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Numerical Solutions of Nine-Dimensional Lorenz System

Dokuz-Boyutlu Lorenz Sisteminin Sayısal Çözümleri

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Abstract

In this study, the performance of the Taylor's decomposition method in the nine-dimensional Lorenz system is investigated. The proposed method is applied to the nine-dimensional Lorenz system for both chaotic and hyperchaotic cases. Phase portraits and time series graphs of the problem are drawn. The results obtained are compared with both the fourth order Runge-Kutta method and multi-domain compact finite difference relaxation method. Comparisons show that the obtained results are consistent with the results of other methods. Thus, the accuracy and efficiency of the method for nonlinear systems were emphasized by using the nine-dimensional Lorenz system.

Keywords: *Taylor's decomposition method, nine-dimensional Lorenz system, chaotic problems, system of non-linear initial value problems*

Öz

Bu çalışmada, Taylor ayrıştırma yönteminin dokuz boyutlu Lorenz sistemi üzerindeki performansı araştırılmıştır. Önerilen yöntem, hem kaotik hem de hiperkaotik durumları için dokuz boyutlu Lorenz sistemine uygulanmıştır. Problemin faz portreleri ve zaman serisi grafikleri çizilmiştir. Elde edilen sonuçlar hem dördüncü mertebe Runge-Kutta yöntemi hem de çok bölgeli kompakt sonlu farklar esnetme yöntemi ile karşılaştırılmıştır. Karşılaştırmalar, elde edilen sonuçların diğer yöntemlerin sonuçlarıyla tutarlı olduğunu göstermiştir. Böylece dokuz boyutlu Lorenz sistemi kullanılarak doğrusal olmayan sistemler için yöntemin doğruluğu ve etkinliği vurgulanmıştır.

Anahtar Kelimeler: *Taylor ayrıştırma yöntemi, dokuz boyutlu Lorenz sistemi, kaotik problemler, doğrusal olmayan başlangıç değer problemleri sistemi*

1. Introduction

The first study in theory of chaos which illustrate the concept of chaos by showing that some systems are capable of producing outputs that seem random, yet are ordered, was introduced by Lorenz [1] in order to demonstrate the chaotic behaviour in a system of ordinary differential equations modelling the athmospheric convection. This epochal discovery has many scientific application area such as electrical circuits, fluid dynamics, mechanical devices, lasers, population growth.

In this work, nine-dimensional (9D) Lorenz system, which is investigated by many authors in recent years, is considered. Some of these works are given in [2-6]. The system derived by

Reiterer et al. [2] who apply a triple Fourier expansion to the Boussinesq-Oberbeck equations governing thermal convection in a 3D spatial domain by using an approach similar to the well known 3D Lorenz system is given by

$$\begin{aligned}
\frac{dx_1}{dt} &= -\sigma b_1 x_1 - \sigma b_2 x_7 - x_2 x_4 + b_4 x_4^2 \\ &\quad + b_3 x_3 x_5 \\
\frac{dx_2}{dt} &= -\sigma x_2 - 0.5\sigma x_9 + x_1 x_4 + x_2 x_5 \\ &\quad + x_4 x_5 \\
\frac{dx_3}{dt} &= -\sigma b_1 x_3 + \sigma b_2 x_8 + x_2 x_4 + b_4 x_2^2 \\ &\quad + b_3 x_1 x_5 \\
\frac{dx_4}{dt} &= -\sigma x_4 + 0.5\sigma x_9 - x_2 x_3 - x_2 x_5 \\ &\quad + x_4 x_5 \\
\frac{dx_5}{dt} &= -\sigma b_5 x_5 + 0.5 x_2^2 + 0.5 x_4^2 \\
\frac{dx_6}{dt} &= -b_6 x_6 + x_2 x_9 - x_4 x_9 \\
\frac{dx_7}{dt} &= -r x_1 - b_1 x_7 + 2 x_5 x_8 - x_4 x_9 \\
\frac{dx_8}{dt} &= -r x_3 - b_1 x_8 - 2 x_5 x_7 - x_2 x_9 \\
\frac{dx_9}{dt} &= -r x_2 + r x_4 - x_9 - 2 x_2 x_6 + 2 x_4 x_6 \\ &\quad + x_4 x_7 - x_2 x_8
\end{aligned} \tag{1}$$

where the constant parameters b_i are the measure for the geometry of the square cell defined by

