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THE DETERMINANTS OF CIRCULANT AND SKEW-CIRCULANT MATRICES WITH TRIBONACCI NUMBERS

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Abstract. Determinant computation has an important role in mathematics. It can be computed by using some different methods but it needs huge amount of operations to compute determinant of a matrix. For instance, using Gauss elimination method, it is neccessary about $2n^3/3$ arithmetic steps of a matrix of order n. Therefore, determinant computation has been considered for special matrices with special entries. Similarly, the permanent of a matrix is an analog of determinant where all the signs in the expansion by minors are taken as positive. This study considers the determinant of circulant matrices whose entries are Tribonacci numbers. Some relations with the permanent are established.

1. Introduction

The Tribonacci sequence is given recursively as

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}\,, \qquad \text{with } T_0 = 0 \text{ and } T_1 = T_2 = 1,$$

for $n > 2$ [8]. This is the sequence $A000073$ in the *The On-Line Encyclopedia of Integer Sequences* [11] and the first few values are

$$1, 1, 2, 4, 7, 13, 24, 44, 81 \ldots .$$

A circulant matrix of order n, $C_n := \operatorname{circ}(c_0, c_1, \ldots, c_{n-1})$, associated with the numbers $c_0, c_1, \ldots, c_{n-1}$, is defined as

$$C_n = \begin{pmatrix} c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & \cdots & c_0 & c_1 \\ c_1 & c_2 & \cdots & c_{n-1} & c_0 \end{pmatrix}.$$

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where each row is a cyclic shift of the row above it [3]. The eigenvalues and eigenvectors of C_n are well known [13]:

$$\lambda_j = \sum_{k=0}^{n-1} c_k \omega^{jk}\,, \quad j = 0, 1, \ldots, n-1\,,$$

where $\omega = \exp(\frac{2\pi i}{n})$ and $i = \sqrt{-1}$, and the corresponding eigenvectors are

$$x_j = (1, \omega^j, \omega^{2j}, \ldots, \omega^{(n-1)j})^T\,, \quad j = 0, 1, \ldots, n-1\,.$$

Therefore, we can write determinant of a nonsingular circulant matrix as:

$$\det C_n = \prod_{j=0}^{n-1} \left(\sum_{k=0}^{n-1} c_k \omega^{jk} \right)$$

where $k = 0, 1, \ldots, n-1$.

A skew circulant matrix with the first row $(a_0, a_1, ..., a_{n-1})$ is meant to be a square matrix of the form

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_{n-2} & a_{n-1} \\ -a_{n-1} & a_0 & \ldots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_2 & -a_3 & \ldots & a_0 & a_1 \\ -a_1 & -a_2 & \ldots & -a_{n-1} & a_0 \end{pmatrix}$$

denoted by $\mathrm{SCirc}(a_0, a_1, ..., a_{n-1})$ [7].

Circulant matrices have applications in many areas such as signal processing, coding theory, image processing and among others. Numerical solutions of elliptic and parabolic partial differential equations with periodic boundary conditions often involve linear systems associated with circulant matrices [2, 14].

Some special matrices involving well known number sequences have been recently investigated in different contexts. For example, Shen et al. [10] defined two circulant matrices whose elements are Fibonacci and Lucas numbers and derived formulas for determinants and inverses of the defined matrices. Similarly, Bozkurt and Tam [1] obtained determinant and inverse formulas for circulant matrices in terms of Jacobsthal and Jacobsthal-Lucas numbers. Alptekin et.al. [15] obtained some results for circulant and semi-circulant matrices. Moreover, there are many interesting relationships between determinants of matrices and such number sequences. For example, in [12] it is investigated the relationship between permanents of one type of Hessenberg matrix and the Pell and Perrin numbers.

