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Research Article

Structure Functions of Cone Curves in $\mathbb{E}_2^3$ and $\mathbb{E}_2^4$

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Abstract
This paper studies cone curves by establishing structure functions. The curvature functions of spacelike and timelike cone curves are expressed in the form of structure functions and the relationship between the defined structure functions is obtined. In addition, some characters and structures for cone curves in $\mathbb{E}_2^3$ and $\mathbb{E}_2^4$ are discussed. Finally, we give some examples.

Keywords and 2010 Mathematics Subject Classification
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MSC: 53A35

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1. Introduction

With the development of Einstein's theory of relativity, we know that the basic outline of black hole event horizon can be reflected through null submanifolds in pseudo Euclidean space [1], therefore more and more geometers and physicists are committed to the study of null submanifolds such as null special curves and lightlike cones [1–7], and have obtained some interesting conclusions. In curve theory, We can divide curves into three types of curves: timelike curves, null curves and spacelike curves. The timelike curves, which are the world lines of massive particles and which come with a preferred parameter, namely proper time; null-geodesics, which are null rays or world lines of massless particles, and spacelike curves, which are everywhere neither timelike nor null. The set of all null-geodesics through a point p generates a lightlike cone, $N(p)$: all timelike curves through p fill the inside of $N(p)$, and all spacelike curves fill the outside of $N(p)$. Lightlike cones are important for both mathematicians and physicists. Many studies have been done on curves in the lightlike cone by many mathematicians. For example, Sun and Pei studied the null curves on $\mathbb{Q}^3$ and unit semi-Euclidean 3-spheres, and the definitions of null Bertrand curves on $\mathbb{Q}^3$ and unit semi-Euclidean 3-spheres are also introduced in [8]. Penrose R. indicated that null curves on null cone were null geodesics [9]. In 2011, Meng [10] firstly proposed the concept of structure functions for cone curves, by which many curves can be visually displayed with the help of computers and lots of pending problems on null submanifolds can be solved. In [11, 12], the author gave the representation formula for null curves in $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$ by using the generalized structure functions which were defined by Meng and studied the properties of those null curves. In 2020, using similar method, Qian discussed pseudo null curves in $\mathbb{E}_1^3$, and got some interesting conclusions [13], which greatly promoted the understanding of pseudo Euclidean space.

Motivated by those ideas, in this paper, we study timelike and spacelike curves in $\mathbb{E}_2^3$ and $\mathbb{E}_2^4$. In Section 3, we first establish the structure functions for spacelike curves in $\mathbb{Q}^3$, then consider some special curves, and give the structure functions and expressions in these cases. In Section 4, we discuss the relationship between the cone curvatures and the structure functions of the timelike curves in $\mathbb{Q}^2$, and discuss some timelike curves for which the curvature functions are constant. In Section 5, we study the expressions for timelike curves in $\mathbb{Q}^3$ and give the relationship between the curvature functions.

2. Preliminaries

We know that the inner product of vectors in $\mathbb{E}_2^m$ space is defined as

$$\langle x,y\rangle=\sum_{i=1}^{m-2}x_iy_i-\sum_{j=m-1}^{m}x_jy_j.$$

A vector v in $\mathbb{E}_2^m$ is called spacelike, timelike or lightlike, if $\langle v,v\rangle>0$, $\langle v,v\rangle<0$, and $\langle v,v\rangle=0$, respectively. In particular, the vector $v=0$ is said to be spacelike. A curve $\xi(s):I\to\mathbb{E}_2^m$ is called spacelike, timelike or null (lightlike) if all of its velocity vectors ξ' satisfy $\langle\xi',\xi'\rangle>0$, $\langle\xi',\xi'\rangle<0$, and $\langle\xi',\xi'\rangle=0$. The norm of a vector v is given by $\parallel v\parallel=\sqrt{|\langle v,v\rangle|}$.

Let c be a fixed point in $\mathbb{E}_q^m$, the pseudo-Riemannian lightlike cone (quadric cone) is defined by [14]

$$\mathbb{Q}_q^m(c)=\{x\in\mathbb{E}_q^{m+1}:\langle x-c,x-c\rangle=0\}.$$

When $c=0$, we simply denote as $\mathbb{Q}^m$ and call it the lightlike cone (or simply the light cone) [15], we assume that the curve

$$x(t):I\to\mathbb{Q}^{m+1}\subset\mathbb{E}_2^{m+2}$$

is a regular curve in $\mathbb{Q}^{m+1}$.

Except for special instructions, the curves we discussed in this paper are all regular curves.

3. Structure functions of spacelike curves in $\mathbb{Q}^3$

Theorem 1. *Let $\xi(s):I\to\mathbb{Q}^3\subset\mathbb{E}_2^4$ be a spacelike curve with parameterized by arc length s. Then $\xi(s)$ can be written as*

$$\xi(s)=\rho(2f,1-f^2+g^2,2g,1+f^2-g^2),$$

where $\rho(s)=\dfrac{1}{2\sqrt{f_s^2-g_s^2}}$.