$b_1 = 4\frac{1+a^2}{1+2a^2}$, $b_2 = \frac{1+2a^2}{2(1+a^2)}$, $b_3 = 2\frac{1-a^2}{1+a^2}$,

$b_4 = \frac{a^2}{1+a^2}$, $b_5 = 8\frac{a^2}{1+2a^2}$ and $b_6 = \frac{4}{1+2a^2}$,

σ is the measure of the relative importance of momentum vorticity diffusion to heat conduction by molecular collisions, r is the reduced Rayleigh and a is the wave number in the horizontal direction.

Taylor's decomposition method is investigated to solve 9D Lorenz system approximately. It is based on the the application of the Taylor's decomposion on two points constructed in [7]. Application of the proposed method to chaotic and hyperchaotic problems is given in [8]. It is also used to approximate the eigenvalus of non-linear problems as in [9] and [10]. One of the advantages of the proposed method is that it is a stable and very efficient method for chaotic problems as it is an implicit one-step method. The most important advantage of the Taylor's decomposition method is that it has high order accuracy for large step sizes with a simple algorithm compared to other methods.

In Section 2, Taylor's decomposition on two points is decribed. In Section 3, obtained numerical results and phase portraits of 9D Lorenz system are given. In the Conclusion, the study is summarized and the numerical observations are dicussed.

2. Material and Method

Consider the initial value problem of the form

$$\boldsymbol{Y}'(t) = \boldsymbol{F}(t, \boldsymbol{Y}(t)), \ \ \boldsymbol{Y}(t_0) = \boldsymbol{Y}_0, \tag{2}$$

where $\boldsymbol{Y}, \boldsymbol{Y}_0 \in \mathbb{R}^n$, n is the number of differential equations in (2). In [3], it is recommended to use the Taylor's decomposition method in order to approximate (2). Since 9D Lorenz system is in the form (2), in this work the Taylor's decomposition method is suggested to find approximate solution of 9D Lorenz system. The proposed method is given by the following one step difference scheme:

$$\begin{aligned}
&\boldsymbol{Y}_k - \boldsymbol{Y}_{k-1} + \sum_{j=1}^{p} \alpha_j \boldsymbol{F}^{(j-1)}(t_k, \boldsymbol{Y}_k) h^j \\
&- \sum_{j=1}^{p} (-1)^j \alpha_j \boldsymbol{F}^{(j-1)}(t_{k-1}, \boldsymbol{Y}_{k-1}) h^j = 0,
\end{aligned} \tag{3}$$

on the uniform grid

$$[t_{0,} t_N]_h = \{t_k = t_0 + k\,h, k = 0,1,\dots,N, \\ N\,h = t_N - t_0\},$$

where

$$\alpha_j = \frac{(2p-j)!\,p!\,(-1)^j}{(p+q)!\,j!\,(p-j)!}, \ \ 1 \le j \le p.$$

The difference scheme (3) has $2p$-order of accuracy and is A-stable proved in [8] by the following lemma and theorem:

Lemma: Taylor's decomposition on two points method is A-stable. (4)

Theorem: If $\boldsymbol{F}^{(j)}$ is Lipschitz in Y with constant $L_j, j = 0, \dots, p-1$,

