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Neumann and Mixed Boundary Value Problems on the Upper Half Plane

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Abstract

We give explicit representation of Neumann boundary value problem for Bitsadze equation on the upper half plane. We will also give solution of the inhomogeneous polyanalytic equation arising from Neumann and (n-1) Dirichlet boundary conditions on the upper half plane $\mathbb{H}$.

Keywords: Dirichlet boundary condition, Schwarz boundary conditions, Cauchy-Pompeiu representation, Gauss theorem.
2010 MSC: 32A30, 30G20, 31A30, 35J55, 30E25, 35J55.

1. Introduction

There are principally two different complex second order partial differential operators, the Laplace $\partial_z \partial_{\overline{z}}$ the Bitsadze operator $\partial_{\overline{z}}^2$. In this article, Neumann boundary problem is solved explicitly for Bitsadze equation on the upper half plane. In case of Neumann boundary value condition, the derivative is taken via outer normal derivative which is defined as $\partial_y = i(\partial_z - \partial_{\overline{z}})$ on Upper half plane. In case of unit disc where the first order solutions are given, the modified Cauchy âĂŞ Pompeiu formula plays a very crucial role and works as starting point while solving BVPs for higher orders, see [1]. The area integral written in Cauchy - Pompeiu formula is known as Pompeiu operator, was studied by Vekuasee [2]. For a regular domain D, if $f \in L_p(D, \mathbb{C})$, $p > 1$, where $L_p(D, \mathbb{C})$ is the space of all equivalence classes of Lebesgue measurable functions f on D for which $|f|^p$ is integrable. Then the Pompeiu operator Tf possesses weak derivatives and

$$\frac{\partial}{\partial_{\overline{z}}}(Tf) = f, \quad \frac{\partial}{\partial_z}(Tf) = \Pi f,$$

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where Πf represents singular integral in the principal value sense. In case of upper half plane if $w : \mathbb{H} \to \mathbb{C}$ satisfies $|w(x) \leq C|x|^{-\epsilon}$ for $|x| > K, \epsilon > 0$ and $w_{\overline{z}} \in L_1(\mathbb{H};\mathbb{C})$, then the Cauchy-Pompeiu formula [3] is given by

$$w(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t-z} - \frac{1}{\pi}\int_{0<Im\zeta} w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

$$w(z) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} w(t)\frac{dt}{t-\overline{z}} - \frac{1}{\pi}\int_{0<Im\zeta} w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\overline{\zeta - z}}}$$

where $z \in \mathbb{H}$.
In case of upper half plane $\mathbb{H}$, the Pompeiu operator T has the following form:

$$Tf(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{\zeta - z}$$

and T satisfies the properties $\frac{\partial}{\partial_{\overline{z}}}(Tf) = f, \quad \frac{\partial}{\partial_z}(Tf) = \Pi f$ where

$$\Pi f(z) = -\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}$$

here derivatives are taken in distributional sense. We observe that for $z = x + iy \in \mathbb{H}$, $\gamma \in L^p(\mathbb{R},\mathbb{C}), p \geq 1$

$$\lim_{z\to t_0} \frac{1}{\pi}\int_{-\infty}^{\infty} \gamma_0(t)\frac{ydt}{|t-z|^2} = \gamma_0(t_0).$$

For regular domains higher order of Pompeiu operators were studied in [4] and for upper half plane in [5]. In last section of this article, we will also present a result for the inhomogeneous polyanalytic equation arising from Neumann and Dirichlet boundary conditions. These types of problems are also studied on regular domains in [1, 6, 7, 8, 9, 10].

2. Neumann Boundary Value Problem for Bitsadze Equation

On the upper half plane, the solution of the inhomogeneous Cauchy-Riemann equations with Neumann boundary conditions is given as:

Theorem 2.1. *[3] The Neumann problem*
$w_{\overline{z}=f}$ in $\mathbb{H}$, $\partial_y w = i\gamma$ on $\mathbb{R}, w(i) = c$ is uniquely solvable for $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C}), \gamma \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C}), c \in \mathbb{C}$ if and only if for $z \in \mathbb{H}$ where $L_{p,2}$ is the space of functions that satisfy the condition f(z) $\in L_p(\mathbb{H} : \mathbb{C})$, $f_2(z) = |z|^{-2}f(\frac{1}{z}) \in L_p(\mathbb{H} : \mathbb{C})$, $p \geq 1$ [2].