Proof. We set $\xi=(\xi_1,\xi_2,\xi_3,\xi_4)$ and obtain

$$\xi_1^2+\xi_2^2-\xi_3^2-\xi_4^2=0.$$

By a direct calculation, we can get

$$\frac{\xi_1+\xi_3}{\xi_4+\xi_2}=\frac{\xi_4-\xi_2}{\xi_1-\xi_3},\quad\text{or}\quad\frac{\xi_1+\xi_3}{\xi_4-\xi_2}=\frac{\xi_4+\xi_2}{\xi_1-\xi_3}.$$

Without loss of generality, we may assume that

$$\begin{cases}\xi_1=2\rho f,\\ \xi_3=2\rho g,\\ \xi_4+\xi_2=2\rho,\end{cases}\tag{1}$$

then, we get

$$\begin{cases}\xi_1=2\rho f,\\ \xi_2=\rho(1-f^2+g^2),\\ \xi_3=2\rho g,\\ \xi_4=\rho(1+f^2-g^2).\end{cases}\tag{2}$$

Therefore, the spacelike curve $\xi(s)$ can be written as

$$\xi=\xi(s)=(\xi_1,\xi_2,\xi_3,\xi_4)=\rho(2f,1-f^2+g^2,2g,1+f^2-g^2).\tag{3}$$

From (3) we have

$$\xi_s=\rho_s(2f,1-f^2+g^2,2g,1+f^2-g^2)+2\rho(f_s,-ff_s+gg_s,g_s,ff_s-gg_s).$$

Since $\langle \xi_s, \xi_s \rangle = 1$, we get

$$4\rho^2(f_s^2 - g_s^2) = 1, \tag{4}$$

by an appropriate transformation if necessary, we have

$$\rho(s) = \frac{1}{2\sqrt{f_s^2 - g_s^2}}. \tag{5}$$

This ends the proof. ∎

Definition 2. *We define the functions $f(s)$ and $g(s)$ as the structure functions of the cone curve $\xi(s) : I \to \mathbb{Q}^3 \subset \mathbb{E}_2^4$ with parameterized by arc length s.*

Putting

$$\gamma(s) = -\xi_{ss} - \frac{1}{2}\langle \xi_{ss}, \xi_{ss} \rangle \xi(s), \tag{6}$$

we have

$$\langle \gamma, \gamma \rangle = \langle \xi, \xi \rangle = \langle \gamma, \xi_s \rangle = 0, \langle \xi, \gamma \rangle = 1. \tag{7}$$

Let $\alpha(s) = \xi_s(s)$, and we select $\beta(s)$ such that

$$\det(\xi(s), \alpha(s), \beta(s), \gamma(s)) = 1. \tag{8}$$

Then from (6) we have

$$\alpha_s(s) = \xi_{ss}(s) = -\frac{1}{2}\langle \xi_{ss}, \xi_{ss} \rangle \xi(s) - \gamma(s) = k(s)\xi(s) - \gamma(s). \tag{9}$$

Thus, the Frenet formulas of the curve $\xi(s)$ are the followings:

$$\begin{cases} \xi_s(s) = \alpha(s), \\ \alpha_s(s) = k(s)\xi(s) - \gamma(s), \\ \beta_s(s) = \tau(s)\xi(s), \\ \gamma_s(s) = -k(s)\alpha(s) - \tau(s)\beta(s). \end{cases} \tag{10}$$

Position 1. *The frame $\{\xi(s), \alpha(s), \beta(s), \gamma(s)\}$ is called the cone Frenet frame of the curve $\xi(s)$ in $\mathbb{Q}^3$.*

Theorem 3. *Let $\xi(s) : I \to \mathbb{Q}^3 \subset \mathbb{E}_2^4$ be a spacelike curve with parameterized by arc length s. $f(s)$ and $g(s)$ are the structure functions of the cone curve $\xi(s)$. Then the curvature functions $k(s)$, $\tau(s)$ can be expressed as*

$$\begin{cases} k(s) = \frac{1}{2}[(\ln\rho)_s]^2 + (\ln\rho)_{ss} + \frac{1}{2}\theta_s^2, \\ \tau^2(s) = -(\theta_s(\ln\rho)_s + \theta_{ss})^2, \end{cases}$$

where $\theta_s = (1 - \frac{g_s^2}{f_s^2})^{-1}(\frac{g_s}{f_s})_s$.

Proof. Assume that

$$\begin{cases} 2f_s = \rho^{-1}\cosh\theta, \\ 2g_s = \rho^{-1}\sinh\theta. \end{cases} \tag{11}$$

where

$$\tanh\theta = \frac{g_s(s)}{f_s(s)}. \tag{12}$$

Differentiating equation (12) with respect to s, we have

$$\theta_s = (1 - \frac{g_s^2}{f_s^2})^{-1}(\frac{g_s}{f_s})_s. \tag{13}$$

By (11), we can obtain

$$\begin{cases} 2f_{ss} = -\rho^{-2}\rho_s \cosh\theta + \rho^{-1}\theta_s \sinh\theta, \\ 2g_{ss} = -\rho^{-2}\rho_s \sinh\theta + \rho^{-1}\theta_s \cosh\theta, \end{cases} \tag{14}$$

and

$$\begin{cases} 2f_{sss} = (2\rho^{-3}\rho_s^2 - \rho^{-2}\rho_{ss} + \rho^{-1}\theta_s^2)\cosh\theta + (\rho^{-1}\theta_{ss} - 2\rho^{-2}\rho_s\theta_s)\sinh\theta, \\ 2g_{sss} = (2\rho^{-3}\rho_s^2 - \rho^{-2}\rho_{ss} + \rho^{-1}\theta_s^2)\sinh\theta + (\rho^{-1}\theta_{ss} - 2\rho^{-2}\rho_s\theta_s)\cosh\theta. \end{cases} \tag{15}$$