$L = \max_{0 \le j \le p-1} L_j$, then the global error for (3) is bounded by

$$||\boldsymbol{Y}(\boldsymbol{t_k}) - \boldsymbol{Y_k}|| \le \mathrm{C}_0 \, ||\mathbf{Y}(0) - \mathbf{Y}_0|| + \mathrm{C}_1 \frac{\mathrm{M}h^{2p}}{(2\mathrm{p})!},$$

where $M = \max_{t \in [t_0, t_N]} || \boldsymbol{F}^{(2p)}(t, \boldsymbol{Y}(t)) ||$, $\boldsymbol{F}^{(j)}(t, \boldsymbol{Y}(t)) = \frac{\partial^j}{\partial t^j} \boldsymbol{F}(t, \boldsymbol{Y}(t))$ for $j = 0,1, \dots ,2p$, $\boldsymbol{Y}'(t) = \boldsymbol{F}(t, \boldsymbol{Y}(t))$, $C_0 = e^{\bar{t} \frac{Lp}{1 - \frac{1}{2} h L p}}$, $C_1 = \frac{C_0}{L}$ for some $\bar{t} > t_0$ and $|| \bullet ||$ denotes $|| \bullet ||_\infty$.

Proof of the lemma and theorem are given in [8].

3. Numerical Results and Discussion

In this section, the results of Taylor's decomposition method (TDM) is presented for the 9D Lorenz system (1) given with the initial condition $\boldsymbol{x}(0) = \{0.01,0,0.01,0,0,0,0,0,0.01\}$ and parameters $\sigma = 0.5$, $a = 0.5$, $r = 14.1$ (for chaotic case) and $r = 55$ (for hyperchaotic case). The reason of these choices is to make comparisons with the solutions of the multidomain compact finite difference relaxation method (MD-CFDRM) [3] and Runge–Kutta method (RK4).

Figure 1 and Figure 2 illustrate the phase projections on the $x_6 - x_7$ and $x_6 - x_9$ planes for varying values of r, respectively. Obtained phase portraits are consistent with those of Reiterer et al. [2] and MD-CFDRM [6]. In Figure 1 and Figure 2 time series solution are showed for the 9D attractor for $r = 14.1$ (the chaotic case) and $r = 55$ (for hyperchaotic case), respectively. They are also consitent with those of mentioned works. Table1 shows the comparison between TDM, MD-CFDRM and RK4 results for the chaotic case, $r = 14.1$, computed over the interval [0,100]. Table 1 also shows that the results of the three methods are in good agreement. Table 2 gives the comparison between TDM, MD-CFDRM and RK4 results for the hyperchaotic case, $r = 55$, on the interval [0,20]. It can be seen in Table 2 that the results obtained by the three methods are also in good agreement.