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t) + 2f(t))\frac{dt}{t-\overline{z}} - \frac{1}{\pi}\int_{\mathbb{H}} f_{\zeta}(\zeta)\frac{d\xi d\eta}{\zeta - \overline{z}} = 0. \quad (1)$$

Then the solution in this case is given by

$$w(z) = c + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t) + 2f(t))log\left(\frac{t-i}{t-z}\right)dt - \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{z-i}{(\zeta - z)(\zeta - i)}d\xi d\eta. \quad (2)$$

Theorem 2.2. *The Neumann boundary problem for the Bitsadze equation*
$w_{\overline{z}\overline{z}} = f$ in $\mathbb{H}$, $w(i) = c_0, w_{\overline{z}}(i) = c_1, \partial_y w = i\gamma_0, \partial_y w_{\overline{z}=i\gamma_1}$ on $\mathbb{R}$ is uniquely solvable for $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C^1(\overline{\mathbb{H}};\mathbb{C}) \cap L_2(\mathbb{R};\mathbb{C}), \gamma_0, \gamma_1 \in L_2(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C}), c \in \mathbb{C}$ if and only if for $z \in \mathbb{H}$

$$c_1 - \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\gamma_0(t)}{t-\overline{z}}dt + \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_1(t) + f(t))\left[\frac{t-z}{t-\overline{z}} + log(t-i)\right]dt$$
$$+ \frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left(\frac{\overline{\zeta} - z}{(\zeta - \overline{z})^2} + \frac{\overline{z} - i}{(\zeta - \overline{z})(\zeta - i)}\right)d\xi d\eta = 0, \quad (3)$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_1(t)+f(t))\frac{dt}{t-\overline{z}}-\frac{1}{\pi}\int_{\mathbb{H}}f_\zeta(\zeta)\frac{d\xi d\eta}{\zeta-\overline{z}}=0. \tag{4}$$

In case equation (4) *is satisfied, the solution of Bitsadze equation is given by*

$$\begin{aligned}w(z)=c_0-c_1\overline{(i-z)}-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_0(t)log\left(\frac{t-z}{t-i}\right)dt\\+\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)+2f(t))[(z-i)+(2t-z-\overline{z})log(t-z)-(2t-i-\overline{z})log(t-i)]dt\\-\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left(\frac{(z-i)\overline{(\zeta-z)}}{(\zeta-z)(\zeta-i)}\right)d\xi d\eta.\end{aligned} \tag{5}$$

Proof. Applying the Cauchy-Pompeiu formula, we have

$$\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left(\frac{(z-i)\overline{(\zeta-z)}}{(\zeta-z)(\zeta-i)}\right)d\xi d\eta=I_1-I_2-c_0+\overline{i-z}c_1+w(z) \tag{6}$$

where

$$I_1=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\partial_{\overline{\zeta}}w(t)\left(\frac{t-\overline{z}}{t-z}-\frac{t-\overline{z}}{t-i}\right)dt.$$

and

$$I_2=\frac{1}{2\pi i}\int_{-\infty}^{\infty}w(t)\left(\frac{1}{t-z}-\frac{1}{t-i}\right)dt.$$

Integrating by parts I_1 and I_2 and using regularity conditions on w and $\partial_{\overline{z}}w$, we have

$$I_1=-\frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma(t)+2f(t))[(i-z)+(z-\overline{z})log(t-z)-(i-\overline{z})log(t-i)]dt,$$

and

$$\begin{aligned}I_2=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_0(t)log\left(\frac{t-z}{t-i}\right)dt\\-\frac{1}{\pi i}\int_{-\infty}^{\infty}(\gamma_1(t)+2f(t))[(z-i)+(t-z)log(t-z)-(t-i)log(t-i)]dt.\end{aligned}$$

Substituting these values in equation (6), we obtain equation (5). Similarly, applying Gauss theorem and Cauchy-Pompei formula and integrating by parts the boundary integrals, the solvability conditions (3), (4) can be obtained.
Using Plemelj-Sokhotzki formula [3] and Theorem (2.1), it can be shown that (5) is indeed a solution.

