Mathematical Sciences And Applications E-Notes
Volume 2 No. 2 pp. 10–21 (2013) ©MSAEN

ANALYSIS OF TWO HETEROGENEOUS SERVER QUEUEING MODEL WITH BALKING, RENEGING AND FEEDBACK

AMINA ANGELIKA BOUCHENTOUF, MOKHTAR KADI, ABBES RABHI

(Communicated by Xiao-Jun YANG)

Abstract. This paper presents an analysis of an $M/M/2/N$ (capacity N) queueing system with balking, reneging, feedback and two heterogeneous servers. By using the Markov process method, we first develop the equations of the steady state probabilities. Then, we give some performance measures of the system. Finally, we present some numerical examples to demonstrate how the various parameters of the model influence the behavior of the system.

1. Introduction

In real life, many queueing situations arise in which there may be a tendency for customers to be discouraged by a long queue. As a result, the customers either decide not to join the queue (i.e. balk) or depart after joining the queue without getting service due to impatience (i.e. renege). Balking and reneging are not only common phenomena in queues arising in daily activities, but also in various machine repair models.

Many practical queueing systems especially those with balking and reneging have been widely applied to many real-life problems, such as the situations involving impatient telephone switchboard customers, the hospital emergency rooms handling critical patients, and the inventory systems with storage of perishable goods [11]. In this paper, we consider an $M/M/2/N$ queueing system with balking, reneging and feedback. Queueing systems with balking, reneging, or both have been studied by many researchers. Haight [6] first considered an $M/M/1$ queue with balking. An $M/M/1$ queue with customers reneging was also proposed by Haight [7]. The combined effects of balking and reneging in an $M/M/1/N$ queue have been investigated by Ancker and Gafarian [2, 3]. Abou-EI-Ata and Hariri [1] considered the multiple servers queueing system $M/M/c/N$ with balking and reneging. Wang and Chang [16] extended this work to study an $M/M/c/N$ queue with balking, reneging and server breakdowns.

Date: Received: September, 16, 2013; Accepted: October, 19, 2013.
2010 *Mathematics Subject Classification.* firstly 60K25, secondly 68M20, thirdly 90B22.
Key words and phrases. Balking, reneging, feedback, heterogeneous servers, steady state solution, performance measures.

Feedback in queueing literature represents customer dissatisfaction because of inappropriate quality of service. In case of feedback, after getting partial or incomplete service, customer retries for service. In computer communication, the transmission of protocol data unit is sometimes repeated due to occurrence of an error. This usually happens because of non-satisfactory quality of service. Rework in industrial operations is also an example of a queue with feedback. Takacs [14] studied queue with feedback to determine the stationary process for the queue size and the first two moments of the distribution function of the total time spent in the system by a customer. In [5] D'Avignon and Disney studied single server queues with state dependent feedback. Santhakumaran and Thangaraj [12] considered a single server feedback queue with impatient and feedback customers, they studied $M/M/1$ queueing model for queue length at arrival epochs and obtained result for stationary distribution, mean and variance of queue length. Thangaraj, and Vanitha [15] obtained transient solution of $M/M/1$ feedback queue with catastrophes using continued fractions, the steady-state solution, moments under steady state and busy period analysis were calculated. Ayyapan et. al [4] studied $M/M/1$ retrial queueing system with loss and feedback under non preemptive priority service by matrix geometric method. Kumar and Sharma [8] studied a single server queueing system with retention of reneged customers. Kumar and Sharma [9] studied a single server queueing system with retention of reneged customers and balking. Sharma and Kumar [13] considered a single server, finite capacity Markovian feedback queue with reneging, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. Mahdy El-Paoumy and Hossam Nabwey [10] studied the $M/M/2/N$ queue with general balk

In this paper, we consider an $M/M/2/N$ queueing system with two heterogeneous servers, finite capacity Markovian feedback queueing system with reneging, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. The reneging times are assumed to be exponentially distributed. After the completion of service, each customer may rejoin the system as a feedback customer for receiving another regular service with probability α or he can leave the system with probability β where $\alpha + \beta = 1$. A reneged customer can be retained in many cases by employing certain convincing mechanisms to stay in the queue for completion of his service. Thus, a reneged customer can be retained in the queueing system with some probability γ and may leave the queue without receiving service with probability $\delta = 1 - \gamma$.