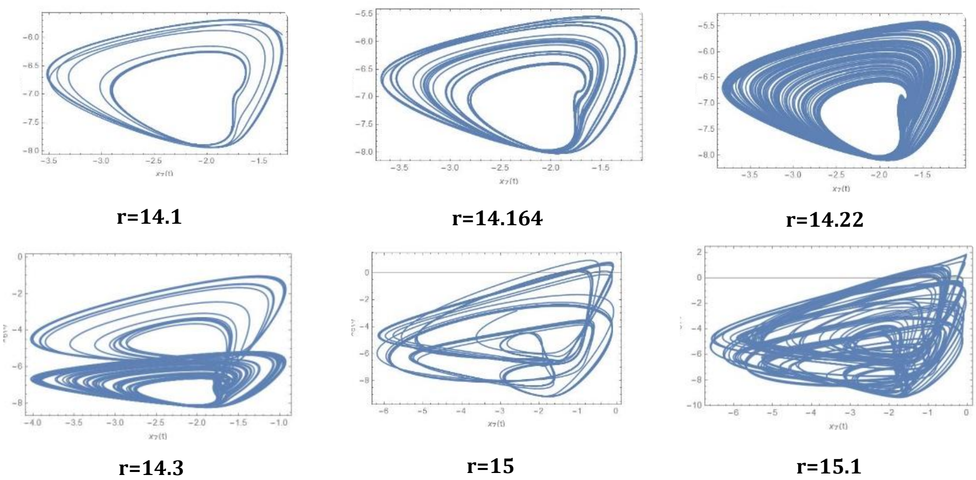
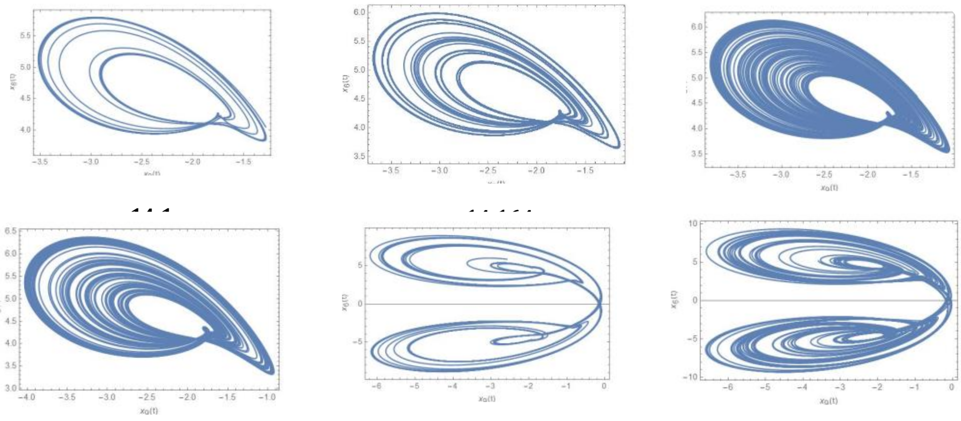


Figure 1. Phase portraits for the 9D attractor on the $x_6 - x_7$ plane fort he variaous values of r.



r=14.3 **r=15** **r=15.1**

Figure 2. Phase portraits for the 9D attractor on the $x_6 - x_9$ plane fort he variaous values of r.

