &\le \|p_{n+1} - p\| + \|p - \big((1-\alpha_n)Tr_n + \alpha_n T^2 r_n\big)\| \\ &\le \|p_{n+1} - p\| + \theta^2(1-\alpha_n(1-\theta))(1-\beta_n(1-\theta))(1-\gamma_n(1-\theta))\|p_n - p\| \\ &\le \|p_{n+1} - p\| + \|p_n - p\|. \end{aligned}$$

By above inequality and $p_n \to p$ as $n \to \infty$, we obtain that $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, the sequence $\{x_n\}$ is T-stable. □

5. Numerical results

Example 5.1. *Consider $\mathbb{R}^2$ with the Euclidean norm. Let $C = [0,1] \times [0,1]$ and $T : C \to C$ be defined as*

$$T((x,y)) = (\sqrt{x^2 - x + \tfrac{1}{2}}, \sin(\cos y)),$$

for all $(x,y) \in C$. Then T is a weak contraction with condition $(*)$. Suppose $\{x_n\}$, $\{s_n\}$, $\{w_n\}$, and $\{h_n\}$ are sequences generated by SR-iteration, SP-iteration, Phuengrattana iteration, and Noor iteration, respectively, Choose $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{1}{2n+1}$, $\gamma_n = \frac{1}{n+2}$, and $\lambda_n = \frac{1}{n^2+1}$, for all $n \in \mathbb{N}$. It is clear that sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\lambda_n\}$ satisfy all the conditions of Theorem 3.1. For the initial point $h_1 = s_1 = w_1 = x_1 = (0,0)$. We obtain the following numerical experiments for fixed point of T, rate of convergence and numerical experiments of the studied methods.

n	Noor	SP	Phuengrattana	SR
	h_n	s_n	w_n	x_n
2	(0.29454,0.41043)	(0.43447,0.62052)	(0.54442,0.80026)	(0.50102,0.68651)
3	(0.37138,0.53183)	(0.47515,0.67722)	(0.51320,0.70256)	(0.50000,0.69441)
4	(0.40647,0.58606)	(0.48741,0.68853)	(0.50595,0.69688)	(0.50000,0.69479)
⋮	⋮	⋮	⋮	⋮
10	(0.46455,0.66564)	(0.49860,0.69459)	(0.50055,0.69487)	(0.50000,0.69482)

Table 1: Numerical experiments of Noor, SP, Phuengrattana, and SR-iterations.

n	Noor	SP	Phuengrattana	SR
	$\|h_n - Th_n\|$	$\|s_n - Ts_n\|$	$\|w_n - Tw_n\|$	$\|x_n - Tx_n\|$
2	0.45548	0.12712	0.16429	0.01215
3	0.26950	0.03610	0.01727	0.00059
4	0.18481	0.01570	0.00664	0.00004
⋮	⋮	⋮	⋮	⋮
10	0.05597	0.00145	0.00055	1.5001e-10

Table 2: Numerical experiments of the studied methods.

n	Noor	SP	Phuengrattana	SR
	$\|h_n - h_{n-1}\|$	$\|s_n - s_{n-1}\|$	$\|w_n - w_{n-1}\|$	$\|x_n - x_{n-1}\|$
2	0.50518	0.75750	0.96789	0.84989
3	0.14368	0.06978	0.10256	0.00797
4	0.06459	0.01669	0.00922	0.00038
⋮	⋮	⋮	⋮	⋮
10	0.00631	0.00042	0.00017	6.0593e-10

Table 3: Numerical errors in Example 5.1.

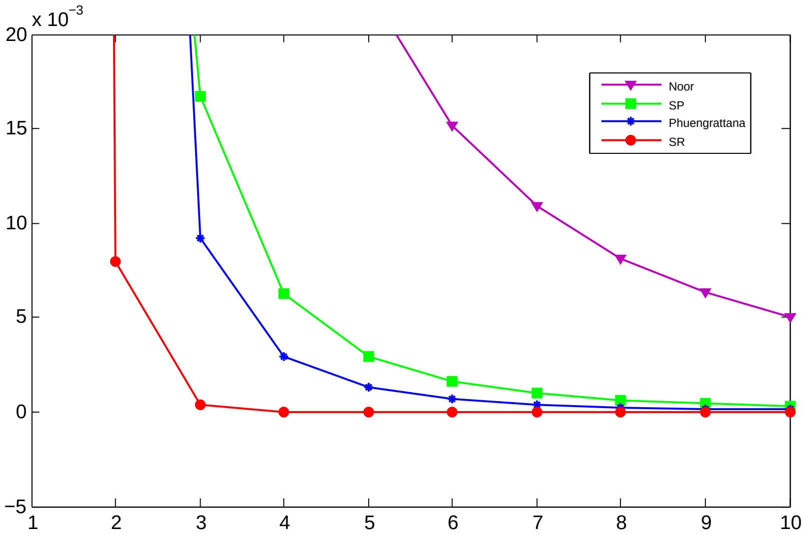

Figure 1: Comparison of errors in Example 5.1.

By Theorem 2.1, we know that the sequence $\{x_n\}$ converges to a unique fixed point p of T faster than that the others. From Tables 2 and 3, we observe that the sequence $\{x_n\}$ converges faster than the others and from Tables 1 and 2, we also note that $p \approx (0.50000, 0.69482)$ with accuracy 9 D.P.