This paper is organized as follows. At first we give a description of the queueing model. Then, we derive the steady-state equations by the Markov process method, and present a procedure for calculating the steady-state probabilities. After that, we give some performance measures of the system. Finally, some numerical examples are presented to demonstrate how the various parameters of the model influence the behavior of the system.

2. Model Description and Notations

Consider an $M/M/2/N$ queue with instantaneous Bernoulli feedback with reneged customers and retention of reneged customers. Capacity of the system is taken as finite (capacity N). Customers arrive at the service station one by one according to Poisson stream with arrival rate λ. There is a two heterogeneous servers which provide service to all arriving customers. Service times are independently and

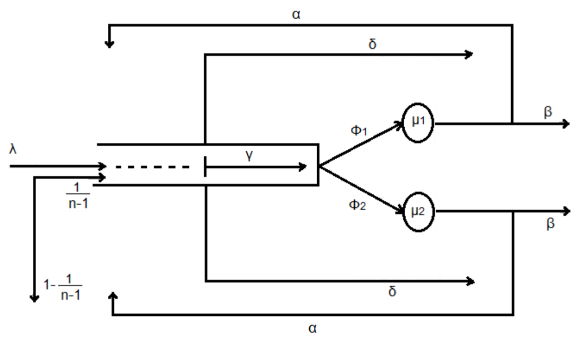


FIGURE 1. The model

identically distributed exponential random variables with rates μ_i, $i = 1, 2$. After completion of each service, the customer can either join at the end of the queue with probability α or he can leave the system with probability β where $\alpha + \beta = 1$. The customers both newly arrived and those that are fed back are served in order in which they join the tail of original queue. We do not distinguish between the regular arrival and feedback arrival. The customers are served according to first come, first served rule. The customer in queue (regular arrival or feedback arrival) may become impatient when the service is not available for a long time. In fact, each customer, upon arrival, activates an individual timer, which follows an exponential distribution with parameter η. This time is reneging time of an individual customer after which customer either decide to abandon the queue with probability $\delta(= 1 - \gamma)$ or retained with complimentary probability γ where $\delta + \gamma = 1$. We suppose that there are n customers in the system $(n > 2)$, the arriving customer joins the system with certain balking probability. It is assumed that an arriving customer balks with probability 1-$\frac{1}{n-1}$, where n is the number of customers in system and therefore joins the system with probability $\frac{1}{n-1}$.

The customers are served according to the following discipline:

• If the two servers are busy the customers wait in the queue.

• If one server is free , the head customer in the queue goes to it.

• If both servers are free, the head customer of the queue chooses server 1 with probability ϕ_1 and server 2 with probability ϕ_1, where $\phi_1 + \phi_2 = 1$

3. The Steady-State Equations and Their Solution

In this part of paper, we derive the steady-state probabilities by the Markov process method. Let P_n be the probability that there are n customers in the system. Applying the Markov process theory, we obtain the following set of steady-state equations

$P_{00} = \mathbb{P}$(there is no customer in the system),

$P_{10} = \mathbb{P}$(there is one customer being served by server 1),

$P_{01} = \mathbb{P}$(there is one customer being served by server 2),

$P_n =$ (there are n customers in the system), $n = 2, 3, ..., N$.

Also, $P_0 = P_{00},\ P_1 = P_{10} + P_{01}$ and $P_2 = P_{11}$

A system of difference differential equations satisfied by the $M/M/2/N$ queue with reneged customers, balking and feedback are modelled as a continuous time Markov chain

$$\lambda P_0 = \beta\mu_1 P_{10} + \beta\mu_2 P_{01},\ \ n = 0 \tag{3.1}$$

$$\begin{cases} (\lambda + \beta\mu_1)P_{10} \ = \ \beta\mu_2 P_{11} + \lambda\phi_1 P_0, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 1 \\ (\lambda + \beta\mu_2)P_{01} \ = \ \beta\mu_1 P_{11} + \lambda\phi_2 P_0, \end{cases} \tag{3.2}$$

(3.3)
$$\left(\tfrac{\lambda}{n} + \beta(\mu_1 + \mu_2) + (n-2)\eta\delta\right) P_n \ = \ \left(\beta(\mu_1 + \mu_2) + (n-1)\eta\delta\right) P_{n+1} + \tfrac{\lambda}{n-1} P_{n-1},$$
$$2 \le n \le N - 1.$$

$$\tfrac{\lambda}{N-1} P_{N-1} = \left(\beta(\mu_1 + \mu_2) + (N-2)\eta\delta\right) P_N,\ \ n = N. \tag{3.4}$$