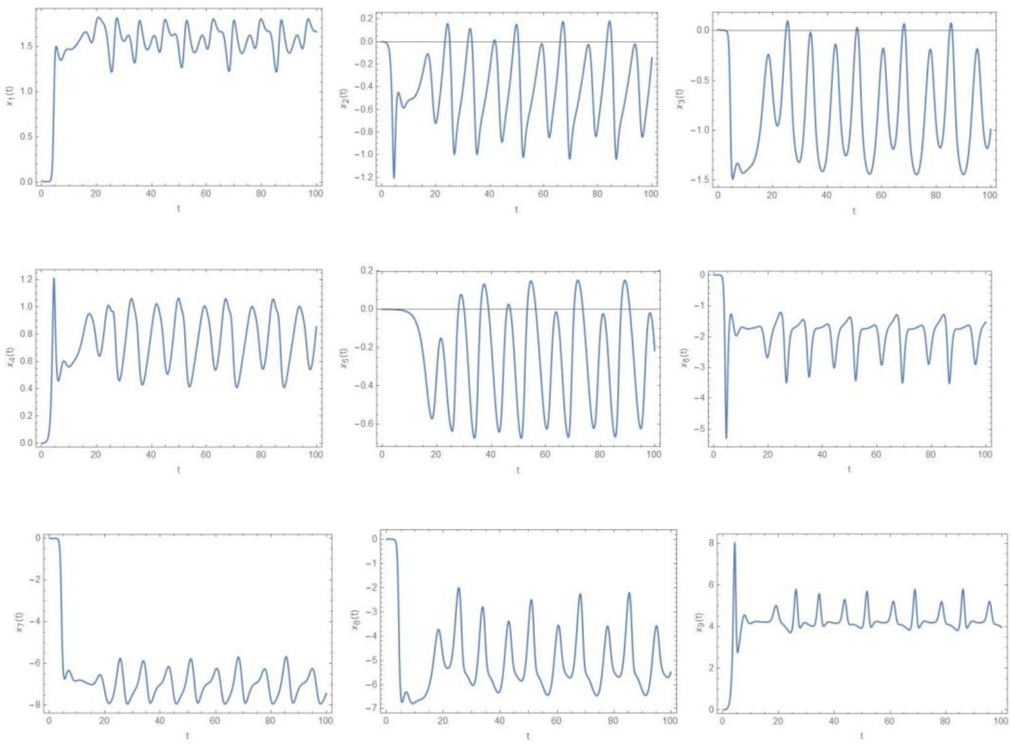


Figure 3. Time series solution for the 9D attractor for $r = 14.1$ (the chaotic case).

Table 1 Comparison results of TDM, RK4 and MD-CFDRM for r=14.1.

	t	TDM	RK4	MD-CFDRM
$x_1(t)$	20	1.7794616020	1.7794616020	1.7794616020
	40	1.5609534149	1.5609534149	1.5609534149
	60	1.4855822940	1.4855822940	1.4855822940
	80	1.7958197536	1.7958197536	1.7958197536
	100	1.6599843283	1.6599843283	1.6599843283
$x_2(t)$	20	-0.7204522789	-0.7204522789	-0.7204522789
	40	-0.2347187741	-0.2347187741	-0.2347187741
	60	-0.1307250241	-0.1307250241	-0.1307250241
	80	-0.7477525490	-0.7477525490	-0.7477525490
	100	-0.1433734338	-0.1433734338	-0.1433734338
$x_3(t)$	20	-0.6362110296	-0.6362110296	-0.6362110296
	40	-1.2023505190	-1.2023505190	-1.2023505190
	60	-0.2401679408	-0.2401679408	-0.2401679408
	80	-0.9536794241	-0.9536794241	-0.9536794241
	100	-0.9914698728	-0.9914698728	-0.9914698728
$x_4(t)$	20	0.7245177009	0.7245177009	0.7245177009
	40	0.8327202872	0.8327202872	0.8327202872
	60	0.9777946806	0.9777946806	0.9777946806
	80	0.5724218362	0.5724218362	0.5724218362
	100	0.8524360371	0.8524360371	0.8524360371
$x_5(t)$	20	-0.3269195959	-0.3269195959	-0.3269195959
	40	-0.1787710447	-0.1787710447	-0.1787710447
	60	-0.6137459610	-0.6137459610	-0.6137459610
	80	-0.1220058785	-0.1220058785	-0.1220058785
	100	-0.2153136081	-0.2153136081	-0.2153136081
$x_6(t)$	20	-2.6127239408	-2.6127239408	-2.6127239408
	40	-1.6858331785	-1.6858331785	-1.6858331785
	60	-1.7451784330	-1.7451784330	-1.7451784330
	80	-2.3524652982	-2.3524652982	-2.3524652982
	100	-1.5321979302	-1.5321979302	-1.5321979302
$x_7(t)$	20	-7.3469555719	-7.3469555719	-7.3469555719
	40	-7.0111342856	-7.0111342856	-7.0111342856
	60	-6.3019708209	-6.3019708209	-6.3019708209
	80	-7.7542533122	-7.7542533122	-7.7542533122
	100	-7.4421079409	-7.4421079409	-7.4421079409
$x_8(t)$	20	-4.9185099929	-4.9185099929	-4.9185099929
	40	-6.2312240135	-6.2312240135	-6.2312240135
	60	-3.7653381446	-3.7653381446	-3.7653381446
	80	-5.4472879550	-5.4472879550	-5.4472879550
	100	-5.4831875435	-5.4831875435	-5.4831875435
$x_9(t)$	20	4.5741481236	4.5741481236	4.5741481236
	40	4.1679032967	4.1679032967	4.1679032967
	60	4.5200856239	4.5200856239	4.5200856239
	80	4.1959347196	4.1959347196	4.1959347196
	100	3.9713031504	3.9713031504	3.9713031504