3. Neumann-Dirichlet Mixed boundary problem in the Upper Half Plane

In this section, we treat the case of inhomogeneous polyanalytic equation on $\mathbb{H}$ where the first factor is having Neumann boundary condition and next (n-1) factors having Dirichlet boundary conditions. In order to solve higher order boundary value problems, we require complex form of Cauchy-Pompeiu formula and Gauss theorem on $\mathbb{H}$. Let $\mathcal{F}_k$ be the space of functions w in $W^{(k,1)}(\mathbb{H},\mathbb{C})$ for which $lim_{R\to\infty}R^\nu M(\partial_{\overline{z}}^\nu,R)=0$, $0\le\nu\le k-1$ where $M(\partial_{\overline{z}}^\nu,R)=\max_{\substack{|z|=R\\0<Imz}}|\partial_{\overline{z}}^\nu w(z)|$ and $z^{-k-2}\partial_{\overline{z}}^k w\in L^1(\mathbb{H},\mathbb{C})$ when $k\ge 2$. Using Theorem 4 from [3], every $w\in\mathcal{F}_k$ can be represented as (for $z\in\mathbb{H}$):

$$w(z)=\sum_{\nu=0}^{k-1}\frac{1}{2\pi i}\frac{1}{\nu!}\int_{-\infty}^{\infty}\partial_{\overline{\zeta}}^\nu w(\zeta)\overline{(z-\zeta)}^\nu\frac{1}{(\zeta-z)}d\zeta-\frac{1}{(k-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\partial_{\overline{\zeta}}^k w(\zeta)\overline{(z-\zeta)}^{k-1}\frac{1}{(\zeta-z)}d\xi d\eta. \tag{7}$$

Theorem 3.1. *The mixed Neumann-$(n-1)$ Dirichlet problem for the inhomogeneous polyanalytic equation in $\mathbb{H}$,*

$$\partial_{\overline{z}}^n w = f \text{ in } \mathbb{H}, \partial_y w(z) = i\gamma_0,\ \partial_{\overline{z}}^{\lambda} w = \gamma_\lambda \text{ on } \mathbb{R}, 1 \le \lambda \le n-1, w(i) = c \tag{8}$$

is uniquely solvable for $f \in L_{p,2}(\mathbb{H};\mathbb{C}) \cap C(\overline{\mathbb{H}};\mathbb{C}), p \ge 2, t^\lambda \gamma_\lambda \in L^p(\mathbb{R};\mathbb{C}) \cap C(\mathbb{R};\mathbb{C})$, if and only if $1 \le \nu \le n-1$,

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\gamma_0(t)+2\gamma_1(t)}{t-\overline{z}}dt - \sum_{\lambda=0}^{n-2} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma_{\lambda+1}(t)\frac{(t-z)^\lambda}{t-\overline{z}}dt + \sum_{\lambda=1}^{n-1} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma_\lambda(t)\frac{(t-z)^\lambda}{(t-\overline{z})^2} - \frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{(\overline{\zeta}-z)^{n-1}}{(\zeta-\overline{z})^2)}d\xi d\eta + \frac{(-1)^{n-2}}{(n-2)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z})^2)}d\xi d\eta = 0 \tag{9}$$

and

$$\sum_{\lambda=\nu}^{n-1} \frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma_\lambda(t)\frac{(t-z)^{\lambda-\nu}}{t-\overline{z}}dt + \frac{(-1)^{n-\nu}}{(n-\nu-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\frac{(\overline{\zeta}-z)^{n-1-\nu}}{\zeta-\overline{z}}d\xi d\eta = 0 \tag{10}$$

In case (9) and (10) are satisfied, then the solution is given by

$$\begin{aligned} w(z) &= c - \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_0(t)+2\gamma_1(t))log\left(\frac{t-z}{t-i}\right)dt \\ &+ \sum_{\lambda=1}^{n-1} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma_\lambda(t)\left(\frac{(t-\overline{z})^\lambda}{t-z} - \frac{(t+i)^\lambda}{t-i}\right)dt \\ &+ \frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-1}}{(\zeta-z)} - \frac{(\overline{\zeta-i})^{n-1}}{(\zeta-i)}\right\}d\xi d\eta. \end{aligned} \tag{11}$$