Solving relation (3.2) and using (3.1) one can easily deduce that:

$$P_{10} = \tfrac{\lambda}{\beta\mu_1}\left(\tfrac{\lambda + \beta(\mu_1+\mu_2)\phi_1}{2\lambda + \beta(\mu_1+\mu_2)}\right) P_0. \tag{3.5}$$

$$P_{01} = \tfrac{\lambda}{\beta\mu_2}\left(\tfrac{\lambda + \beta(\mu_1+\mu_2)\phi_2}{2\lambda + \beta(\mu_1+\mu_2)}\right) P_0. \tag{3.6}$$

Therefore

$$P_1 = \tfrac{\lambda}{\beta\mu_1\mu_2}\left(\tfrac{\lambda(\mu_1+\mu_2) + \beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda + \beta(\mu_1+\mu_2))}\right) P_0. \tag{3.7}$$

Solving iteratively equations (3.2)-(3.4), we get

$$P_n = \tfrac{1}{(n-1)!}\prod_{k=2}^{n}\left(\frac{\lambda}{\beta(\mu_1+\mu_2) + (k-2)\eta\delta}\right) P_1,\ \ 2 \le n \le N. \tag{3.8}$$

Consequently

$$\begin{aligned} P_n \ = \ & \tfrac{\lambda}{\beta\mu_1\mu_2}\left(\tfrac{\lambda(\mu_1+\mu_2) + \beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda + \beta(\mu_1+\mu_2))}\right) \\ & \tfrac{1}{(n-1)!}\prod_{k=2}^{n}\left(\frac{\lambda}{\beta(\mu_1+\mu_2) + (k-2)\eta\delta}\right) P_0,\ \ 2 \le n \le N. \end{aligned} \tag{3.9}$$

Using the normalization condition, $\displaystyle\sum_{n=0}^{N} P_n = 1$, we get

$$\begin{aligned} P_0 \ = \ & \Bigg[1 + \left(\tfrac{\lambda}{\beta\mu_1\mu_2}\left(\tfrac{\lambda(\mu_1+\mu_2) + \beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda + \beta(\mu_1+\mu_2))}\right)\right) \\ & \left(1 + \sum_{n=2}^{N}\frac{1}{(n-1)!}\prod_{k=2}^{n}\left(\frac{\lambda}{\beta(\mu_1+\mu_2) + (k-2)\eta\delta}\right)\right)\Bigg]^{-1} \end{aligned} \tag{3.10}$$

4. Performance Measures

Using the steady state probability given above, we can derive some important performance measures, such as the expected number of customers in the system L_s, the expected number of customers in the queue L_q, the expected waiting time of customers in the System W_s, the expected waiting time of customers in the queue W_q and the expected number of customers served $\mathbb{E}$(Customer Served).

- The expected system size.

$$\begin{aligned} L_s &= \sum_{n=1}^{N} nP_n \\ &= \left[\frac{\lambda}{\beta\mu_1\mu_2} \left(\frac{\lambda(\mu_1+\mu_2)+\beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda+\beta(\mu_1+\mu_2))} \right) \right. \\ &\left. \left(1 + \sum_{n=2}^{N} \frac{n}{(n-1)!} \prod_{k=2}^{n} \left(\frac{\lambda}{\beta(\mu_1+\mu_2)+(k-2)\eta\delta} \right) \right) \right] P_0 \end{aligned}$$

- The expected queue length.

$$\begin{aligned} L_q &= \sum_{n=3}^{N} (n-2)P_n \\ &= \left[\frac{\lambda}{\beta\mu_1\mu_2} \left(\frac{\lambda(\mu_1+\mu_2)+\beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda+\beta(\mu_1+\mu_2))} \right) \right. \\ &\left. \left(\sum_{n=3}^{N} \frac{n-2}{(n-1)!} \prod_{k=2}^{n} \left(\frac{\lambda}{\beta(\mu_1+\mu_2)+(k-2)\eta\delta} \right) \right) \right] P_0 \end{aligned}$$

- The expected waiting time in the System.

$$\begin{aligned} W_s &= L_s/\lambda \\ &= \left[\frac{1}{\beta\mu_1\mu_2} \left(\frac{\lambda(\mu_1+\mu_2)+\beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda+\beta(\mu_1+\mu_2))} \right) \right. \\ &\left. \left(1 + \sum_{n=2}^{N} \frac{n}{(n-1)!} \prod_{k=2}^{n} \left(\frac{\lambda}{\beta(\mu_1+\mu_2)+(k-2)\eta\delta} \right) \right) \right] P_0 \end{aligned}$$