By (3), we can calculate that

$$\begin{aligned} \xi_{ss} = {} & \rho_{ss}(2f, 1-f^2+g^2, 2g, 1+f^2-g^2) + 4\rho_s(f_s, -ff_s+gg_s, g_s, ff_s-gg_s) \\ & + 2\rho(f_{ss}, -f_s^2-ff_{ss}+g_s^2+gg_{ss}, g_{ss}, f_s^2+ff_{ss}-g_s^2-gg_{ss}), \end{aligned}$$

and

$$\begin{aligned} \xi_{sss} = {} & \rho_{sss}(2f, 1-f^2+g^2, 2g, 1+f^2-g^2) + 6\rho_{ss}(f_s, -ff_s+gg_s, g_s, ff_s-gg_s) \\ & + 6\rho_s(f_{ss}, -f_s^2-ff_{ss}+g_s^2+gg_{ss}, g_{ss}, f_s^2+ff_{ss}-g_s^2-gg_{ss}) \\ & + 2\rho(f_{sss}, -3f_sf_{ss}-ff_{sss}+3g_sg_{ss}+gg_{sss}, g_{sss}, 3f_sf_{ss}+ff_{sss}-3g_sg_{ss}-gg_{sss}). \end{aligned} \tag{16}$$

From (11), (14) and (15) we have

$$\begin{aligned} \langle \xi_{ss}, \xi_{ss} \rangle &= 16\rho_s^2(f_s^2-g_s^2) + 4\rho^2(f_{ss}^2-g_{ss}^2) + 8\rho\rho_{ss}(g_s^2-f_s^2) + 16\rho\rho_s(f_sf_{ss}-g_sg_{ss}) \\ &= \rho^{-2}\rho_s^2 - \theta_s^2 - 2\rho^{-1}\rho_{ss}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} \langle \xi_{sss}, \xi_{sss} \rangle = {} & 36\rho_{ss}^2(f_s^2-g_s^2) + 36\rho_s^2(f_{ss}^2-g_{ss}^2) + 4\rho^2(f_{sss}^2-g_{sss}^2) \\ & + 24\rho_s\rho_{sss}(g_s^2-f_s^2) + 24\rho\rho_{ss}(f_sf_{sss}-g_sg_{sss}) \\ & + 24\rho\rho_s(f_{ss}f_{sss}-g_{ss}g_{sss}) + 24\rho\rho_{sss}(g_sg_{ss}-f_sf_{ss}) \\ & + 72\rho_s\rho_{ss}(f_sf_{ss}-g_sg_{ss}) \\ = {} & 4\rho^{-2}\rho_{ss}^2 + \rho^{-4}\rho_s^4 - 3\rho^{-2}\rho_s^2\theta_s^2 - 4\rho^{-3}\rho_s^2\rho_{ss} \\ & + 4\rho^{-1}\rho_{ss}\theta_s^2 + \theta_s^4 - 2\rho^{-1}\rho_s\theta_s\theta_{ss} - \theta_{ss}^2. \end{aligned} \tag{18}$$

From (9) and (17), we have

$$\begin{aligned} k(s) &= -\frac{1}{2}\langle \xi_{ss}, \xi_{ss} \rangle \\ &= -\frac{1}{2}\rho^{-2}\rho_s^2 + \frac{1}{2}\theta_s^2 + \rho^{-1}\rho_{ss} \\ &= \frac{1}{2}[(\ln\rho)_s]^2 + (\ln\rho)_{ss} + \frac{1}{2}\theta_s^2. \end{aligned} \tag{19}$$

From (9), (10) and $\langle \xi, \xi_{sss} \rangle = 0$, $\langle \xi_s, \xi_{sss} \rangle = -\langle \xi_{ss}, \xi_{ss} \rangle$, we have

$$\tau(s)\beta(s) = \xi_{sss} + \langle \xi_{ss}, \xi_{ss} \rangle \xi_s + \langle \xi_{ss}, \xi_{sss} \rangle \xi \tag{20}$$

and then

$$\begin{aligned} \tau^2(s) &= \langle -\xi_{sss} + k_s\xi + k\xi_s, -\xi_{sss} + k_s\xi + k\xi_s \rangle - k^2 \\ &= -(\rho^{-1}\rho_s\theta_s + \theta_{ss})^2 \\ &= -(\theta_s(\ln\rho)_s + \theta_{ss})^2. \end{aligned} \tag{21}$$

This ends the proof. ■

Some examples of spacelike curves in $\mathbb{Q}^3$ are given below.