In this note, we considered circulant matrices whose entries are Tribonacci numbers. We obtain a formula for the determinant of these matrices. Then we analyze a family of matrices with permanents equal to the Tribonacci numbers. First, we will give a lemma to simplify the derivations. We remind that the Chebyshev polynomials of second kind satisfying $\{U_n(x)\}_{n\geqslant 0}$, where each $U_n(x)$ is of degree n, satisfy the three-term recurrence relations

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)\,, \quad \text{for all } n = 1, 2, \ldots\,,$$

with initial conditions $U_0(x) = 1$ and $U_1(x) = 2x$, or, equivalently,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}\,, \quad \text{with } x = \cos\theta \quad (0 \leqslant \theta < \pi),$$

for all $n = 0, 1, 2 \ldots$. It is also standard (see e.g. [5]) that

$$\det \begin{pmatrix} a & b & & \\ c & \ddots & \ddots & \\ & \ddots & \ddots & b \\ & & c & a \end{pmatrix}_{n\times n} = \left(\sqrt{bc}\right)^n U_n\left(\frac{a}{2\sqrt{bc}}\right).$$

Lemma 1.1. *If*

$$A_n = \begin{pmatrix} d_1 & d_2 & d_3 & \cdots & d_{n-1} & d_n \\ a & b & & & & \\ c & a & b & & & \\ & c & a & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & a & b \end{pmatrix} \tag{1.1}$$

then

$$\det A_n = \sum_{k=1}^{n} d_k b^{n-k} \left(-\sqrt{bc}\right)^{k-1} U_{k-1}\left(\frac{a}{2\sqrt{bc}}\right), \tag{1.2}$$

where $U_k(x)$ is the kth Chebyshev polynomial of second kind.

Proof. It is clear that

$$\det A_n = b \det A_{n-1} + (-)^{n-1} d_n \left(\sqrt{bc}\right)^{n-1} U_{n-1}\left(\frac{a}{2\sqrt{bc}}\right). \tag{1.3}$$

Applying now (1.3) recursively to $\det A_{n-1}, \det A_{n-2}, \ldots, \det A_1$, one gets (1.2). □

1.1. **Determinant of Circulant matrices with Tribonacci numbers.** In this section, we consider the n-square circulant matrix

$$\mathcal{T}_n := \operatorname{circ}(T_1, T_2, \ldots, T_n)$$

where T_n is the nth Tribonacci number. Then we give a formula for determinants of these matrices. First, let us define n-square matrix

$$Q_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & t^{n-2} & 0 & \ldots & 0 & 1 \\ 0 & t^{n-3} & 0 & \ldots & 1 & 0 \\ 0 & t^{n-4} & 0 & & & 0 \\ \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & t & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{1.4}$$

here t is the positive root of the characteristic equation $x_n t^2 + y_n t + z_n = 0$, i.e.,

$$t = \frac{-y_n + \sqrt{y_n^2 - 4x_n z_n}}{2x_n}$$

where

$$x_n = 1 - T_{n+1},\ y_n = -(T_n + T_{n-1}), \text{ and } z_n = -T_n.$$

Then, let us consider n-square matrix P_n as below:

$$P_n = \left(\begin{array}{r|rrrrrrr|rrr}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\
\vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & \vdots & \vdots & \vdots \\
0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

It can be seen that for all $n > 3$,

$$\det(P_n) = \det(Q_n) = \left\{ \begin{array}{rll} 1, & n \equiv 1 \text{ or } 2 & \mod 4 \\ -1, & n \equiv 0 \text{ or } 3 & \mod 4. \end{array} \right.$$

where Q_n is defined in (1.4) and $\det(P_n Q_n) = 1$. By matrix multiplication, we get;

$$S_n = P_n \mathcal{T}_n Q_n \tag{1.5}$$

i.e.,

$$S_n = \left(\begin{array}{cc|cccccc}
1 & f_n & T_{n-1} & T_{n-2} & T_{n-3} & \cdots & T_3 & T_2 \\
0 & g_n & T_n - T_{n-1} & T_{n-1} - T_{n-2} & T_{n-2} - T_{n-3} & \cdots & T_4 - T_3 & T_3 - T_2 \\
0 & h_n & y_n + 1 & T_{n-3} & T_{n-4} & \cdots & T_2 & T_1 \\
\hline
0 & 0 & y_n & x_n & & & & \\
0 & 0 & z_n & y_n & x_n & & & \\
\vdots & \vdots & & \ddots & \ddots & \ddots & & \\
0 & 0 & & & z_n & y_n & x_n & \\
0 & 0 & & & & z_n & y_n & x_n
\end{array}\right)$$