- The expected waiting time in the queue.

$$\begin{aligned} W_q &= L_q/\lambda \\ &= \left[\frac{1}{\beta\mu_1\mu_2} \left(\frac{\lambda(\mu_1+\mu_2)+\beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda+\beta(\mu_1+\mu_2))} \right) \right. \\ &\left. \left(\sum_{n=3}^{N} \frac{n-2}{(n-1)!} \prod_{k=2}^{n} \left(\frac{\lambda}{\beta(\mu_1+\mu_2)+(k-2)\eta\delta} \right) \right) \right] P_0 \end{aligned}$$

• The expected number of customers served.

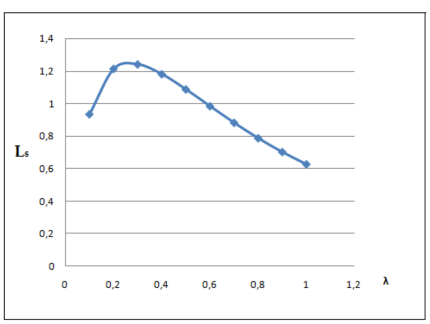
$$\begin{aligned}\mathbb{E}(\text{Customer served}) &= \sum_{n=1}^{N} n(\mu_1+\mu_2)\beta P_n \\ &= \tfrac{\lambda(\mu_1+\mu_2)}{\mu_1\mu_2}\Bigg[\left(\tfrac{\lambda(\mu_1+\mu_2)+\beta(\mu_1+\mu_2)(\mu_1\phi_2+\mu_2\phi_1))}{(2\lambda+\beta(\mu_1+\mu_2))}\right) \\ &\quad \left(1+\sum_{n=2}^{N}\frac{n}{(n-1)!}\prod_{k=2}^{n}\left(\frac{\lambda}{\beta(\mu_1+\mu_2)+(k-2)\eta\delta}\right)\right)\Bigg]P_0\end{aligned}$$

5. Numerical Results

In this part of this paper, we present some numerical examples to show the impact of different parameters and its relationship with the expected system size, the expected queue length, the expected waiting time in the system ,the expected waiting time in the queue and the expected number of customers served.

Firstly, Let us present the evolution of the system when λ varies from 0.1 to 1, $N=6$, $\delta=0.2$, $\mu_1=7$, $\mu_2=4$, $\beta=0.01$, $\eta=0.1$, $\phi_1=0.3$, $\phi_2=0.7$

λ	L_s	L_q	W_s	W_q	$\mathbb{E}(CustomerServed)$
0.1	0.933977962	0.431967748	9.33977962	4.319677481	0.146387405
0.2	1.213010966	0.34877451	6.065054832	1.743872551	0.181817771
0.3	1.240585123	0.309719988	4.135283745	1.032399959	0.210161695
0.4	1.180298512	0.285826868	2.950746281	0.714567169	0.243151735
0.5	1.086840648	0.265819453	2.173681296	0.531638905	0.278733228
0.6	0.983377705	0.24701997	1.638962842	0.411699951	0.314391502
0.7	0.881352598	0.229088382	1.25907514	0.327269118	0.34835838
0.8	0.786437572	0.212222905	0.983046965	0.265278631	0.379619449
0.9	0.701064439	0.196642663	0.778960487	0.218491847	0.407736074
1	0.625821636	0.18246406	0.625821636	0.18246406	0.432651425



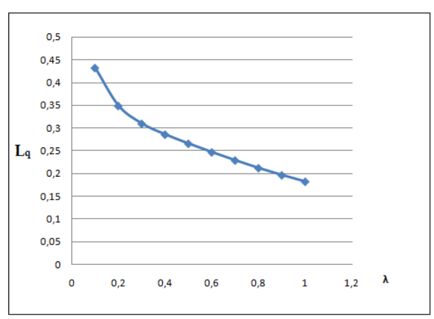
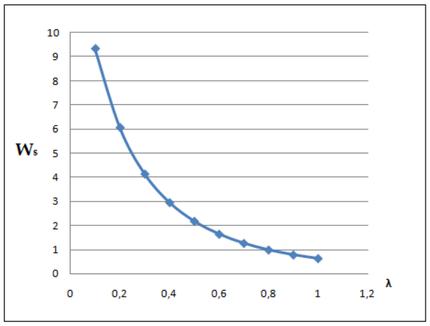


