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Remarks on General Principally Injective Rings

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ABSTRACT

In [3], Chen and Li proved that every left CS and left p-injective ring is a QF-ring. In this study, we show that a right Noetherian, left CS and left GP-injective ring is right Artinian. We also prove that, if every singular simple right R-module is GP-injective, then $J(R)\cap Z_r = 0$. This gives a partially answer to a question of Ming [5].

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Key Words: CS-rings, GP-injective rings

GENELLEŞTİRİLMİŞ TEMEL İNJEKTİF HALKALAR ÜZERİNE

ÖZET

[3] de, Chen ve Li her sol CS ve sol p-injektif halkanın bir QF-halka olduğunu ispatladı. Bu çalışmada, sağ Noetherian, sol CS ve sol GP-injektif halkanın sağ Artinian olduğunu gösterdik. Daha sonra, her singüler basit sağ R-modüle GP-injektif ise $J(R)\cap Z_r = 0$ olduğunu ispatladık. Bu [5] de Ming'in sorusuna kısmen de olsa bir cevaptır.

Anahtar Kelimeler: CS-halkalar, GP-injektif halkalar, Noetherian, Artinian

1. INTRODUCTION

Throughout this paper, we assume that R is an associative ring (not necessarily commutative) with unity and M_R (resp., ${}_RM$) is unital right (resp. left) R-module. The notions, "$\leq$"will denote a submodule "$\leq_e$" an essential submodule and $l_R(X)$ (resp., $r_R(X)$) the left (resp. right) annihilator of a subset X of R, respectively. We also write "J", "Z_R" ("Z_l") and "S_R" ("S_l") for the Jacobson radical, the right (left) singular ideal and the right (left) socle of R, respectively. The texts by Anderson and Fuller [1] and [6] are the general references for notions of rings and modules not defined in this work.

A module M is called principally injective (*p-injective* for short) if every R-homomorphism from a principal right ideal aR to M extends to one from R_R to M, i.e., is given by left multiplication by an element of M. This is equivalent to saying that $l_M r_R(a) = Ma$ for all $a \in R$. R is called *right P-injective ring*, if R_R is a p-injective module. A ring R is said to be *general right principally injective* (briefly right GP-injective) if, for any $0 \neq a \in R$, there exists a positive integer $n = n(a)$ such that $a^n \neq 0$ and any right R-homomorphism from $a^n R$ to R extends to an endomorphism of R (see [10]).
A module M is called *extending* (or *CS*) if, for all $N \leq M$, there exists a direct summand $N' \leq_d M$ such that $N \leq_e N'$ and a ring R is called *right* (*resp., left*) *CS* if R_R (resp., ${}_R R$) is CS (see [6]). Examples of extending modules are injective modules, quasi-injective modules and uniform modules. The notions of p-injective rings, CS rings and GP-injective rings have been the focus of a number of research papers.

A right R-module M_R is called *mininjective* if, for each simple right ideal K of R, every R-morphism $\alpha : K \to M$ extends to R; equivalently if $\alpha : m$ is left multiplication by some element *m* of M. Hence the ring R is right mininjective if R_R is mininjective [8]. By [8, Lemma1.1], R is right mininjective if and only if, for $a \in R$, $l_r(a) = Ra$ where Ra is simple right ideal of R. A ring R is called right *simple injective* if for some R-homomorphism γ with $\gamma(I)$ simple extends to R. So we have the following strict hierarchy.

$$\{right\ self\ injective\} \subset \{right\ simple\ injective\} \subset \{right\ mininjective\}$$

A ring R is called a *right generalized V-ring* if every singular simple right R-module is injective.

In this paper, by using a method due to Chen and Li [3], we obtain that if R is a right Noetherian, left CS and left GP-injective ring, then R is right Artinian. We also prove that a right CF, right GP-injective and semi regular ring is a QF-ring.