where

$$\begin{aligned}
f_n &= -\sum_{i=2}^{n} z_i t^{n-i}, \\
g_n &= \sum_{i=2}^{n-1} (T_{i+1} - T_i) t^{n-i} + x_{n-1}, \\
h_n &= -\sum_{i=2}^{n-2} z_{i-1} t^{n-i} + (y_n + 1)t + z_n\,.
\end{aligned}$$

A simple application of the Laplace expansion in (1.5), determinant will be

$$|S_n| = \left|\begin{array}{c|cccccc}
g_n & T_n - T_{n-1} & T_{n-1} - T_{n-2} & T_{n-2} - T_{n-3} & \cdots & T_4 - T_3 & T_3 - T_2 \\
h_n & y_n + 1 & T_{n-3} & T_{n-4} & \cdots & T_2 & T_1 \\
\hline
0 & y_n & x_n & & & & \\
0 & z_n & y_n & x_n & & & \\
\vdots & & \ddots & \ddots & \ddots & & \\
0 & & & z_n & y_n & x_n & \\
0 & & & & z_n & y_n & x_n
\end{array}\right|$$

Multiplying the first row with $-m = \dfrac{h_n}{g_n}$ and adding it to the second row in S_n, we obtain

$$|S_n| = \left|\begin{array}{c|cccccc} g_n & T_n - T_{n-1} & T_{n-1} - T_{n-2} & & \cdots & & T_3 - T_2 \\ 0 & m(T_n - T_{n-1}) + y_n + 1 & m(T_{n-1} - T_{n-2}) + T_{n-3} & & \cdots & & m(T_3 - T_2) + T_1 \\ \hline 0 & y_n & x_n & & & & \\ 0 & z_n & y_n & x_n & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & z_n & y_n & x_n & \\ 0 & & & & z_n & y_n & x_n \end{array}\right|$$

Using Laplace expansion on the first column, we get a Hessenberg matrix in the form of (1.1). Applying Lemma 1.1, we obtain for $n \geqslant 3$

$$\begin{aligned} \det \mathcal{T}_n &= g_n[m(T_n - T_{n-1}) + y_n + 1]x_n^{n-1} + \\ &\quad + g_n \sum_{k=2}^{n-2} \left([m(T_{n-k+1} - T_{n-k}) + T_{n-k-1}]x_n^{n-k}(-\sqrt{x_n z_n})^{k-1} U_{k-1}\left(\frac{y_n}{2\sqrt{x_n z_n}}\right)\right). \end{aligned}$$

Determinant of Skew-Circulant matrices with Tribonacci numbers. In this section, we consider n-square skew circulant matrix

$$K_n := \mathrm{SCirc}\,(T_1, T_2, \ldots, T_n)$$

here T_n denotes nth Tribonacci number. Then we give a formula for determinants of these matrices. In the first step, let us define the $n \times n$ matrix

$$M_n = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & m^{n-2} & 0 & \ldots & 0 & 1 \\ 0 & m^{n-3} & 0 & \ldots & 1 & 0 \\ 0 & m^{n-4} & 0 & & & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & m & 1 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

here m is the positive root of the characteristic equation $a_n m^2 + b_n m + c_n = 0, i.e.,$

$$m := \frac{-b_n + \sqrt{b_n^2 - 4a_n c_n}}{2a_n}$$

here $a_n = 1 + T_{n+1}$, $b_n = (T_n + T_{n-1})$ and $c_n = T_n$.