Proof. Using Cauchy-Pompeiu formula and expression, (7) we obtain

$$\begin{aligned} &\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}} f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-1}}{(\zeta-z)} - \frac{(\overline{\zeta-i})^{n-1}}{(\zeta-i)}\right\}d\xi d\eta \\ =\ & w(z) - w(i) - \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \partial_{\overline{\zeta}}^\lambda w(\zeta)\left(\frac{(t-\overline{z})^\lambda}{t-z} - \frac{(t+i)^\lambda}{t-i}\right)dt \\ =\ & w(z) - c - \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \partial_{\overline{\zeta}}^\lambda w(\zeta)\left(\frac{(t-\overline{z})^\lambda}{t-z} - \frac{(t+i)^\lambda}{t-i}\right)dt \\ &- \frac{1}{2\pi i}\int_{-\infty}^{\infty} w(\zeta)\left(\frac{1}{t-z} - \frac{1}{t-i}\right)dt \\ =\ & w(z) - c - \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \partial_{\overline{\zeta}}^\lambda w(\zeta)\left(\frac{(t-\overline{z})^\lambda}{t-z} - \frac{(t+i)^\lambda}{t-i}\right)dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}[(\partial_\zeta w(\zeta) - \partial_{\overline{\zeta}} w(\zeta)) + 2\partial_{\overline{\zeta}} w(\zeta)][log(t-z) - log(t-i)]dt \\ =\ & w(z) - c - \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty} \gamma_\lambda(t)\left(\frac{(t-\overline{z})^\lambda}{t-z} - \frac{(t+i)^\lambda}{t-i}\right)dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty}[\gamma_0(t)+2\gamma_1(t)]\left[log\frac{(t-z)}{(t-i)}\right]dt. \end{aligned}$$

Therefore

$$\begin{aligned} w(z) &= c - \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_0(t)+2\gamma_1(t))log\left(\frac{t-z}{t-\overline{z}}\right)dt \\ &+\sum_{\lambda=1}^{n-1}\frac{(-1)^\lambda}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_\lambda(t)\left(\frac{(t-\overline{z})^\lambda}{t-z}-\frac{(t+i)^\lambda}{t-i}\right)dt \\ &+\frac{(-1)^n}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left\{\frac{\overline{(\zeta-z)^{n-1}}}{\zeta-z}-\frac{\overline{(\zeta-i)^{n-1}}}{\zeta-i}\right\}d\xi d\eta. \end{aligned}$$

Which is (11). Now,

$$\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\frac{(\overline{\zeta}-z)^{n-1}}{(\zeta-\overline{z})^2}d\xi d\eta-\frac{(-1)^{n-2}}{(n-2)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\frac{(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z})}d\xi d\eta$$

$$\begin{aligned} &= \frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\partial_{\overline{\zeta}}^{n-1}w(\zeta)\frac{(\overline{\zeta}-z)^{n-1}}{(\zeta-\overline{z})^2}\right)d\xi d\eta \\ &+\frac{(-1)^{n-2}}{(n-2)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\partial_{\overline{\zeta}}^{n-1}w(\zeta)\frac{(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z})^2}\right)d\xi d\eta \\ &-\frac{(-1)^{n-2}}{(n-2)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\partial_{\overline{\zeta}}^{n-1}w(\zeta)\frac{(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z})^2}\right)d\xi d\eta \\ &-\frac{(-1)^{n-3}}{(n-3)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial\overline{\zeta}}\left(\partial_{\overline{\zeta}}^{n-1}w(\zeta)\frac{(\overline{\zeta}-z)^{n-3}}{(\zeta-\overline{z}}\right)d\xi d\eta \end{aligned}$$