2. RESULTS

Lemma 2.1. *Let R be a right Noetherian, left GP-injective and left finite dimensional ring. Then R is right Artinian.*

Proof. By [2, Theorem 4.6], every left GP-injective and left finite dimensional ring is semilocal. Note that in [2, Theorem 4.6], the reader is referred to [7, Theorem 3.3]. Now, because R is right Noetherian, there exists $n \geq 1$ such that $l(J^n) = l(J^{n+1}) = \cdots$. We claim that J is nilpotent. If not, there exists a maximal element $r(a)$ in nonempty set $\{r(b) : bJ^n \neq 0\}$. Assume that $J^{n+1} \neq 0$ and we get a contradiction. Since $l(J^n) = l(J^{2n})$, we have $J^{2n} \neq 0$. This implies that there exists an element $x \in J^n$ such that $axJ^n \neq 0$. Because of GP-injectivity of R, $l(J) \leq_{e\ R} R$ and so $l(J^n) \leq_{e\ R} R$ since $l(J) \leq l(J^n)$. Therefore there exists an element $y \in J^n$ such that $0 \neq yax \in l(J^n)$, and so $r(a) \leq r(ya)$. This is a contradiction of the maximality of $r(a)$. Hence J is nilpotent by Hopkin's Theorem [1], so R is a right Artinian ring.

In [3], Chen and Li proved that every right Noetherian, left CS and left p-injective ring is QF.

Theorem 2.2. *If R is a right Noetherian, left CS and left GP-injective ring, then R is right Artinian.*

Proof. Let R be a right Noetherian, left CS and left GP-injective ring. By [3, Theorem 2.11], R is a left finite dimensional ring. Hence R is a right Artinian ring by Lemma 2.1.

Hence one may ask the following question**.**

Question: Let R be a right Noetherian, left CS and left GP-injective ring. Is R left Artinian?

If the answer is true, then R is a QF-ring by [9, Theorem 3.4] because Soc (Re) is simple for any local idempotent $e \in R$.

Recall that a ring R said to be *right Kasch ring* if every simple right R-module embeds in R and R said to be a *semiregular ring* if R/J is von Neumann regular and idempotents can be lifted modulo J.

Theorem 2.3. [9, Theorem 3.31] *Suppose that R is a semilocal, left and right mininjective ring with ACC on right annihilators in which $S_r \leq_e R_R$. Then R is a QF-ring.*

Theorem 2.4. *Let R be a left GP-injective, left CS-ring with $S_l \leq_e R_R$ and right mininjective ring with ACC on right annihilators in which $S_r \leq_e R_R$. Then R is QF-ring.*

Proof. Let *e* be any primitive idempotent of R. It is easy to see that *Re* is uniform. This follows that *Soc*(*Re*) is simple and so R is left mininjective ring by [8]. Since R is a left GP-injective ring, we have $J(R) = Z({}_R R)$. By [9, Lemma 8.1], R is a right Kasch ring and so R is semiperfect by [9, Theorem 4.10]. By [9, Theorem 3.24] and [2, Theorem 2.3], R is a left Kasch ring with $S_r = S_l$. Therefore $S_r \leq_e R_R$ by [2, Theorem 2.3]. Hence R is a QF-ring by Theorem 2.3.

Remark: A ring R said to be a *CF-ring* if every cyclic right R-module embeds in R. In [3], they shown that;
(1) If R is right CF, semiregular and $J \leq Z_r$, then R is a right Artinian ring.
(2) A right CF, semiregular and right p-injective ring is QF.

Lemma 2.5. *Let R be a left Kasch and right CF-ring. Then R is a right Kasch, right Artinian (and so right Noetherian) and semilocal ring with $J = Z_R$.*

Proof. See [4, Theorem 2.6].