Then consider n-square matrix N_n as below:

$$N_n = \left(\begin{array}{c|ccccccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 \\ \vdots & \vdots & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & \vdots & \vdots & \vdots \\ 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

It can be seen that $\det(M_n) = \det(N_n) = \begin{cases} 1, & n \equiv 1 \text{ or } 2 \mod 4 \\ -1, & n \equiv 0 \text{ or } 3 \mod 4. \end{cases}$ for $n > 3$.
Using matrix multiplication, we get

$$W_n := N_n K_n M_n$$

i.e.,
(1.6)

$$W_n = \left(\begin{array}{cc|cccccc} 1 & f_n^* & T_n & T_{n-1} & T_{n-2} & \cdots & T_3 & T_2 \\ 0 & g_n^* & T_{n-1}-T_n & T_{n-2}-T_{n-1} & T_{n-3}-T_{n-2} & \cdots & T_3-T_4 & T_2-T_3 \\ 0 & h_n^* & b_n+1 & -T_{n-3} & -T_{n-4} & \cdots & -T_2 & -T_1 \\ \hline 0 & 0 & b_n & a_n & & & & \\ 0 & 0 & c_n & b_n & a_n & & & \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \\ 0 & 0 & & & c_n & b_n & a_n & \\ 0 & 0 & & & & c_n & b_n & a_n \end{array}\right)$$

here

$$\begin{aligned} f_n^* &= \sum_{i=2}^{n} T_i m^{n-i}, \\ g_n^* &= \sum_{i=2}^{n-1} (T_i - T_{i+1}) m^{n-i} + T_n + 1, \\ h_n^* &= -\sum_{i=2}^{n-2} T_{i-1} m^{n-i} + (b_n+1)m + c_n\,. \end{aligned}$$

Using Laplace expansion in (1.6),

$$|W_n| = \left|\begin{array}{c|cccccc} g_n^* & T_{n-1}-T_n & T_{n-2}-T_{n-1} & T_{n-3}-T_{n-2} & \cdots & T_3-T_4 & T_2-T_3 \\ h_n^* & b_n+1 & -T_{n-3} & -T_{n-4} & \cdots & -T_2 & -T_1 \\ \hline 0 & b_n & a_n & & & & \\ 0 & c_n & b_n & a_n & & & \\ \vdots & & \ddots & \ddots & \ddots & & \\ 0 & & & c_n & b_n & a_n & \\ 0 & & & & c_n & b_n & a_n \end{array}\right|$$

Multiplying the first row with $s := -\dfrac{h_n^*}{g_n^*}$ and adding it to the second row in W_n

$$|W_n| = \left|\begin{array}{c|ccccc} g_n^* & T_{n-1}-T_n & T_{n-2}-T_{n-1} & \cdots & T_3-T_4 & T_2-T_3 \\ 0 & s(T_{n-1}-T_n)+b_n+1 & s(T_{n-2}-T_{n-1})-T_{n-3} & \cdots & s(T_3-T_4)-T_2 & s(T_2-T_3)-T_1 \\ \hline 0 & b_n & a_n & & & \\ \vdots & c_n & b_n & \ddots & & \\ 0 & & \ddots & \ddots & a_n & \\ 0 & & & c_n & b_n & a_n \end{array}\right|$$

whose determinant is equal to

$$= g_n^* \begin{vmatrix} s(T_{n-1}-T_n)+b_n+1 & s(T_{n-2}-T_{n-1})-T_{n-3} & \cdots & s(T_3-T_4)-T_2 & s(T_2-T_3)-T_1 \\ b_n & a_n & & & \\ c_n & b_n & \ddots & & \\ & \ddots & \ddots & a_n & \\ & & c_n & b_n & a_n \end{vmatrix}.$$

Using Lemma 1.1, we get

$$\begin{aligned} \det W_n &= g_n^* \left([s(T_{n-1}-T_n)+b_n+1]b_n^{n-1}\right) + \\ &\quad + g_n^* \sum_{k=2}^{n-2} \left([s(T_{n-k}-T_{n-k+1})-T_{n-k-1}] b_n^{n-k} (-\sqrt{b_n c_n})^{k-1} U_{k-1}\left(\frac{a_n}{2\sqrt{b_n c_n}}\right)\right). \end{aligned}$$