Example 4. *Let* $\xi_1(s) = (s, \frac{1}{2}s^2 - 1, 1, \frac{1}{2}s^2)$ *be a spacelike curve in* $\mathbb{Q}^3$ *with curvature* $k(s) = 0$, $\tau = 0$. *Then the structure functions can be written as*

$$\begin{cases} f(s) = \frac{s}{s^2-1}, \\ g(s) = \frac{1}{s^2-1}, \\ \rho(s) = \frac{s^2-1}{2}. \end{cases}$$

Example 5. *Let* $\xi_2(s) = (m, \sqrt{m^2-n^2}\sinh\frac{s}{\sqrt{m^2-n^2}}, n, \sqrt{m^2-n^2}\cosh\frac{s}{\sqrt{m^2-n^2}})$ *be a spacelike curve in* $\mathbb{Q}^3$ *with curvature* $k(s) > 0$, $\tau = 0$. *Then the structure functions can be written as*

$$\begin{cases} f(s) = \frac{m}{2\rho}, \\ g(s) = \frac{n}{2\rho}, \\ 2\rho(s) = \sqrt{m^2-n^2}(\cosh\frac{s}{\sqrt{m^2-n^2}} + \sinh\frac{s}{\sqrt{m^2-n^2}}), \end{cases}$$

where $m^2 - n^2 > 0$.

Example 6. *Let* $\xi_3(s) = (\sqrt{m^2-n^2}\sinh\frac{s}{\sqrt{m^2-n^2}}, m, \sqrt{m^2-n^2}\cosh\frac{s}{\sqrt{m^2-n^2}}, b)$ *be a spacelike curve in* $\mathbb{Q}^3$ *with curvature* $k(s) < 0$, $\tau = 0$. *Then the structure functions can be written as*

$$\begin{cases} f(s) = \frac{\sqrt{m^2-n^2}}{m+n}\sinh\frac{s}{\sqrt{m^2-n^2}}, \\ g(s) = \frac{\sqrt{m^2-n^2}}{m+n}\cosh\frac{s}{\sqrt{m^2-n^2}}, \\ 2\rho(s) = m+n, \end{cases}$$

where $m^2 - n^2 > 0$.

4. Structure functions of timelike curves in $\mathbb{Q}^2$

Theorem 7. *Let* $\xi(s) : I \to \mathbb{Q}^2 \subset \mathbb{E}^3_2$ *be a timelike curve with parameterized by arc length s. Then* $\xi(s)$ *can be written as*

$$\xi(s) = \frac{1}{2}f_s^{-1}(f^2+1, f^2-1, 2f),$$

where $f_s = \frac{df}{ds} \neq constant$.

Proof. We set $\xi(s) = (\xi_1, \xi_2, \xi_3)$ and obtain

$$\xi_1^2 - \xi_2^2 - \xi_3^2 = 0.$$

By a direct calculation, we can get

$$\frac{\xi_1 + \xi_2}{\xi_3} = \frac{\xi_3}{\xi_1 - \xi_2}.$$

Without loss of generality, we may assume that

$$\frac{\xi_1 + \xi_2}{\xi_3} = \frac{\xi_3}{\xi_1 - \xi_2} = f(s), \quad \xi_3 = 2\rho(s), \tag{22}$$

so that

$$\begin{cases} \xi_1 = \rho(f + f^{-1}), \\ \xi_2 = \rho(f - f^{-1}), \\ \xi_3 = 2\rho. \end{cases} \tag{23}$$

Therefore, the timelike curves $\xi(s)$ can be written as

$$\xi=\xi(s)=(\xi_1,\xi_2,\xi_3)=\rho(f+f^{-1},f-f^{-1},2). \tag{24}$$

From (24) we have

$$\xi_s=\rho_s(f+f^{-1},f-f^{-1},2)+\rho f_s(1-f^{-2},1+f^{-2},0).$$

Since $\langle\xi_s,\xi_s\rangle=-1$, we get

$$4\rho^2 f_s^2 f^{-2}=1, \tag{25}$$

by an appropriate transformation if necessary, we have

$$\rho(s)=\frac{f(s)}{2f_s(s)}. \tag{26}$$

Thus, we have

$$\xi(s)=\frac{f}{2f_s}(f+f^{-1},f-f^{-1},2)=\frac{1}{2}f_s^{-1}(f^2+1,f^2-1,2f). \tag{27}$$

This ends the proof. ■

Definition 8. *We define the function $f(s)$ as the structure function of the cone curve $\xi(s):I\to\mathbb{Q}^2\subset\mathbb{E}_2^3$ with arc length parameter s.*

Putting

$$\gamma(s)=-\xi_{ss}(s)+\frac{1}{2}\langle\xi_{ss}(s),\xi_{ss}(s)\rangle\xi(s). \tag{28}$$

By (24) we can obtain

$$\begin{aligned}\gamma(s)=&-\xi_{ss}+\frac{1}{2}\langle\xi_{ss}(s),\xi_{ss}(s)\rangle\xi(s)\\=&\frac{1}{4}f_s^{-3}f_{ss}^2(f^2+1,f^2-1,2f)-f_s^{-1}f_{ss}(f,f,1)+f_s(1,1,0)\\=&-\frac{1}{2}f_s^{-2}f_{ss}^2\xi+f_s^{-1}f_{ss}(f,f,1)-f_s(1,1,0),\end{aligned} \tag{29}$$

and

$$\langle\gamma,\gamma\rangle=\langle\xi,\xi\rangle=\langle\gamma,\xi_s\rangle=0,\langle\xi,\gamma\rangle=-1. \tag{30}$$