Table 2 Comparison results of TDM, RK4 and MD-CFDRM for r=55.

	t	TDM	RK4	MD-CFDRM
$x_1(t)$	4	2.264802	2.264802	2.264802
	8	-0.254814	-0.254814	-0.254814
	12	-4.177500	-4.177500	-4.177500
	16	5.971705	5.971705	5.971705
	20	−0.458099	-0.458099	-0.458099
$x_2(t)$	4	5.374258	5.374258	5.374258
	8	-4.806631	-4.806631	-4.806631
	12	3.236132	3.236132	3.236132
	16	-3.474094	-3.474095	-3.474095
	20	1.194282	1.194282	1.194282
$x_3(t)$	4	1.663534	1.663534	1.663534
	8	5.133758	5.133758	5.133758
	12	4.609824	4.609824	4.609824
	16	-2.599292	-2.599292	-2.599292
	20	1.946396	1.946396	1.946396
$x_4(t)$	4	-5.680465	-5.680465	-5.680465
	8	-5.680465	-5.680465	-5.680465
	12	1.230726	1.230726	1.230726
	16	-3.455944	-3.455944	-3.455944
	20	-4.536999	-4.536999	-4.536999
$x_5(t)$	4	3.295077	3.295077	3.295077
	8	2.900768	2.900768	2.900768
	12	-0.997081	-0.997081	-0.997081
	16	0.276657	0.276657	0.276657
	20	-0.265393	-0.265394	-0.265394
$x_6(t)$	4	44.575344	44.575344	44.575344
	8	23.072529	23.072529	23.072529
	12	-8.179131	-8.179131	-8.179131
	16	-8.746022	-8.746022	-8.746022
	20	23.120068	23.120074	23.120074
$x_7(t)$	4	1.558516	1.558516	1.558516
	8	19.065008	19.065008	19.065008
	12	25.129832	25.129832	25.129832
	16	55.935018	55.935017	55.935017
	20	-2.946509	-2.946509	-2.946509
$x_8(t)$	4	-0.874860	-0.874860	-0.874860
	8	9.809716	9.809716	9.809716
	12	36.175991	36.175991	36.175991
	16	17.578243	17.578250	17.578250
	20	17.728233	17.728235	17.728235
$x_9(t)$	4	-8.160863	-8.160863	-8.160863
	8	3.606984	3.606984	3.606984
	12	26.497018	26.497018	26.497018
	16	35.292191	35.292195	35.292195
	20	39.708822	39.708821	39.708821