FIGURE 2. λ versus L_s and λ versus L_q



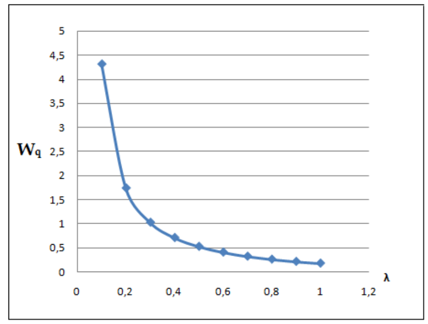


FIGURE 3. λ versus W_s and λ versus W_q

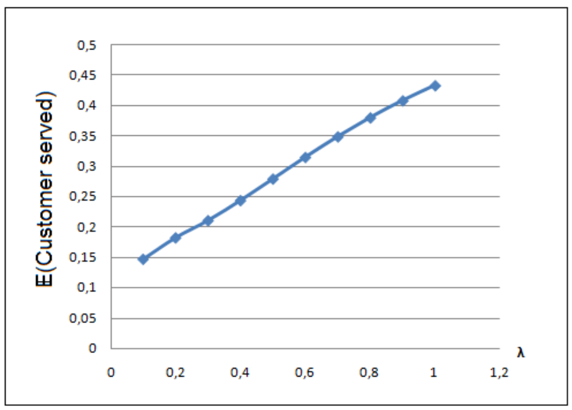


FIGURE 4. λ versus $\mathbb{E}$(Customer Served)

Figure 2, shows that along the increase of λ the expected number of customers in the system increases, then from the value of $\lambda = 0.4$, it starts to decrease,while the queue length, the expected waiting time in the queue and in the system decrease along the increase of λ, figure 3, because of balking and reneging discipline and finally the expected number of customers served increases with the increase of λ, figure 4. All these results agree absolutely with our intuition.

Now, let us present the relationship between δ (the probability to leave the queue without receiving service) and different performance measures of the system.

At first, we fix the maximum number of customers in the system $N = 5$, the arrival rate $\lambda = 0.5$, the service rate at server 1, $\mu_1 = 7$, the service rate at server 2, $\mu_2 = 4$, the probability that the customer leave the system without having a feedback $\beta = 0.7$, reneging time rate $\eta = 0.1$, the probability that customer chooses server 1, $\phi_1 = 0.3$, the probability that customer chooses server 2, $\phi_2 = 0.7$. Then we vary δ from 0 to 1 in increments of 0.1. The numerical results are summarized in the following table:

δ	L_s	L_q	W_s	W_q	$\mathbb{E}(CustomerServed)$
0	0,135312293	0,132785988	0,270624587	0,265571976	1,158397488
0.1	0,1353097	0,132786683	0,2706194	0,265573366	1,158399427
0.2	0,135307114	0,132787376	0,270614228	0,265574752	1,158401363
0.3	0,135304535	0,132788068	0,270609069	0,265576136	1,158403294
0.4	0,135301962	0,132788758	0,270603924	0,265577516	1,158405222
0.5	0,135299397	0,132789446	0,270598793	0,265578892	1,158407146
0.6	0,135296838	0,132790133	0,270593676	0,265580265	1,158409066
0.7	0,135294286	0,132790818	0,270588572	0,265581635	1,158410982
0.8	0,135291741	0,132791501	0,270583481	0,265583002	1,158412894
0.9	0,135289202	0,132792182	0,270578404	0,265584365	1,158414803
1	0,13528667	0,132792862	0,270573341	0,265585725	1,158416708

For the second result, for each value of $N = 5$, $\lambda = 0.5$, $\mu_1 = 7$, $\mu_2 = 4$, $\beta = 0.25$, $\eta = 0.1$, $\phi_1 = 0.3$, $\phi_2 = 0.7$ selected, we vary δ from 0 to 1 in increments of 0.1. The numerical results are summarized in the following table:

δ	L_s	L_q	W_s	W_q	$\mathbb{E}(CustomerServed)$
0	0,331603365	0,288830411	0,663206731	0,577660822	1,090550257
0.1	0,331480297	0,288859098	0,662960594	0,577718195	1,090644402
0.2	0,331358181	0,28888763	0,662716363	0,57777526	1,090737894
0.3	0,331237006	0,288916009	0,662474012	0,577832018	1,090830738
0.4	0,331116759	0,288944235	0,662233518	0,57788847	1,090922942
0.5	0,330997428	0,288972308	0,661994856	0,577944616	1,091014511
0.6	0,330879002	0,289000229	0,661758003	0,578000459	1,091105453
0.7	0,330761468	0,289027999	0,661522937	0,578055999	1,091195773
0.8	0,330644817	0,289055618	0,661289634	0,578111237	1,091285477
0,9	0.330529037	0,289083087	0,661058074	0,578166175	1,091374571
1	0.330414117	0,289110407	0,660828234	0,578220813	1,091463062

These two tables show that L_s and W_s decrease with the increase of δ, while L_q and W_q increase along the increase of δ, the expected number of customers served is increasing.

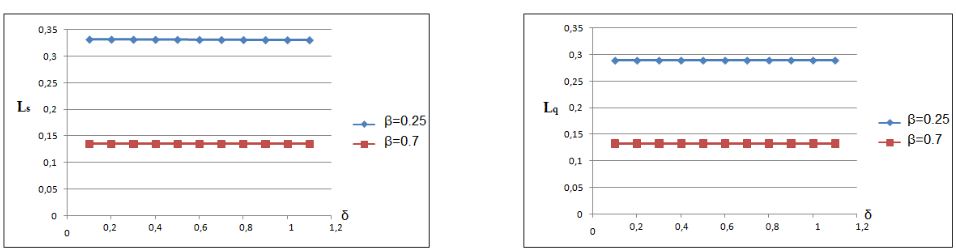


FIGURE 5. δ versus L_s and δ versus L_q

Compared results are shown in figures 5, 6, 7, these later show that when the probability of non-feedback is big, $\beta = 0.7$, the number of customers in the system, in the queue is less than that when β is small, $\beta = 0.25$, and consequently the

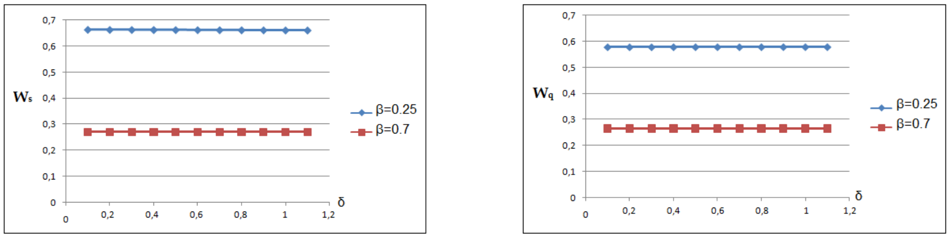


FIGURE 6. δ versus W_s and δ versus W_q

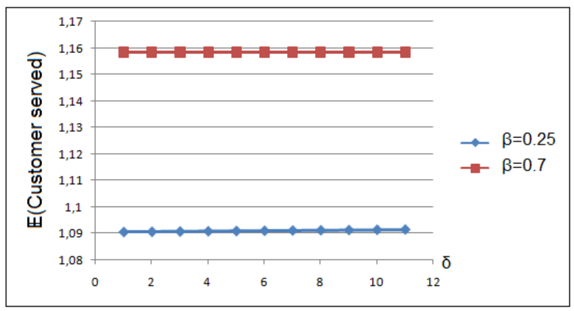


FIGURE 7. δ versus $\mathbb{E}$(Customer Served)

expected waiting time in the system, the expected waiting time in the queue, in the first case is less than the second one, the expected number of customers served in the first case is bigger than the second one. These results agree nicely with our intuition.

Finally, we present the effect of of non-feedback probability β, we evaluate different measure performances at its different values while $N = 7$, $\lambda = 2$, $\delta = 0.75$ $\mu_1 = 7$, $\mu_2 = 4$, $\eta = 0.1$, $\phi_1 = 0.3$, $\phi_2 = 0.7$. See the following table:

β	L_s	L_q	W_s	W_q	$\mathbb{E}(CustomerServed)$
0,1	1,181243839	0,334005347	0,59062192	0,16700267	1,88176221
0,2	0,940511222	0,413229209	0,47025561	0,2066146	2,90935318
0,3	0,737499147	0,436879606	0,36874957	0,2184398	3,52937514
0,4	0,604826516	0,426653359	0,30241326	0,21332668	3,87112936
0,5	0,515591814	0,403652243	0,25779591	0,20182612	4,06933007
0,6	0,451791792	0,377567405	0,2258959	0,1887837	4,1938374
0,7	0,403652144	0,352145624	0,20182607	0,17607281	4,27813706
0,8	0,365778891	0,32866691	0,18288945	0,16433345	4,33888292
0,9	0,335022606	0,307435757	0,1675113	0,15371788	4,38488782
1	0,309431653	0,288383991	0,15471583	0,144192	4,42112069