Let $\alpha(s)=\xi_s(s)$, then

$$\begin{cases}\xi_s(s)=\alpha(s),\\\alpha_s(s)=k(s)\xi(s)-\gamma(s),\\\gamma_s(s)=-k(s)\alpha(s).\end{cases} \tag{31}$$

Position 2. *The frame $\{\xi(s),\alpha(s),\gamma(s)\}$ is called the cone Frenet frame of the curve $\xi(s)$ in $\mathbb{Q}^2$.*

Theorem 9. *Let $\xi(s):I\to\mathbb{Q}^2\subset\mathbb{E}_2^3$ be a timelike curve with parameterized by arc length s. $f(s)$ is the structure function of the cone curve $\xi(s)$. Then the curvature function $k(s)$ can be expressed as*

$$k(s)=\frac{1}{2}[(\ln f_s)_s]^2-[(\ln f_s)_s]_s.$$

Proof. From (27) we have

$$\langle \xi_{ss}, \xi_{ss} \rangle = -f_s^{-2} f_{ss}^2 + 2f_s(2f_s^{-3} f_{ss}^2 - f_s^{-2} f_{sss}) = 3f_s^{-2} f_{ss}^2 - 2f_s^{-1} f_{sss}. \tag{32}$$

From (28), (31) and (32), we have

$$k(s) = \frac{1}{2}\langle \xi_{ss}, \xi_{ss} \rangle = \frac{3}{2} f_s^{-2} f_{ss}^2 - f_s^{-2} f_{sss} = \frac{1}{2}[(\ln f_s)_s]^2 - [(\ln f_s)_s]_s. \tag{33}$$

This ends the proof. ■

By solving above Riccati equation (33) with vanishing linear term, we obtain some special solutions and have the following classifications for special curves in $\mathbb{Q}^2$.

Remark 10. *In particular, when the curve $\xi(s)$ is planar, the structure function $f(s)$ satisfies*

$$((\ln f_s)_s)^2 - 2(\ln f_s)_{ss} = 2k(s) = c = constant,$$

and we have the following classifications:

1. *when $-m^2 = c < 0$, $f(s) = \frac{2}{m}\tan\frac{ms}{2}$, $\xi(s)$ is an ellipse, and can be written as*
$$\xi(s) = (\frac{2}{m^2}\sin^2\frac{ms}{2} + \frac{1}{2}\cos^2\frac{ms}{2}, \frac{2}{m^2}\sin^2\frac{ms}{2} - \frac{1}{2}\cos^2\frac{ms}{2}, \frac{1}{m}\sin ms).$$
2. *when $c = 0$, $f(s) = -\frac{m}{s}$, $\xi(s)$ is a parabola, and can be written as*
$$\xi(s) = (\frac{m}{2} + \frac{s^2}{2m}, \frac{m}{2} - \frac{s^2}{2m}, -s).$$
3. *when $m^2 = c > 0$, $f(s) = \frac{2}{m}\tanh\frac{ms}{2}$, $\xi(s)$ is a hyperbola, and can be written as*
$$\xi(s) = (\frac{2}{m^2}\sinh^2\frac{ms}{2} + \frac{1}{2}\cosh^2\frac{ms}{2}, \frac{2}{m^2}\sinh^2\frac{ms}{2} - \frac{1}{2}\cosh^2\frac{ms}{2}, \frac{1}{m}\sinh ms).$$

Remark 11. *In particular, when the curve $\xi(s)$ is a non planar helix, the structure function $f(s)$ satisfies*

$$((\ln f_s)_s)^2 - 2(\ln f_s)_{ss} = 2k(s) = m(s+n)^{-2}. \tag{34}$$

Moreover, we have

1. *$f(s) = s^c$ or $f(s) = s^{-c}$, for $c \neq 0, \pm 1$ and $m = c^2 - 1$;*
2. *$f(s) = \frac{c}{\ln s}$ or $f(s) = \frac{\ln s}{c}$, for $c \neq 0$ and $m = -1$;*
3. *$f(s) = \frac{2}{c}\tan(\frac{c}{2}\ln s)$ or $f(s) = -\frac{2}{c}\tan^{-1}(\frac{c}{2}\ln s)$, for $c \neq 0, \pm 1$ and $m = -c^2 - 1$.*

Proof. When $2k(s) = m(s+n)^{-2}$, we can assume $\eta(s) = (\ln f_s)_s$. The equation (34) can be written as $\eta^2 - 2\eta_s = m(s+n)^{-2}$. By solving this equation, we can get the conclusions of Remark 11. ■

Some examples of timelike curves in $\mathbb{Q}^2$ are given below.