$$\begin{aligned} &= \frac{(-1)^{n-1}}{(n-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\partial}{\partial\overline{\zeta}}\left(\partial_{\overline{\zeta}}^{n-1}w(t)\frac{(\overline{t}-z)^{n-1}}{(t-\overline{z})^2}\right)dt \\ &-\frac{(-1)^{n-2}}{(n-2)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{\partial_{\overline{\zeta}}^{n-1}w(t)(t-z)^{n-2}}{t-\overline{z}}\right)dt \\ &+\frac{(-1)^{n-2}}{(n-2)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial_{\overline{\zeta}}}\left(\frac{\partial_{\overline{\zeta}}^{n-2}w(\zeta)(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z}})^2\right)d\xi d\eta \\ &-\frac{(-1)^{n-3}}{(n-3)!}\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{\partial_{\overline{\zeta}}^{n-2}w(\zeta)(\overline{\zeta}-z)^{n-3}}{\zeta-\overline{z}}\right)d\xi d\eta \\ &-\frac{(-1)^{n-3}}{(n-3)!}\frac{1}{\pi}\int_{\mathbb{H}}\frac{\partial}{\partial_{\overline{\zeta}}}\left(\frac{\partial_{\overline{\zeta}}^{n-3}w(\zeta)(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z}})\right)d\xi d\eta \\ &-\frac{(-1)^{n-4}}{(n-4)!}\frac{1}{\pi}\int_{\mathbb{H}}\left(\frac{\partial_{\overline{\zeta}}^{n-2}w(\zeta)(\overline{\zeta}-z)^{n-4}}{\zeta-\overline{z}}\right)d\xi d\eta \end{aligned}$$

$$= \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\partial_\zeta^\lambda w(\zeta) \frac{(t-z)^\lambda}{(t-\overline{z})^2} \right) dt$$
$$- \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{\partial_\zeta^{\lambda+1} w(\zeta)(t-z)^\lambda}{t-\overline{z}} \right) dt$$

$$= \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\gamma_\lambda(t) \frac{(t-z)^\lambda}{(t-\overline{z})^2} \right) dt$$
$$- \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{\gamma_{\lambda+1}(t)(t-z)^\lambda}{t-\overline{z}} \right) dt + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{w(t)}{(t-\overline{z})^2} \right) dt$$

$$= \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\gamma_\lambda(t) \frac{(t-z)^\lambda}{(t-\overline{z})^2} \right) dt$$
$$- - \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{\gamma_{\lambda+1}(t)(t-z)^\lambda}{t-\overline{z}} \right) dt + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{\gamma_0(t) + 2\gamma_1(t)}{(t-\overline{z})} \right) dt$$

Therefore

$$\frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\pi} \int_{\mathbb{H}} f(\zeta) \frac{(\overline{\zeta}-z)^{n-1}}{(\zeta-\overline{z})^2} d\xi d\eta - \frac{(-1)^{n-2}}{(n-2)!} \frac{1}{\pi} \int_{\mathbb{H}} f(\zeta) \frac{(\overline{\zeta}-z)^{n-2}}{(\zeta-\overline{z})} d\xi d\eta$$
$$- \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{\gamma_0(t) + 2\gamma_1(t)}{(t-\overline{z})} \right) dt - \sum_{\lambda=0}^{n-1} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\gamma_\lambda(t) \frac{(t-z)^\lambda}{(t-\overline{z})^2} \right) dt$$
$$+ \sum_{\lambda=0}^{n-2} \frac{(-1)^\lambda}{\lambda!} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\gamma_{\lambda+1}(t) \frac{(t-z)^\lambda}{(t-\overline{z})} \right) dt = 0$$

which is (9). Applying Gauss theorem repeatedly, we have

$$\frac{(-1)^{n-\nu}}{(n-\nu-1)!} \frac{1}{\pi} \int_{\mathbb{H}} \partial_\zeta^n w(\zeta) \frac{(\overline{\zeta}-z)^{n-\nu-1}}{(\zeta-\overline{z})} d\xi d\eta$$
$$= - \sum_{\lambda=\nu}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!} \int_{-\infty}^{\infty} \partial_\zeta^\nu w(t) \frac{(t-z)^{\lambda-\nu}}{t-\overline{z}} dt$$
$$= - \sum_{\lambda=\nu}^{n-1} \frac{1}{2\pi i} \frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)!} \int_{-\infty}^{\infty} \gamma_\nu(t) \frac{(t-z)^{\lambda-\nu}}{t-\overline{z}} dt$$

which is (10).