Theorem 2.6. *Assume that R is a right CF-ring. Then R is a QF-ring if the following are satisfied:*
(1) *R is semiregular and right GP-injective ring or;*
(2) *R is left Kasch ring or;*
(3) *R is semiregular and right mininjective ring with $S_r \leq_e R_R$*

Proof. (1) and (3) If R is a right GP-injective and semiregular ring with $S_r \leq_e R_R$, then $J = Z_R$. By Remark, R is right Artinian. Because of right mininjectivity of R, we have R is a QF-ring by Theorem 2.3.
(2) It follows from Lemma 2.5 and Theorem 2.2.

Theorem 2.7. *Assume that R is a right CF-ring and right mininjective ring. Then the following are equivalent:*
(1) *R is QF*
(2) *S_l is finitely generated as left R-module*
(3) *R is semilocal*

Proof. (1)⇒(2) Clear.
(2)⇒(3) By assumption, R is a left p-injective and right Kasch ring, and so $S_l = S_r$. It is enough to show that $J = J(S_l)$ and R/J is semisimple. Let $x \in J(S_l)$. For maximal left ideal I of R and simple left ideal *A* of R, we consider the isomorphism $f: R/I \to A$. Clearly, $f(R/I)x = f((R/I)x) = 0$, that is $Ax = 0$. This implies that $(R/I)x = 0$ and so $x \in I$. The other side is obvious. Hence $J = J(S_l)$. Now, since S_l is finitely generated as left R-module, we write $S_l = Rx_1 \oplus Rx_2 \oplus \cdots \oplus Rx_n$, where each Rx_i is a simple left ideal of R. Note that

$$J = r(S_l) = \bigcap_{i=1}^{n} r(x_i)$$

and

$$g: R/J = R/r(S_l) = R/\bigcap_{i=1}^{n} r(x_i) \to R/\bigoplus_{i=1}^{n} r(x_i)$$

is a monomorphism. Therefore R/J is semisimple.
(3)⇒(1) If R is a semilocal, right mininjective and right CF-ring, then R is quasi-Frobenius by [9, Theorem 8.11].

Lemma 2.8. *Assume that R is a right simple injective ring, $M \neq \oplus_n R$ and $M_R \neq R_R$. If M is a finitely generated right R-module then M is semisimple.*

Proof. Let $M = m_1R + m_2R + \cdots + m_nR$ be a finitely generated R-module and F be a free R-module. Then we have the epimorphism $g: F \cong \oplus_n R \to M \cong \oplus_n R/Ker(f)$ defined by $(x_i) = \sum_{i=1}^{n} m_i(x_i)$ where $f: F \to M$ is an epimorphism. Since R is a right simple injective ring, there exists $h: \oplus_n R \to \oplus_n a_i R$. Then $\oplus_n a_i R$ is semisimple and $Ker(h) \subseteq Ker(g)$. Since $\oplus_n a_i R$ is semisimple and $\alpha: \oplus_n a_i R \to M$ is an epimorphism, we can say that M is semisimple.

Theorem 2.9. *Assume that R is a right (left) self-*

injective ring, $M \neq \oplus_n R$ *and* $M_R \neq R_R$ *. Then,*

(1) *Every finitely generated right (left)* R *-module is a right (left) Artinian and right (left) Noetherian module of finite length.*

(2) *Every finitely generated right* R *-module is injective and projective*.

Proof. (1) By Lemma 2.8.

(2) Let R be a right self-injective ring and M be a finitely generated R-module. By Lemma 2.8, M is semisimple. This implies that every submodule of M is a direct summand.

Corollary 2.10. *Assume that* R *is a right perfect and two sided self injective ring such that* $Soc(eR) \neq 0$ *for every local idempotent* e *of* R*. Let ,* $M \neq \oplus_n R$ *and* $M_R \neq R_R$*. Then is a QF-ring.*

Proof. By [9, Theorem 6.16], R is right and left Kasch ring. By Theorems 6.19 and 6.20 in [9], the ring R is finitely cogenerated. Now, by Theorem 2.9, R is left Artinian. This implies that R *is a QF-ring.*

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