Example 12. *Let $\xi_4(s)$ be a timelike curve in $\mathbb{Q}^2$ with curvature $k(s) = 0$. Then $\xi_4(s)$ can be written as $\xi_4(s) = (\frac{1+s^2}{2}, \frac{1-s^2}{2}, -s)$(see Figure 1).*

Example 13. *Let $\xi_5(s)$ be a timelike curve in $\mathbb{Q}^2$ with curvature $k(s) = -\frac{1}{2s^2}$. Then $\xi_5(s)$ can be written as $\xi_5(s) = (\frac{s\ln^2 s + s}{2}, \frac{s\ln^2 s - s}{2}, s\ln s)$, (s>0)(see Figure 2).*

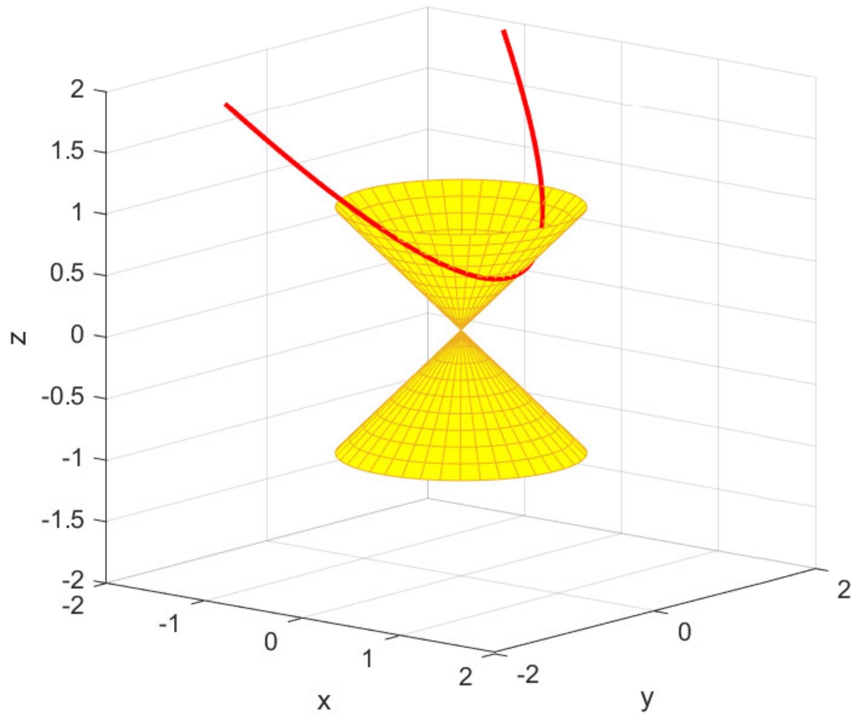

Fig. 1. $\xi_4(s)$ shown with cone

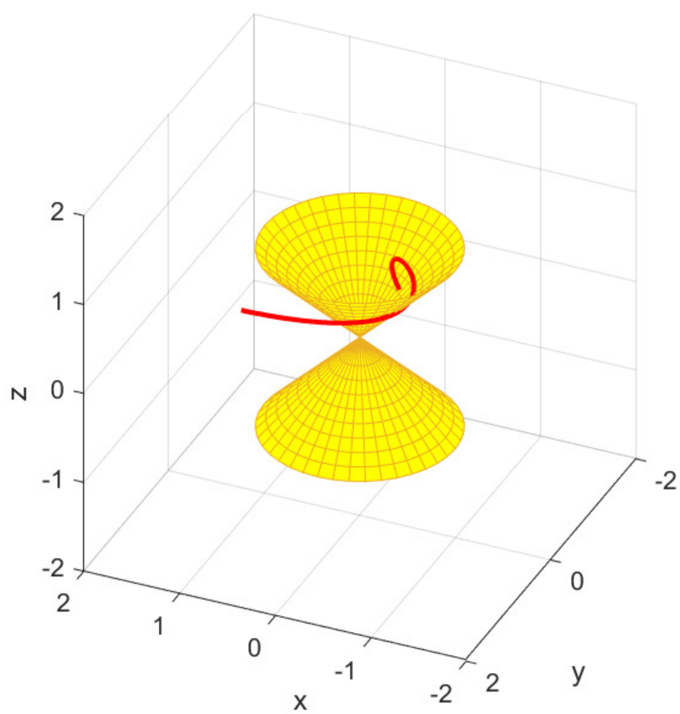

Fig. 2. $\xi_5(s)$ shown with cone

5. Structure functions of timelike curves in $\mathbb{Q}^3$

Theorem 14. *Let $\xi(s): I \to \mathbb{Q}^3 \subset \mathbb{E}^4_2$ be a timelike curve with arc length parameter s. Then $\xi(s)$ can be written as*

$$\xi(s) = \rho(2f, 1 - f^2 + g^2, 2g, 1 + f^2 - g^2),$$

where $\rho(s) = \frac{1}{2\sqrt{g_s^2 - f_s^2}}$.