Let $T_{0,n}$ be higher order Cauchy Pompei operator on $\mathbb{H}$ [7, 10]. Since $\partial_{\overline{z}} T_{0,n} w = T_{0,n-1} w$ it follows that (11) indeed satisfy $\partial_{\overline{z}}^n w = f$. Now, differentiating (11) w.r.t z and $\overline{z}$, we have

$$\begin{aligned}
\partial_z w(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(\gamma_0(t)+2\gamma_1(t))\frac{1}{t-z}dt \\
&+\sum_{\lambda=1}^{n-1}\frac{(-1)^{\lambda}}{\lambda!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_\lambda(t)\left(\frac{(t-\overline{z})^{\lambda}}{(t-z)^2}\right)dt \\
&+\frac{(-1)^{n-1}}{(n-1)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-1}}{(\zeta-z)^2}\right\}d\xi d\eta \\
\partial_z w(z) &= \sum_{\lambda=1}^{n-1}\frac{(-1)^{\lambda-1}}{(\lambda-1)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_\lambda(t)\left(\frac{(t-\overline{z})^{\lambda-1}}{(t-z)}\right)dt \\
&+\frac{(-1)^{n-1}}{(n-2)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-2}}{(\zeta-z)}\right\}d\xi d\eta
\end{aligned} \tag{12}$$

Since,

$$lim_{z\to t_0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\gamma_0(t)ydt}{|t-z|^2}=\gamma_0(t_0).$$

Therefore, subtracting (9) from $\partial_z w(z)-\partial_{\overline{z}}w(z)$ and using Plemelj-Sokhotzki formula [3], we obtain $\partial_y w(z)=\gamma_0$ on $\mathbb{R}$. For fixed k, $1\le k\le n-1$, (11) results into

$$\begin{aligned}
\partial_{\overline{z}}^k w(z) &= \sum_{\lambda=k}^{n-2}\frac{(-1)^{\lambda-k}}{(\lambda-k)!}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_\lambda(t)\left(\frac{(t-\overline{z})^{\lambda-k}}{(t-z)}\right)dt \\
&-\frac{(-1)^{n-k-2}}{(n-k-2)!}\frac{1}{\pi}\int_{\mathbb{H}}f(\zeta)\left\{\frac{(\overline{\zeta-z})^{n-k-2}}{(\zeta-z)}\right\}d\xi d\eta
\end{aligned} \tag{13}$$

Let $T_{0,n}$ be the operators defined as above, then (13) can be expressed as

$$\partial_{\overline{z}}^k w(z) = T_{0,n-k}f(z)+\sum_{\lambda=k+1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-k}}{(\lambda-k)}\int_{-\infty}^{\infty}\gamma_\lambda(t)\left(\frac{(t-\overline{z})^{\lambda-k+1}}{(t-z)}\right)dt$$

and (10) the solvability condition as

$$T_{0,n-\nu}f(\overline{z})+\sum_{\lambda=\nu+1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-\nu}}{(\lambda-\nu)}\int_{-\infty}^{\infty}\gamma_\nu(t)\left(\frac{(t-\overline{z})^{\lambda-\nu+1}}{(t-\overline{z})}\right)dt = 0.$$

Now, making use of Lemma 1.1 from [5], and computing difference of above two equations for $\nu=k$, we have

$$\begin{aligned}
\partial_{\overline{z}}^k w(z) &= T_{0,n-k}(f(z)-f(\overline{z})+\frac{1}{\pi}\int_{-\infty}^{\infty}\gamma_k(t)\frac{ydt}{|t-z|^2}dt \\
&\sum_{\lambda=k+1}^{n-1}\frac{1}{2\pi i}\frac{(-1)^{\lambda-k}}{(\lambda-k)}\int_{-\infty}^{\infty}\gamma_k(t)\left[\frac{(t-\overline{z})^{\lambda-k+1}-(t-z)^{\lambda-k+1}}{|t-z|^2}\right]dt.
\end{aligned}$$

Hence, $lim_{z\to t_0}\partial_{\overline{z}}^k w(z)=\gamma_k(t_0)$.

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