Now, we evaluate different measure performances at different values of non-feedback probability β, while $N = 7$, $\lambda = 2$, $\delta = 0.75$ $\mu_1 = 14$, $\mu_2 = 10$, $\eta = 0.1$,

$\phi_1 = 0.3$, $\phi_2 = 0.7$. The numerical results are summarized in the following table:

β	L_s	L_q	W_s	W_q	$\mathbb{E}(CustomerServed)$
0,1	0,877125577	0,411818314	0,43856279	0,20590916	2,98166869
0.2	0,547837429	0,403817466	0,27391871	0,20190873	3,81108064
0,3	0,404824653	0,34705907	0,20241233	0,17352953	4,05151848
0,4	0,32522401	0,296927802	0,162612	0,1484639	4,14885349
0,5	0,273399553	0,257559104	0,13669978	0,12877955	4,19974158
0,6	0,236487835	0,22675818	0,11824392	0,11337909	4,23111321
0,7	0,208663127	0,202268759	0,10433156	0,10113438	4,25262326
0,8	0,186850741	0,182427073	0,09342537	0,09121354	4,26845414
0,9	0,169250844	0,166064974	0,08462542	0,08303249	4,2806916
1	0,154729651	0,152359846	0,07736483	0,07617992	4,29049251

The above tables show the effect of non-feedback probability β for our model with balking, reneging and feedback. When the probability β of non-feedback or returning to the system increases, L_s, L_q, W_s, W_q decrease, while $\mathbb{E}(CustomerServed)$ increases, which agrees perfectly with the intuition.

Compared results of the above two tables are shown in the following figures

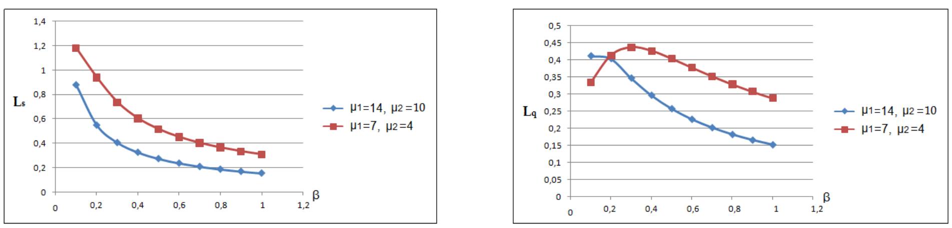


FIGURE 8. β versus L_s and β versus L_q

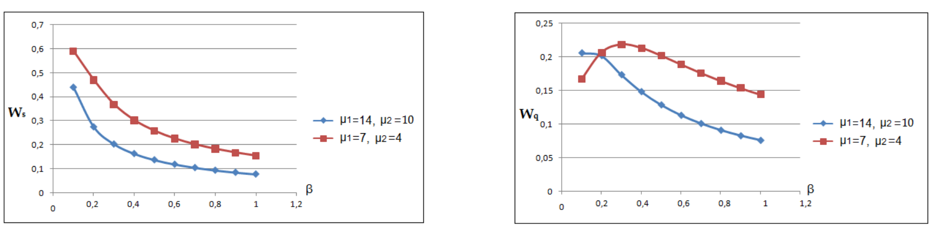


FIGURE 9. β versus W_s and β versus W_q

Figures 8, 9, 10 give a significative result for our system. When the mean service times are small, $\mu_1 = 14$, $\mu_2 = 10$ the expected number of customer in the system, in the queue is less than that when the mean service times is big $\mu_1 = 7$, $\mu_2 = 4$, the expected waiting time in the system, in the queue in the first case is less than

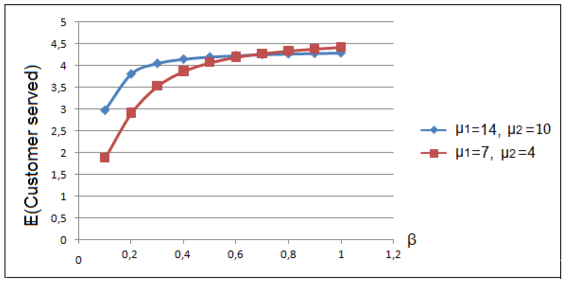


FIGURE 10. β versus $\mathbb{E}$(Customer Served)

in the second one, and finally the expected number of customers served is large in the first case than the second one.