Proof. We set $\xi(s) = (\xi_1, \xi_2, \xi_3, \xi_4)$ and obtain

$$\xi_1^2 + \xi_2^2 - \xi_3^2 - \xi_4^2 = 0.$$

By a direct calculation, we can get

$$\frac{\xi_1 + \xi_3}{\xi_4 + \xi_2} = \frac{\xi_4 - \xi_2}{\xi_1 - \xi_3}, \quad \text{or} \quad \frac{\xi_1 + \xi_3}{\xi_4 - \xi_2} = \frac{\xi_4 + \xi_2}{\xi_1 - \xi_3}.$$

Without loss of generality, we may assume that

$$\begin{cases} \xi_1 = 2\rho f, \\ \xi_4 + \xi_2 = 2\rho(s), \\ \xi_3 = 2\rho g, \end{cases} \tag{35}$$

then, we get

$$\begin{cases} \xi_1 = 2\rho f, \\ \xi_2 = \rho(1 - f^2 + g^2), \\ \xi_3 = 2\rho g, \\ \xi_4 = \rho(1 + f^2 - g^2). \end{cases} \tag{36}$$

Therefore, the timelike curves $\xi(s)$ can be written as

$$\xi = \xi(s) = (\xi_1, \xi_2, \xi_3, \xi_4) = \rho(2f, 1 - f^2 + g^2, 2g, 1 + f^2 - g^2). \tag{37}$$

From (37) we have

$$\xi_s = \rho_s(2f, 1 - f^2 + g^2, 2g, 1 + f^2 - g^2) + 2\rho(f_s, -ff_s + gg_s, g_s, ff_s - gg_s).$$

Since $\langle \xi_s, \xi_s \rangle = -1$, we get

$$4\rho^2(g_s^2 - f_s^2) = 1, \tag{38}$$

by an appropriate transformation if necessary, we have

$$\rho(s) = \frac{1}{2\sqrt{g_s^2 - f_s^2}}. \tag{39}$$

This ends the proof. ∎

Definition 15. *We define the functions $f(s)$ and $g(s)$ as the structure functions of the cone curve $\xi(s): I \to \mathbb{Q}^3 \subset \mathbb{E}^4_2$ with arc length parameter s.*

Putting

$$\gamma(s) = -\xi_{ss} + \frac{1}{2}\langle \xi_{ss}, \xi_{ss} \rangle \xi, \tag{40}$$

we have

$$\langle \gamma, \gamma \rangle = \langle \xi, \xi \rangle = \langle \gamma, \xi_s \rangle = 0, \langle \xi, \gamma \rangle = 1. \tag{41}$$

Let $\alpha(s) = \xi_s(s)$, and we select $\beta(s)$ such that

$$\det(\xi(s), \alpha(s), \beta(s), \gamma(s)) = 1. \tag{42}$$

Then from (40) we have

$$\alpha_s(s) = \xi_{ss}(s) = \frac{1}{2}\langle \xi_{ss}, \xi_{ss}\rangle \xi(s) - \gamma(s). \tag{43}$$

Therefore, the Frenet formulas of the curve $\xi(s)$ are the followings:

$$\begin{cases} \xi_s(s) = \alpha(s), \\ \alpha_s(s) = k(s)\xi(s) - \gamma(s), \\ \beta_s(s) = \tau(s)\xi(s), \\ \gamma_s(s) = -k(s)\alpha(s) - \tau(s)\beta(s). \end{cases} \tag{44}$$

Position 3. *The frame $\{\xi(s), \alpha(s), \beta(s), \gamma(s)\}$ is called the cone Frenet frame of the curve $\xi(s)$ in $\mathbb{Q}^3$.*

Theorem 16. *Let $\xi(s) : I \to \mathbb{Q}^3 \subset \mathbb{E}^4_2$ be a timelike curve with parameterized by arc length s. $f(s)$ and $g(s)$ are the structure functions of the cone curve $\xi(s)$. Then the curvature functions $k(s)$, $\tau(s)$ can be expressed as*

$$\begin{cases} k(s) = \frac{1}{2}[(\ln\rho)_s]^2 + (\ln\rho)_{ss} + \frac{1}{2}\theta_s^2, \\ \tau^2(s) = (\theta_s(\ln\rho)_s + \theta_{ss})^2, \end{cases}$$

where $\theta_s = (1 - \dfrac{f_s^2}{g_s^2})^{-1}(\dfrac{f_s}{g_s})_s$.

Proof. We can assume

$$\begin{cases} 2g_s = \rho^{-1}\cosh\theta, \\ 2f_s = \rho^{-1}\sinh\theta. \end{cases} \tag{45}$$

where

$$\tanh\theta = \frac{f_s(s)}{g_s(s)}. \tag{46}$$

Differentiating equation (46) with respect to s, we have

$$\theta_s = (1 - \frac{f_s^2}{g_s^2})^{-1}(\frac{f_s}{g_s})_s. \tag{47}$$

By (45) we can obtain

$$\begin{cases} 2g_{ss} = -\rho^{-2}\rho_s\cosh\theta + \rho^{-1}\theta_s\sinh\theta, \\ 2f_{ss} = -\rho^{-2}\rho_s\sinh\theta + \rho^{-1}\theta_s\cosh\theta, \end{cases} \tag{48}$$

and

$$\begin{cases} 2g_{sss} = (2\rho^{-3}\rho_s^2 - \rho^{-2}\rho_{ss} + \rho^{-1}\theta_s^2)\cosh\theta + (\rho^{-1}\theta_{ss} - 2\rho^{-2}\rho_s\theta_s)\sinh\theta, \\ 2f_{sss} = (2\rho^{-3}\rho_s^2 - \rho^{-2}\rho_{ss} + \rho^{-1}\theta_s^2)\sinh\theta + (\rho^{-1}\theta_{ss} - 2\rho^{-2}\rho_s\theta_s)\cosh\theta. \end{cases} \tag{49}$$