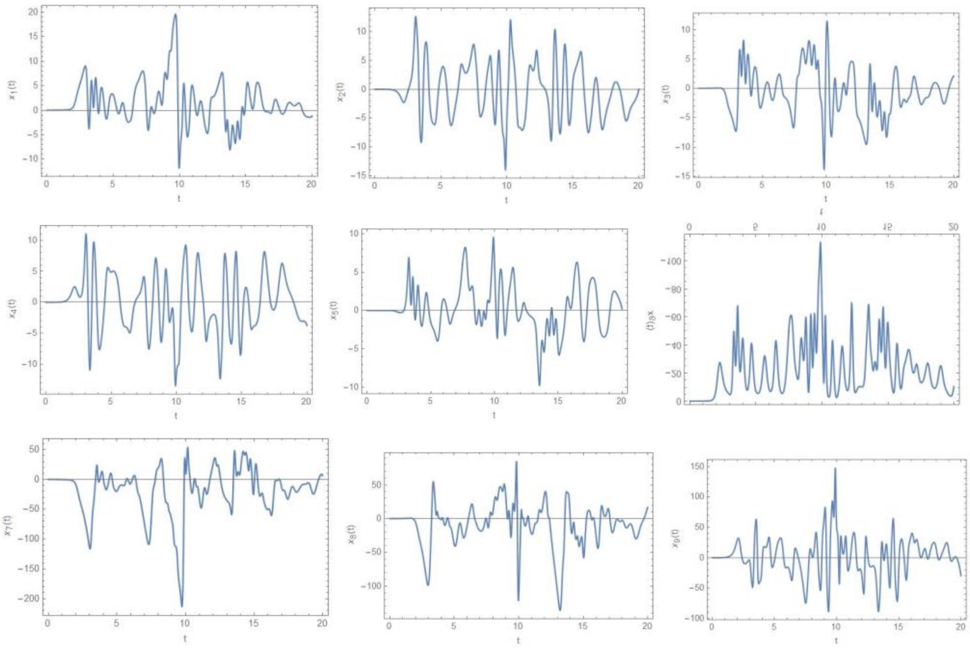


Figure 4. Time series solution for the 9D attractor for $r = 55$ (the hyperchaotic case)

4. Conclusion

In this study, the performance of Taylor's decomposition method on the nine-dimensional Lorenz system is investigated. The obtained numerical results are compared with other high-accuracy methods. From the comparisons and the obtained graphs it is shown that the proposed method gives high order accuracy results for both chaotic and hyperchaotic cases of the nine-dimensional Lorenz system. As a result, Taylor's decomposition method is also recommended for numerical solutions of other higher order systems.

References

[1] Lorenz, E. 1963. Deterministic nonperiodic flow, J. Atmos. Sci., Vol. 20, No. 2 pp. 130-141. DOI: 10.1007/978-0-387-21830-4_2.

[2] Reitere P., Lainscsek, C.,Schuerrer F., Maquet J. 1998. A nine-dimensional Lorenz system to study high dimensional chaos, J. Phys. A: Math. Gen, Vol. 31, pp. 7121-7139. DOI: 10.1088/0305-4470/31/34/015.

[3] Kouagou J.N., Dlamini,P. G., Simelane S. M. 2020. On the multi-domain compact finite difference relaxation method for high dimensional chaos: The nine-dimensional Lorenz system, Alexandria Engineering Journal, Vol. 59, pp. 2617-2625. DOI: 10.1016/j.aej.2020.04.025.

[4] Mahmoud, E. E., Higazy, M., Al-Harthi, T. M. 2019. A new nine-dimensional chaotic Lorenz system with quaternion Variables: Complicated dynamics, electronic circuit design, anti-anticipating synchronization, and chaotic masking communication application, Mathematics, Vol. 7, pp. 877. DOI: 10.3390/math7100877.

[5] Dlamini, P., and Simelane, S. 2021. An Efficient Spectral Method-based Algorithm for Solving a High-dimensional Chaotic Lorenz System, Journal of Applied and Computational Mechanics, Vol 7, pp. 225-234. DOI: 10.22055/JACM.2020.34364.2393.

[6] Shen, B. W., Reyes, T., Faghih-Naini, S. 2018. Coexistence of chaotic and non-chaotic orbits in a new nine-dimensional Lorenz model In Chaotic Modeling and Simulation International Conference, Springer, June, Cham, 239-255.

[7] Ashyralyev A. and Sobolevskii P. E. 2004. New Difference Schemes for Partial Differential Equations. ss 1-443. Springer Science & Business Media, Birkhauser, Basel.

[8] Adiyaman M. E. High Order Aprroach for Solving Chaotic and Hyperchaotic Problems, Hacettepe Journal of Mathematics and Statistics, Article in Press, DOI:10.15672/hujms.88515.

[9] Adiyaman M. E. and Somali S. 2010. Taylor's Decomposition on two points for one-dimensional Bratu problem, Numer. Methods Partial Differential Eq., Vol. 26, pp. 412-425. DOI: 10.1002/num.20443.

[10] Adiyaman M. E. and Somali S. 2012. A new approach for linear eigenvalue problems and nonlinear Euler buckling problem, Abstract and Applied Analysis, Vol. 2012, Article ID 697013.