6. Application to mixed type Volterra-Fredholm functional nonlinear integral equation

In this section, we use the SR-iteration to approximate the unique solution of mixed type Volterra-Fredholm functional nonlinear integral equation which is in the following from (see [14, 16]):

$$x(t) = F\Big(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds\Big), \tag{8}$$

where $[a_1, b_1]\times\cdots\times[a_m, b_m]$ be an interval in $\mathbb{R}^m$, $K, H : [a_1, b_1]\times\cdots\times[a_m, b_m]\times[a_1, b_1]\times\cdots\times[a_m, b_m]\times\mathbb{R} \to \mathbb{R}$ continuous functions and $F : [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3 \to \mathbb{R}$.

Theorem 6.1. *Let $X = C([a_1, b_1] \times \cdots \times [a_m, b_m])$ be the Banach space with the Cebyshev's norm. Assume that $T : X \to X$ is a mapping defined by*

$$T(x)(t) = F\Big(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds\Big), \tag{9}$$

for all $x \in X$. Suppose that the following condition holds:

$(C4)$ $K, H \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times [a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R})$;

$(C5)$ $F \in C([a_1, b_1] \times \cdots \times [a_m, b_m] \times \mathbb{R}^3)$;

$(C6)$ *there exist nonnegative constants κ, ζ, η with $\kappa < 1$ such that*

$$|F(t, u_1, v_1, w_1) - F(t, u_2, v_2, w_2)| \le \kappa|u_1 - u_2| + \zeta|v_1 - v_2| + \eta|w_1 - w_2|,$$

for all $t \in [a_1, b_1] \times \cdots \times [a_m, b_m], u_i, v_i, w_i \in \mathbb{R}, i = 1, 2$;

$(C7)$ *there exist nonnegative constants L_K, L_H such that*

$$|K(t,s,x(s)) - K(t,s,y(s))| \le L_K \min\big\{|y(s) - T(x)(s)|, |y(s) - T(y)(s)|\big\},$$
$$|H(t,s,x(s)) - H(t,s,y(s))| \le L_H \min\big\{|y(s) - T(x)(s)|, |y(s) - T(y)(s)|\big\},$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m], x, y \in X$.

Then T is a weak contraction with condition $(*)$.

Proof. Assume that $x, y \in X$ and $t, s \in [a_1, b_1] \times \cdots \times [a_m, b_m]$. Then

$$\begin{aligned}
&\|Tx - Ty\| = |T(x)(t) - T(y)(t)| \\
&= \Big|F\Big(t, x(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds\Big) \\
&\quad - F\Big(t, y(t), \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,y(s))ds, \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,y(s))ds\Big)\Big| \\
&\le \kappa|x(t) - y(t)| + \zeta\Big|\int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,x(s))ds - \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} K(t,s,y(s))ds\Big| \\
&\quad + \eta\Big|\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,x(s))ds - \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} H(t,s,y(s))ds\Big| \\
&\le \kappa|x(t) - y(t)| + \zeta \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} \big|K(t,s,x(s)) - K(t,s,y(s))\big|ds \\
&\quad + \eta \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \big|H(t,s,x(s)) - H(t,s,y(s))\big|ds \\
&\le \kappa|x(t) - y(t)| + \zeta \int_{a_1}^{t_1} \cdots \int_{a_m}^{t_m} L_K \min\big\{|y(s) - T(x)(s)|, |y(s) - T(y)(s)|\big\}ds \\
&\quad + \eta \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} L_H \min\big\{|y(s) - T(x)(s)|, |y(s) - T(y)(s)|\big\}ds \\
&\le \kappa\|x - y\| + (\zeta L_K + \eta L_H)(b_1 - a_1)\cdots(b_m - a_m)\min\big\{\|y - Tx\|, \|y - Ty\|\big\}.
\end{aligned}$$

By our assumptions, we can conclude that T is a weak contraction with condition $(*)$. □

The following result shows that the mixed type Volterra-Fredholm functional nonlinear integral equation (8) has a unique solution.

Theorem 6.2. *Let $X = C([a_1, b_1] \times \cdots \times [a_m, b_m])$ be the Banach space with the Cebyshev's norm. Assume that $T : X \to X$ is a mapping defined by (9). Suppose that the conditions $(C4) - (C7)$ in Theorem 6.1 hold. Then the equation (8) has a unique solution, say x^*, in X, and the Picard iteration converges to x^*.*

Proof. By Theorem 6.1, we know that T is a weak contraction with condition $(*)$. By Proposition 1.7, there exists a unique solution x^* of the equation (8) and the Picard iteration converges to x^*. □

Theorem 6.3. *Let $X = C([a_1, b_1] \times \cdots \times [a_m, b_m])$ be the Banach space with the Cebyshev's norm. Suppose that the sequence $\{x_n\}$ is defined by the SR-iteration and the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $[0, 1]$ which satisfy one of the conditions $(C1)$, $(C2)$, $(C3)$ in Theorem 2.1. Assume that $T : X \to X$ is a mapping defined by (9). Suppose that the conditions $(C4)-(C7)$ in Theorem 6.1 hold. Then the SR-iteration converges to a unique solution of the equation (8).*

Proof. By Theorem 2.1, Theorem 6.1, and Theorem 6.2, we can conclude that the SR-iteration converges to a unique solution of the equation (8). □

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