We can also calculate that

$$\begin{aligned} \langle \xi_{ss}, \xi_{ss}\rangle &= 16\rho_s^2(f_s^2 - g_s^2) + 4\rho^2(f_{ss}^2 - g_{ss}^2) \\ &\quad + 8\rho\rho_{ss}(g_s^2 - f_s^2) + 16\rho\rho_s(f_sf_{ss} - g_sg_{ss}) \\ &= -\rho^{-2}\rho_s^2 + \theta_s^2 + 2\rho^{-1}\rho_{ss}, \end{aligned} \tag{50}$$

and

$$\begin{aligned}\langle \xi_{sss}, \xi_{sss}\rangle =&36\rho_{ss}^2(f_s^2-g_s^2)+36\rho_s^2(f_{ss}^2-g_{ss}^2)+4\rho^2(f_{sss}^2-g_{sss}^2)\\ &+24\rho\rho_{ss}(f_sf_{sss}-g_sg_{sss})+24\rho\rho_s(f_{ss}f_{sss}-g_{ss}g_{sss})\\ &+24\rho\rho_{sss}(g_sg_{ss}-f_sf_{ss})+72\rho_s\rho_{ss}(f_sf_{ss}-g_sg_{ss})\\ &+24\rho_s\rho_{sss}(g_s^2-f_s^2)\\ =&-4\rho^{-2}\rho_{ss}^2-\rho^{-4}\rho_s^4+3\rho^{-2}\rho_s^2\theta_s^2+4\rho^{-3}\rho_s^2\rho_{ss}\\ &-4\rho^{-1}\rho_{ss}\theta_s^2-\theta_s^4+2\rho^{-1}\rho_s\theta_s\theta_{ss}+\theta_{ss}^2.\end{aligned} \tag{51}$$

From (43) and (50) , we have

$$\begin{aligned}k(s)=&\frac{1}{2}\langle \xi_{ss}, \xi_{ss}\rangle\\ =&-\frac{1}{2}\rho^{-2}\rho_s^2+\frac{1}{2}\theta_s^2+\rho^{-1}\rho_{ss}\\ =&\frac{1}{2}[(\ln\rho)_s]^2+(\ln\rho)_{ss}+\frac{1}{2}\theta_s^2,\end{aligned} \tag{52}$$

From (43), (44) and $\langle \xi, \xi_{sss}\rangle=0$, $\langle \xi_s, \xi_{sss}\rangle=-\langle \xi_{ss}, \xi_{ss}\rangle$, we have

$$\tau(s)\beta(s)=\xi_{sss}-\langle \xi_{ss}, \xi_{ss}\rangle\xi_s-\langle \xi_{ss}, \xi_{sss}\rangle\xi \tag{53}$$

and then

$$\begin{aligned}\tau^2(s)=&\langle -\xi_{sss}+k_s\xi+kx_s, -\xi_{sss}+k_s\xi+k\xi_s\rangle-k^2\\ =&\frac{1}{2}\langle \xi_{ss}, \xi_{ss}\rangle^2+\langle \xi_{sss}, \xi_{sss}\rangle\\ =&(\rho^{-1}\rho_s\theta_s+\theta_{ss})^2\\ =&(\theta_s(\ln\rho)_s+\theta_{ss})^2.\end{aligned}$$

This ends the proof. ■

Two examples of timelike curves in $\mathbb{Q}^3$ are given below.

Example 17. *Let $\xi_7(s)=(1, \frac{1}{2}s^2, s, \frac{1}{2}s^2-1)$ be a timelike curve in $\mathbb{Q}^3$ with curvature $k(s)=0$, $\tau(s)=0$. Then the structure functions can be written as*

$$\begin{cases} f(s)=\frac{1}{s^2-1},\\ g(s)=\frac{s}{s^2-1},\\ \rho(s)=\frac{s^2-1}{2}.\end{cases}$$

Example 18. *Let $\xi_8(s)=(n, \sqrt{m^2-n^2}\cosh\dfrac{s}{\sqrt{m^2-n^2}}, m, \sqrt{m^2-n^2}\sinh\dfrac{s}{\sqrt{m^2-n^2}})$ be a timelike curve in $\mathbb{Q}^3$ with curvature $k(s)>0$, $\tau(s)=0$. Then the structure functions can be written as*

$$\begin{cases} f(s)=\frac{n}{2\rho},\\ g(s)=\frac{m}{2\rho},\\ 2\rho(s)=\sqrt{m^2-n^2}(\cosh\frac{s}{\sqrt{m^2-n^2}}+\sinh\frac{s}{\sqrt{m^2-n^2}}),\end{cases}$$

where $m^2-n^2>0$.

6. Conclusions

In this paper, we discussed the cone curves in $\mathbb{E}_2^3$ and $\mathbb{E}_2^4$. By putting forward the concept of structure function of cone curve, the relationship between curvature function and structure function was obtained. By using the results in this study, cone curves could be displayed intuitively via drawing software on computer, which will greatly promote the widespread application of cone curves.

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