1.1.1. *Permanent and Tribonacci numbers.* There are many references establishing determinantal representations of well-known number sequences. For example, in [9] Lee defined the matrix

$$\mathcal{L}_n = \begin{pmatrix} 1 & 0 & 1 & & & \\ 1 & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & 1 & 1 \end{pmatrix}$$

and showed that $\operatorname{per}\mathcal{L}_n = L_{n-1}$, where L_n is the nth Lucas number. The authors [12] associated permanents of Hessenberg matrices with Pell and Perrin numbers. Recall that the permanent of a square matrix is similar to the determinant but lacks the alternating signs, i.e., for an $n \times n$ matrix $A = (a_{ij})$, $\operatorname{per}A = \sum_\pi \prod_{i=1}^n a_{i,\pi(i)}$, where the sum is over all permutations π.

In this section, we will relate the permanent of a Hessenberg matrix and the Tribonacci numbers. Let us consider the n-square lower-Hessenberg matrix:

$$H_n = \begin{cases} 2\,, & \text{if } j = i, \text{ for } i,j = 2,\dots,n-1; \\ 1\,, & \text{if } j = i+1 \text{ and } i = j = 1; \\ -1\,, & \text{if } j = i-3; \\ 0\,, & \text{otherwise,} \end{cases} \tag{1.7}$$

i.e.,

$$H_n = \begin{pmatrix} 1 & 1 & & & & & \\ 0 & 2 & 1 & & & & \\ 0 & 0 & 2 & 1 & & & \\ -1 & 0 & 0 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 0 & 2 & 1 \\ & & & -1 & 0 & 0 & 2 \end{pmatrix}.$$

Then we have the following theorem.

Theorem 1.1. *Let us consider the matrix defined in* (1.7)*. Then*

$$\operatorname{per} H_n = T_{n+1}\,, \tag{1.8}$$

where T_n is nth Tribonacci number.

Proof. The result is clear for $n = 1, 2, 3, 4$. It is not hard to see that, for $n > 4$, we have

$$\operatorname{per} H_n = 2 \operatorname{per} H_{n-1} - \operatorname{per} H_{n-4}\,.$$

This means that

$$\begin{aligned}
\operatorname{per} H_n &= \operatorname{per} H_{n-1} + 2\operatorname{per} H_{n-2} - \operatorname{per} H_{n-4} - \operatorname{per} H_{n-5} \\
&= \operatorname{per} H_{n-1} + \operatorname{per} H_{n-2} + 2\operatorname{per} H_{n-3} - \operatorname{per} H_{n-4} - \operatorname{per} H_{n-5} - \operatorname{per} H_{n-6} \\
&= \operatorname{per} H_{n-1} + \operatorname{per} H_{n-2} + \operatorname{per} H_{n-3} + \\
&\qquad + \operatorname{per} H_{n-4} - \operatorname{per} H_{n-5} - \operatorname{per} H_{n-6} - \operatorname{per} H_{n-7} \\
&\ \vdots \\
&= \operatorname{per} H_{n-1} + \operatorname{per} H_{n-2} + \operatorname{per} H_{n-3} + \operatorname{per} H_4 - \operatorname{per} H_3 - \operatorname{per} H_2 - \operatorname{per} H_1 \\
&= \operatorname{per} H_{n-1} + \operatorname{per} H_{n-2} + \operatorname{per} H_{n-3}\,.
\end{aligned}$$

Therefore, we get (1.8). □

This theorem can also be proved by using *contraction* method defined by Brualdi and Gibson in [16].

Notice that according to [4, 6], we also have

$$\det \begin{pmatrix}
1 & -1 & & & & & \\
0 & 2 & -1 & & & & \\
0 & 0 & 2 & -1 & & & \\
-1 & 0 & 0 & 2 & -1 & & \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \\
& & -1 & 0 & 0 & 2 & -1 \\
& & & -1 & 0 & 0 & 2
\end{pmatrix}_{n\times n} = T_{n+1}\,.$$

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