As a conclusion, we conclude that all the figures indicate that the phase-merging algorithm is reasonably effective in approximating L_s, L_q, W_s, W_q, and $\mathbb{E}$ (Customer Served) for all values of λ, μ_1, μ_2, δ and β.

6. Conclusion

The aim of this paper was to obtain the analytical solution of the $M/M/2/N$ queue (capacity N) with balking, reneging, feedback and two heterogeneous servers. The steady state probabilities and some measures of effectiveness of the system were obtained. Moreover, some numerical values of the analytical results were also obtained.

References

[1] Abou El-Ata, M. O. and Hariri, A. M. A. The $M/M/c/N$ queue with balking and reneging. Computers and Operations Research. 19 (1992), No. 13 713-716.

[2] Ancker, Jr., C. J. and Gafarian, A. V. Some queuing problems with balking and reneging: I. Operations Research. 11 (1963), No. 1, 88-100.

[3] Ancker, Jr., C. J. and Gafarian, A. V. Some queuing problems with balking and reneging: II. Operations Research. 11 (1963), No. 6, 928-937.

[4] Ayyapan, G., Muthu Ganapathi Subramanian, A. and Sekar, G.$M/M/1$ Retrial Queueing System with Loss and Feedback under Non-pre-emptive Priority Service by Matrix Geometric Method, Applied Mathematical Sciences.4,(2010), No. 48, 2379-2389.

[5] D'Avignon, G.R. and Disney, R.L., Single Server Queue with State Dependent Feedback,INFOR. 14 (1976), 71-85.

[6] Haight, F. A. Queueing with balking. Biometrika. 44 (1957), 360-369.

[7] Haight, F. A. Queueing with reneging. Metrika. 2 (1959) 186-197.

[8] Kumar, R. and Sharma, S.K. M/M/1/N Queuing System with Retention of Reneged Customers, Pakistan Journal of Statistics and Operation Research. 8(2012), 859-866.

[9] Kumar, R. and Sharma,S.K. An M/M/1/N Queuing Model with Retention of reneged customers and Balking, American Journal of Operational Research. 2(2012), No. 1, 1-5.

[10] Mahdy S. El-Paoumy and Hossam A. Nabwey, The Poissonian Queue with Balking Function, Reneging and two Heterogeneous Servers. International Journal of Basic & Applied Sciences IJBAS-IJENS. 11 (2011), No. 6, 149-152.

[11] Robert, E. Reneging phenomenon of single channel queues. Mathematics of Operations Research. 4 (1979), No. 2, 162-178.

[12] Santhakumaran, A. and Thangaraj, V. A Single Server Queue with Impatient and Feedback Customers,Information and Management Science. 11, (2000), No. 3, 71-79.

[13] Sharma S. K and Kumar. R, A Markovian Feedback Queue with Retention of Reneged Customers and Balking. AMO-Advanced Modeling and Optimization. 14 (2012), No. 3, 681-688.

[14] Takacs, L. A Single Server Queue with Feedback,The Bell System Tech. Journal. 42 (1963), 134-149.

[15] Thangaraj, V. and Vanitha, S. On the Analysis of M/M/1 Feedback Queue with Catastrophes using Continued Fractions,International Journal of Pure and Applied Mathematics. 53(2009), No. 1, 131-151.

[16] Wang, K-H. and Chang, Y-C. Cost analysis of a finite M/M/R queueing system with balking, reneging and server breakdowns. Mathematical Methods of Operations Research 56 (2002), No.2, 169-180.

Department of Mathematics, Djillali Liabes University of Sidi Bel Abbes, B. P. 89, Sidi Bel Abbes 22000, Algeria.

E-mail address: bouchentouf_amina@yahoo.fr

Department of Mathematics and Computer Sciences, Moulay Tahar University, Saida, B.P. 138, En-Nasr Saida 20 000, Algeria.

E-mail address: kadi1969@yahoo.fr

Department of Mathematics, Djillali Liabes University of Sidi Bel Abbes, B. P. 89, Sidi Bel Abbes 22000, Algeria.

E-mail address: rabhi_abbes@yahoo.fr